Friedrich-Schiller-Universität Jena Summer term 2018
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Exercises to „Symmetries in Physics“

Sheet 6

Problem 22: Universal covering of the group $\mathbf{SO}_+^\uparrow(1,3)$

The group SU(2) is the universal double covering group of SO(3). This is relevant for non-relativistic quantum mechanics. There is a similar and related double covering of the proper Lorentz group and this is relevant in relativistic quantum mechanics. Fermion fields do not transform with Lorentz transformations but rather with transformations from SL(2,$\mathbb{C}$). In this exercise we investigate the relation between the classical Lorentz group and its universal covering.

The Pauli-matrices are

$$\mathbb{1} \equiv \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

An arbitary hermitean matrix is a real linear combination of these matrices,

$$(h,\sigma) = h^\mu \sigma_\mu = \begin{pmatrix} h^0 + h^3 & h^1 - \mathrm{i}h^2 \\ h^1 + \mathrm{i}h^2 & h^0 - h^3 \end{pmatrix}.$$

- Show that $h^\mu = \frac{1}{2}\mathrm{tr}[\bar{\sigma}^\mu (h,\sigma)]$. Here we used

  $$\bar{\sigma}_0 = \sigma_0, \quad \bar{\sigma}_i = -\sigma_i, \quad i = 1,2,3.$$

  This means, that $h^\mu \to (h,\sigma)$ is a bijective map from $\mathbb{R}^4 \mapsto \{H \in \mathrm{Mat}(2,\mathbb{C}) | H = H^\dagger\}$.
- Calculate the determinant $\det(h,\sigma)$.
- Let $A \in \mathrm{SL}(2,\mathbb{C})$ be an arbitrary 2−dimensional complex matrix with determinant 1. Why is the matrix $A(h,\sigma)A^\dagger$ again a linear combination of the form (h',σ)?
- Argue that the map $h \to h'$, defined by $(h',\sigma) = A(h,\sigma)A^\dagger$, is linear and thus can be written as $h'^\mu = \Lambda^\mu{}_\nu h^\nu$. Prove that Λ is a Lorentz-transformation.
- Show that the (non-linear) map $A \to \Lambda(A)$, given by $(\Lambda h,\sigma) = A(h,\sigma)A^\dagger$, is a group homorphism $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)_+^\uparrow$. What is the kernel of this map? Which coset is then equals the Lorentz group $\mathrm{SO}(1,3)_+^\uparrow$?

Problem 23: Invariant integration

There are many way to compute the unique invariant Haar measure on a (compact) Lie group. You may choose any of these methods to solve the following problem.

Compute the Haar measure for the integration over SU(2) in the parametrization

$$U = \begin{pmatrix} \cos\vartheta\ \mathrm{e}^{\mathrm{i}\zeta} & -\sin\vartheta\ \mathrm{e}^{\mathrm{i}\eta} \\ \sin\vartheta\ \mathrm{e}^{-\mathrm{i}\eta} & \cos\vartheta\ \mathrm{e}^{-\mathrm{i}\zeta} \end{pmatrix}.$$

Normalize the integration measure, such that Vol(SU(2))= 1.

Problem 24: Relation between O(4) and SU(2)×SU(2)

Groups may look different but are actually (almost) identical. We have seen examples of finite groups at the beginning of the lecture. Also, up to global „subtleties“ we have $\mathrm{SU}(2) \sim \mathrm{SO}(3)$ or $\mathrm{SO}(1,3)^{\uparrow} \sim \mathrm{SL}(2,\mathbb{C})$ or $\mathrm{SO}(6) \sim \mathrm{SU}(4)$. Here we relate the Lie groups SO(4) and SU(2)×SU(2). Since SU(2) is well-known and SO(4) is the Lorentzgroup in 4 Euclidean dimensions, this homomorphism is extremely useful to construct all representations of the Lorentz group.

We begin with studying the set of matrices

$$x = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a, b \in \mathbb{C}$$

which form a linear space $\mathbb{C}^2 \cong \mathbb{R}^4$ with scalar product

$$\langle x, y \rangle = \mathrm{Sp}\left(x^\dagger y\right) .$$

The matrices with $|a|^2 + |b|^2 = 1$ belong to SU(2) and form the unit sphere in $\mathbb{R}^4$.

- Let g_1, g_2 be two matrices in SU(2) and x a matrix of the above form. Show that the map $R(g_1, g_2)$ defined by
$$R(g_1, g_2)x = g_1 x g_2^{-1}$$
is linear and preserves the scalar product,
$$\langle R(g_1, g_2)x, R(g_1, g_2)y \rangle = \langle x, y \rangle .$$
Hence $R(g_1, g_2)$ can be considered as a linear map $\mathbb{R}^4 \to \mathbb{R}^4$ which preserves the lengths of vectors.
- Show that this map defines a group homomorphism $\mathrm{SU}(2)\times\mathrm{SU}(2) \to \mathrm{SO}(4)$, i.e. that
$$R(g_1', g_2')R(g_1, g_2) = R(g_1' g_1, g_2' g_2) .$$
- Is this homomorphism faithful? If not, can you identify SO(4) with $\mathrm{SU}(2)\times\mathrm{SU}(2)/Z$ for some discrete normal subgroup Z of SU(2)×SU(2) ?