Dirac Operators and Supersymmetry – From the Coulomb Problem to Field Theories

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Zusammenfassung

Gegenstand dieser Dissertation ist die Untersuchung einer Reihe von speziellen Eigenschaften supersymmetrischer Theorien. In einem ersten Schritt werden Modelle in der Quantenmechanik analysiert, die eine solche Symmetrie besitzen. Aus dem Dirac-Operator $\mathrm{i}\not{D}$, der auf einer – zunächst beliebigen – Riemannschen Mannigfaltigkeit $\mathcal{M}$ im Hintergrund möglicher Eichfeldkonfigurationen definiert ist, kann man einen supersymmetrischen Hamilton-Operator H gewinnen, indem man $H = (\mathrm{i}\not{D})^2$ definiert. H ist invariant unter Transformationen, welche von $\mathrm{i}\not{D}$ erzeugt werden. Hier haben wir den Dirac-Operator als eine spezielle Superladung von H identifiziert. Unter bestimmten Voraussetzungen gestattet H die Existenz weiterer Superladungen. In der vorliegenden Arbeit werden diese Voraussetzungen analysiert. Zum Beispiel besitzt H eine erweiterte $\mathcal{N} = 2$ Supersymmetrie, d.h. es existiert eine weitere Superladung neben $\mathrm{i}\not{D}$, falls $\mathcal{M}$ eine Kähler-Mannigfaltigkeit ist und falls die Eichkrümmung F mit der komplexen Struktur kommutiert. Ferner wird gezeigt, dass $\mathcal{N} = 4$ der Tatsache entspricht, dass $\mathcal{M}$ eine Hyper-Kähler-Mannigfaltigkeit ist und F mit allen drei komplexen Strukturen vertauscht. Theorien mit $\mathcal{N} = 8$ existieren nur auf Mannigfaltigkeiten, deren Dimension ein Vielfaches von acht ist. In acht Dimensionen besitzt nur der flache Raum $\mathbb{R}^8$ ohne Eichfelder diese hohe Symmetrie.

$\mathcal{N} = 2$ Supersymmetrie ist hinreichend, um Anzahloperatoren und Superpotentiale zu definieren. Mit Hilfe dieser Superpotentiale können die Superladungen – insbesondere auch der Dirac-Operator – auf ihre freien Gegenstücke (im flachen Raum und ohne Eichfelder) abgebildet werden. Anschließend kann man auf diese Weise Nullmoden des Dirac-Operators konstruieren, d.h. man findet Lösungen der Gleichung $\mathrm{i}\not{D}\psi = 0$. Als ein konkretes Beispiel berechnen wir die Nullmoden des Dirac-Operators $\mathrm{i}\not{D}$ auf $\mathbb{C}P^n$.

Im nachfolgenden Schritt werden zwei weitere bekannte quantenmechanische Systeme untersucht: das Coulomb-Problem und der harmonische Oszillator. Diese Potentiale sind ausgezeichnet unter den typischen Problemen, da sie eine Symmetriealgebra besitzen, die größer ist als die Algebra der von den Drehimpulsoperatoren erzeugten Rotationen. In d Dimensionen sind dies $\mathfrak{so}(d+1)$ bzw. $\mathfrak{su}(d)$ statt der zu erwartenden $\mathfrak{so}(d)$. Wir definieren die entsprechenden supersymmetrischen Erweiterungen dieser Modelle und konstruieren die supersymmetrischen Analoga des Laplace-Runge-Lenz-Vektors und des entsprechenden Tensors zweiter Stufe für den Oszillator. Diese werden anschließend verwendet, um die Eigenwertprobleme der zugeordneten Hamilton-Operatoren H auf algebraischem Weg zu lösen. Wir zeigen, dass man H durch den quadratischen Casimir-operator der jeweiligen dynamischen Symmetriealgebra ausdrücken kann. Die Darstellungstheorie der $\mathfrak{so}(d+1)$- bzw. $\mathfrak{su}(d)$-Algebren legt dann die Eigenwerte von H und den jeweiligen Grad der Entartung fest.

Die Quadrate spezieller Dirac-Operatoren im Hintergrund abelscher Eichfelder können mit den Hamilton-Operatoren wechselwirkender Wess-Zumino-Modelle (auf einem räumlichen Gitter) identifiziert werden. Dies erlaubt den Übergang von der supersymmetrischen Quantenmechanik zu Feldtheorien.

Ein konkretes Beispiel für eine solche supersymmetrische Feldtheorie wird im zweiten Teil der Dissertation untersucht. Dabei handelt es sich um $\mathcal{N}=2$ Eichtheorie mit abelschen oder nicht-abelschen Eichfeldern, die mit Materiefeldern gekoppelt sind. Insbesondere wird die Theorie in drei euklidischen Dimensionen betrachtet. Ziel dieser Untersuchungen ist es, Lösungen der Dirac-Gleichung zu konstruieren. In geraden Dimensionen kann man Index-Theoreme verwenden, um die Dimension des Kerns von $\mathrm{i}\not{D}$ nach unten abzuschätzen. In ungeraden Dimensionen existieren solche Theoreme nicht, und über die Existenz und die Dimension dieses Kerns ist nur wenig bekannt.

Wir erzeugen Nullmoden als Supersymmetrievariationen bosonischer Hintergrundfelder. In der Arbeit wird gezeigt, dass auf diese Weise Nullmoden des Dirac-Operators konstruiert werden können, welche jedoch auf $\mathbb{R}^3$ nicht normierbar sind. Ferner werden Nullmoden in Anwesenheit von Vortex-Feldern bestimmt und eine analoge Konstruktion auch für nicht-abelsche Theorien durchgeführt. Hier liefert eine Kompaktifizierung auf den Torus $\mathbb{T}^3$ normierbare Lösungen. Anschließend werden Nullmoden im Hintergrund von Yang-Mills-Higgs-Monopolen bestimmt. Diese entsprechen gerade den Jackiw-Rebbi-Moden, die aus der vierdimensionalen Theorie bekannt sind.

Contents

1. Introduction

In all modern physical theories, symmetries are ingredients of outstanding importance. Already the solution of higher-dimensional problems in classical or quantum mechanics relies heavily on the presence of symmetries. For instance, exploiting rotational invariance reduces the problem of motion in a central potential to an ordinary differential equation for the radial coordinate. In this way, one can determine the orbits of planets and the spectra of hydrogen atoms and isotropic harmonic oscillators.

In General Relativity, spherical and axial symmetry allow for the construction of exact solutions to Einsteins equations, like black holes [1] or the rigidly rotating disk of dust [2]. Analytical solutions for generic situations – with less symmetry – are almost inaccessible.

Today, we distinguish two kinds of symmetries: those, that are related to spacetime (or space and time in the nonrelativistic setting) and internal symmetries like gauge symmetries in the Standard Model.

Sensible physical theories of elementary particles must admit invariance under Poincaré transformations. The representation theory of the Poincaré group predicts the basic properties of all possible particles and classifies them according to their mass and spin. All fundamental particles are either bosons, if they have integer spin, or fermions, if their spin is half-integer.

The symmetry group $SU(3) \times SU(2) \times U(1)$ of the Standard Model of particle physics admits the classification of elementary particles in terms of irreducible representations: particles with similar properties arrange themselves into multiplets. Out of fundamental building blocks (quarks, leptons) all matter fields can be constructed. Associated with the symmetry group are carriers of forces, eight gluons for $SU(3)$, three W/Z bosons for $SU(2)$ and the $U(1)$ photon.

Indeed, even *approximate* symmetries have been successfully applied to physical problems. In 1962, Gell-Mann and Ne'eman [3] used the approximate $SU(3)_f$ flavor symmetry, to predict the Ω^-, an unstable particle, the existence of which was verified only later in 1964. Similarly, $SU(2)_f$ isospin symmetry explains, why the proton and neutron mass are almost identical.

In all examples given above, the generators B of symmetry transformations form a Lie algebra. Since the B satisfy certain *commutation relations*, they are called bosonic generators. Here, B are the generators T^A of the gauge group for internal symmetries, or they correspond to generators of translations P_μ or Lorentz transformations $M_{\mu\nu}$, depending on the symmetry one is considering. In these cases, they transform states of different internal quantum numbers into each other, or they relate states in different inertial frames.

For B being the generators of the Poincaré group, a theorem by Coleman and Mandula [4] states, that there is no way of extending that algebra as to include operators which change the spin of the states. So all states in a given multiplet have to have the same spin, in particular all of them must be either bosons or fermions.

Supersymmetry extends the idea of bosonic symmetries to the case of a $\mathbb{Z}_2$-graded Lie algebra. Here, one introduces fermionic operators F in addition to the bosonic generators B. The bracket $[.,.\}$ between generators respects this grading, and is given by

$$[B,B\} = [B,B], \qquad [F,F\} = \{F,F\}, \qquad [F,B\} = [F,B],$$

where $[.,.]$ denotes the commutator, and $\{.,.\}$ the anticommutator.

In particular, if B denotes the generators of the Poincaré group, the graded Lie algebra is a way to overcome the Coleman-Mandula theorem. This has been pointed out by Haag, Łopuszanski and Sohnius [5] in 1975. The Poincaré algebra can be extended to a super-Poincaré algebra, including fermionic generators, which we will call supercharges henceforth. The representation theory of the super-Poincaré algebra predicts the existence of supermultiplets, which contain bosonic states as well as fermionic ones. An important consequence of the supersymmetry algebra is that all states in a given multiplet have the same mass.

Usually, the number of independent supercharges is called $\mathcal{N}$. In four dimensions, theories with (global) supersymmetry possess up to $\mathcal{N} = 4$ supercharges, whereas supergravity can incorporate up to $\mathcal{N} = 8$ of them.

First examples of field theories, which describe the dynamics of such supermultiplets were constructed more than three decades ago by different groups, Wess and Zumino [6], Gol'fand and Likhtman [7], Volkov and Akulov [8], and, in the context of string theory, Ramond [9] and Neveu and Schwarz [10]. Also gravity has been generalized to *supergravity*, incorporating a local version of supersymmetry [11].

Since these seminal works, supersymmetric field theories have become a well-studied subject in theoretical and mathematical physics, because they possess a remarkably rich structure. For instance, in such theories ultra-violet divergences are softened out or are even absent. In Seiberg-Witten theory ($\mathcal{N} = 2$ super-Yang-Mills theory) [12] the low-energy effective action can be calculated exactly, and invariants in that theory can be used to determine properties of three- and four-manifolds. The Maldacena conjecture [13] states a duality between $\mathcal{N} = 4$ superconformal Yang-Mills theories on $D3$-branes and supergravity theories on AdS_5. A deep relation between supersymmetric models and index theorems for differential operators on manifolds has been pointed out by Álvarez-Gaumé [14]. Consistent string theories in ten dimensions require supersymmetry in order to be free of tachyonic states [15]. Today, string theories provide the most promising candidate of a *Theory of Everything*.

The supersymmetric extension of the Standard Model, the (minimal) supersymmetric Standard Model, has been constructed. It contains all known particles, together with their superpartners. Until now, no such superpartner has ever been observed in nature. Nevertheless, many physicists believe that supersymmetry is a symmetry of nature, because *it is so beautiful that it must be true* [16].

One explanation why we haven't observed those superpartners yet, is that supersymmetry is spontaneously broken at present-day energy scales. If this is the case, then supersymmetry can still be a symmetry of nature, even though the supersymmetry multiplets do not reflect that fact. In particular, states in a given multiplet need no longer be of the same mass. So the masses of the unobserved superpartners could be too large to observe them in current experiments.

This raises the question of spontaneous supersymmetry breaking in field theories. Usually, such effects are difficult to address, since they necessitate a nonperturbative treatment of the problem. In the beginning of the eighties, *supersymmetric quantum mechanics* was proposed as a toy model for analyzing such breakings by Witten [17, 18]. He defined a quantity, the Witten index, that measures the difference between the number of bosonic and fermionic states with vanishing energy. If that index is nonzero, supersymmetry is unbroken for the theory at hand.

Supersymmetric quantum mechanics itself has become a field of active research since. Many seemingly unrelated problems in quantum mechanics turned out to be related by a supersymmetry transformation. One prominent example of such a pairing is the one-dimensional infinite square well and the $(\sin x)^{-2}$ potential. Supersymmetry together with shape invariance can be used to determine the complete spectra of certain Hamiltonians algebraically. For a review of these issues and a list of problems that can be solved this way, see [19].

It is the aim of this thesis to study some particular examples of supersymmetric quantum mechanics and field theories in detail.

First of all, in Chapter 2, we note that the Dirac operator $\mathrm{i}\not{D}$, defined on a Riemannian manifold $\mathcal{M}$ in the background of some gauge field configuration, can be used as a fundamental supercharge. Here, the corresponding Hamiltonian is given by the square of the Dirac operator, $H = (\mathrm{i}\not{D})^2$. In this way, a quantum system with $\mathcal{N} = 1$ supersymmetry can be formulated. We raise the question, under which conditions additional supercharges (i.e. square-roots of H) can be defined.

Whereas $\mathcal{N} = 1$ can be realized without further restrictions on even-dimensional $\mathcal{M}$, the existence of $\mathcal{N} = 2$ supersymmetry requires $\mathcal{M}$ to be a Kähler manifold and the gauge curvature F to commute with the complex structure defined on (the tangent bundle of) $\mathcal{M}$. We will show how the existence of three supercharges, $\mathcal{N} = 3$, already implies $\mathcal{N} = 4$ supersymmetry. This in turn is equivalent to $\mathcal{M}$ being a Hyper-Kähler manifold and to F commuting with the three complex structures on $\mathcal{M}$. In four dimensions, this implies that F must be either selfdual or anti-selfdual. Our analysis can be extended to arbitrary values of $\mathcal{N}$.

It turns out that $\mathcal{N} = 2$ supersymmetry is sufficient to define a *number operator* and a *superpotential*, which can be used to deform the supercharges into their free counterparts. In particular, the Dirac operator $\mathrm{i}\not{D}$ can be deformed and simplified this way. This deformation à la Witten can be used to construct zero modes of the Dirac operator. We outline this construction in the general setting and present the zero modes on $\mathbb{C}P^n$ as a particular application.

In a subsequent step, we analyze properties of two prominent quantum mechanical systems with $\mathcal{N} = 2$ supersymmetry. They arise after a particular dimensional reduction of Dirac operators in Abelian backgrounds. In this case, the superpotential reduces to a scalar function, the $1/r$-potential of the Coulomb or Kepler problem, and the r^2-potential of the isotropic harmonic oscillator. For a variety of reasons, the non-supersymmetric counterparts of these models have been studied since the very beginning of modern physics, and we list a few of them now.

There are eleven coordinate systems for which the Hamilton-Jacobi equation may separate in $\mathbb{R}^3$ [20]. Superintegrable Hamiltonians in three degrees of freedom possess – by definition – more than three functionally independent, globally defined and single-valued integrals of motion. The system is called *maximally superintegrable* if it admits five such integrals, then all finite trajectories are closed, so they fill a surface of less dimensions (here one) than the number of degrees of freedom (here two). The system is called *minimally superintegrable* if it admits four globally defined and single-valued integrals of motion. In this case, the trajectories are restricted to a two-dimensional surface. Superintegrability implies the existence of separable solutions to the Hamilton-Jacobi equation in at least two coordinate systems, cf. [21] and references therein. The isotropic harmonic oscillator in three dimensions is separable in eight coordinate systems[1]. The five isolating integrals in this case are the energy E, the energies of two subsystems, say E_1 and E_2, angular momentum squared $\boldsymbol{L}^2$ and its third component L_3. The Coulomb problem separates in three coordinate systems[2], and isolating integrals are the energy E, angular momentum squared $\boldsymbol{L}^2$, its third component squared L_3^2, its second component squared L_2^2, and the third component of the Laplace-Runge-Lenz vector. So both, oscillator and Coulomb problem, possess five isolating integrals and are therefore superintegrable.

A theorem by Bertrand [22] states that the $1/r$-potential of the Coulomb problem and the r^2-potential of the harmonic oscillator are the only spherically symmetric potentials such that every admissible (bound-state) trajectory is closed.

It is well-known, that these two potentials exhibit unique properties also in the quantum mechanical setting. There has been no difficulty in exploiting ostensible geometric symmetry, which is manifested by a group of linear operators commuting with the Hamiltonian of the system. Schur's lemmas describe the limitations imposed on the

[1] These are rectangular cartesian, spherical polar, cylindrical polar, elliptic cylindrical, confocal ellipsoidal, conical, oblate spheroidal and prolate spheroidal coordinates.

[2] Rotational parabolic, conical and spherical polar coordinates.

Hamiltonian, which are that there be no matrix element connecting wave functions of different symmetry types, and that all the eigenvalues belonging to one irreducible representation of the symmetry group be equal. This last mentioned requirement is, of course, the well-known relationship between symmetry and degeneracy. Every symmetric system will show characteristic degeneracies, the multiplicity of which is prescribed by the dimensions of the irreducible representations of its symmetry group. Yet, there is no restriction arising from group theoretical reasoning which prevents the existence of a higher multiplicity of degeneracy than required by Schur's lemmas. Any degeneracy of this kind is commonly called *accidental* due to a presumption as to its unlikelihood [23].

A *hidden* symmetry is not necessarily of a geometric nature, but together with the geometric symmetries already known, yields a group of symmetries large enough to exactly account for all the observed degeneracies of the system. Classical mechanics actually contains a reasonable source of hidden symmetries, the phase space having twice the dimension of the configuration space in which the geometric symmetries are evident. In other words, it is possible that there are additional symmetries of the phase space as a whole which comprise the desired hidden-symmetry group.

It is well-known that the Hamiltonian associated with the $1/r$ potential in d dimensions is in fact invariant under $\mathfrak{so}(d+1)$ transformations instead of $\mathfrak{so}(d)$ ones. For the harmonic oscillator, $\mathfrak{so}(d)$ is extended to $\mathfrak{su}(d)$. To be more precise, the hidden-symmetry algebra for the $1/r$-potential consists of $\mathfrak{so}(d+1)$ for states with energy $E < 0$, $\mathfrak{so}(d,1)$ for $E > 0$, and the algebra of the Euclidean group (of rotations and translations) in $d+1$ dimensions for $E = 0$.

In the present thesis, we construct the $\mathcal{N} = 2$ supersymmetric extensions of these two models. We define the supersymmetric analogues of the Laplace-Runge-Lenz vector and of the corresponding tensor for the oscillator. Using them, we express the Hamiltonians in terms of the quadratic Casimir operators of the associated hidden-symmetry algebras and obtain the spectra algebraically. Eigenvalues of H and the corresponding degeneracies are fixed by the representation theory of those algebras.

We conclude the first part of the thesis with an attempt to construct a relationship between these two potentials. In the non-supersymmetric case, such relations are known under the name of Euler-, Levi-Civitá-, Kustaanheimo-Stiefel- or Hurwitz-transformations, depending on the dimensions d. For the supersymmetric case, we propose an algebraic version of these transformations by rewriting the Laplace-Runge-Lenz vector in terms of creation and annihilation operators.

We finally show how the Hamiltonian $H = (\mathrm{i}\not{D})^2$ on $\mathbb{R}^N$, in the presence of particular Abelian gauge potentials, can be identified with the Hamiltonian of the two-dimensional Wess-Zumino model on a spatial lattice. Here, the dimension N grows with the number of lattice points. In this way, we bridge the gap from supersymmetric quantum mechanics to field theories.

The remainder of the thesis is concerned with the application of supersymmetry to field theories in three Euclidean dimensions. Again, we consider $\mathcal{N} = 2$ extended supersymmetry. We construct a particular field theory as a dimensionally reduced theory from $(3+1)$-dimensional Minkowski space. This allows us to construct zero modes of Dirac operators in three dimensions. The existence of such zero modes has been obscure until 1986 [24, 25], when first examples were explicitly constructed. Only recently, a whole class of examples was given by Adam, Muratori and Nash [26]. Matters are complicated in three dimensions due to the absence of an index theorem à la Atiyah and Singer, which, if valid, would give a lower bound on the number of zero modes.

We construct zero modes of the Dirac operator as supersymmetry variations around bosonic field configurations. We prove that, for instanton-like configurations, normalizable solutions cannot be obtained in flat space $\mathbb{R}^3$. After compactification to a three-torus, $\mathbb{R}^3 \to \mathbb{T}^3$, however, such solutions can be constructed, and we present some examples: 't Hooft's constant-curvature solutions and the associated zero modes. For non-Abelian theories we derive the three-dimensional analogue of the Jackiw-Rebbi modes. These are zero modes in the presence of a Yang-Mills-Higgs monopole system. Similar reasoning is applied to a further reduced theory in two Euclidean dimensions, where vortex configurations are shown to yield zero modes of the Dirac operator, too.

2. Supersymmetric Quantum Mechanics

In this Chapter we define the notion of a supersymmetric quantum mechanical system, identify the Dirac operator as a particular supercharge and construct additional supercharges. Their existence – corresponding to a higher amount of supersymmetry – is shown to put strong restrictions on our theory. Subsequently we focus on the case of $\mathcal{N} = 2$ extended supersymmetry. We show, how this symmetry can be used to construct zero modes of Dirac operators and apply our formalism to Dirac operators defined on $\mathbb{C}P^n$. The results of Sections 2.1–2.5 have been published in [AK2].

2.1. The Setting

Supersymmetric quantum mechanics describes systems with nonnegative Hamiltonians H that can be written as[1]

$$\delta_{ij} H = \frac{1}{2}\{Q_i, Q_j\}, \qquad i, j = 1, \ldots \mathcal{N}, \tag{2.1}$$

with Hermitian supercharges Q_i that anticommute with an involutary operator Γ,

$$\{Q_i, \Gamma\} = 0, \qquad \Gamma^\dagger = \Gamma, \qquad \Gamma^2 = \mathbb{1}. \tag{2.2}$$

Γ possesses eigenvalues ± 1, and we choose the convention, to call the $(+1)$–eigenspace the *bosonic* and the (-1)–eigenspace the *fermionic* subsector of our theory. Accordingly,

[1]Note that in the literature [19, 27] various definitions of supersymmetric quantum mechanics exist. For an account on their mutual relationship cf. [28].

the Hilbert space $\mathcal{H}$ decomposes as

$$\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F, \qquad \mathcal{H}_B = \mathcal{P}_+\mathcal{H}, \qquad \mathcal{H}_F = \mathcal{P}_-\mathcal{H}, \qquad \mathcal{P}_\pm = \frac{1}{2}\left(\mathbb{1} \pm \Gamma\right). \tag{2.3}$$

A theory with $\mathcal{N} = 1$ is said to have (simple) supersymmetry and, for $\mathcal{N} \geq 2$, to have $\mathcal{N}$-extended supersymmetry. The supercharges Q_i map $\mathcal{H}_B$ into $\mathcal{H}_F$ and vice versa, so they transform bosonic states into fermionic ones and back. The superalgebra (2.1) implies that they commute with the Hamiltonian, $[Q_i, H] = 0$, and thus generate supersymmetries of the theory. First examples of such structures were studied by Nicolai [29] and Witten [17, 30]. They considered one-dimensional systems with all operators realized as 2×2-matrix operators. The Q_i are matrices of first-order differential operators, and the matrix entries of H are ordinary Schrödinger operators.

In the case of $\mathcal{N} = 1$, every eigenstate of $H = Q_1^2 \geq 0$ with positive energy is paired by the action of Q_1: for example, if $|B\rangle$ is a bosonic eigenstate with positive energy, then $|F\rangle \sim Q_1\,|B\rangle$ is a fermionic eigenstate with the same energy. Eigenstates with zero energy are annihilated by the supercharge, and hence have no superpartner. In a basis, where $\Gamma = \sigma_3 \otimes \mathbb{1}$, Q_1 has the form

$$Q_1 = \mathcal{P}_- Q_1 \mathcal{P}_+ + \mathcal{P}_+ Q_1 \mathcal{P}_- \equiv \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}. \tag{2.4}$$

The index of Q_1 counts the difference of bosonic and fermionic zero modes,

$$\operatorname{ind} Q_1 = \dim\ker A - \dim\ker A^\dagger = n_B^0 - n_F^0. \tag{2.5}$$

Supersymmetry is spontaneously broken if and only if there exists no state which is left invariant by Q_1, or equivalently, if there are no zero-energy states. Thus, $\operatorname{ind} Q_1 \neq 0$ is sufficient for supersymmetry to be unbroken.

For $\mathcal{N} = 2$ there are two roots of H,

$$H = Q_1^2 = Q_2^2, \qquad \{Q_1, Q_2\} = 0, \qquad Q_i^\dagger = Q_i. \tag{2.6}$$

This will be the most important case in our subsequent studies of the hydrogen atom and the harmonic oscillator. Later on, we will also use the complex nilpotent supercharge

$Q = \frac{1}{2}(Q_1 + \mathrm{i}Q_2)$ and its adjoint $Q^\dagger$. The superalgebra in that case reads

$$H = \{Q, Q^\dagger\}, \qquad Q^2 = Q^{\dagger 2} = 0, \qquad [Q, H] = 0. \tag{2.7}$$

If there are $\mathcal{N} = 4$ supercharges present, $H = Q_1^2 = Q_2^2 = Q_3^2 = Q_4^2$, we find the following nontrivial anticommutation-relations,

$$\{Q, Q^\dagger\} = \{\tilde{Q}, \tilde{Q}^\dagger\} = H, \tag{2.8}$$

where

$$Q = \frac{1}{2}\left(Q_1 + \mathrm{i}Q_2\right), \qquad \tilde{Q} = \frac{1}{2}\left(Q_3 + \mathrm{i}Q_4\right). \tag{2.9}$$

2.2. Supersymmetry and the Euclidean Dirac Operator

In this Section, we identify $\mathrm{i}\not{D}$ as a distinguished supercharge of the Hamiltonian $H = (\mathrm{i}\not{D})^2$, and we try to enlarge the superalgebra by constructing additional charges. Here, $\mathrm{i}\not{D}$ is the Dirac operator on a d-dimensional Riemannian manifold $\mathcal{M}$ in the background of certain gauge field configurations. We will study the consequences of imposing a given amount of supersymmetry (in particular, $\mathcal{N} = 1, 2, 4$ and 8). A higher degree of supersymmetry puts stronger restrictions on the theory. The geometry of the underlying space on which our theories are defined, as well as admissible gauge field configurations must satisfy certain conditions. Obviously, $\mathcal{N} = 1$ supersymmetry is always realized in this setting ($Q_1 = \mathrm{i}\not{D}$). $\mathcal{N} = 2$ can be realized on Kähler manifolds of real dimension $d = 2n$ and for background fields that commute with the complex structure. Hyper-Kähler manifolds with $d = 4n$ plus gauge fields that commute with three complex structures correspond to $\mathcal{N} = 4$.

Let g_{mn} be the metric on $\mathcal{M}$. We will use vielbeine e^a_m, which have (flat) Lorentz indices $a, b = 1, 2, \ldots d$ and (curved) coordinate indices, $m, n = 1, 2, \ldots d$. g_{mn} and the flat metric δ_{ab} are related as follows,

$$g_{mn} = e^a_m e^b_n \delta_{ab}, \qquad \delta^{ab} = g^{mn} e^a_m e^b_n, \tag{2.10}$$

where g^{mn} is the inverse of g_{mn}. The Clifford algebra is generated by the Hermitian matrices γ^a (or $\gamma^m \equiv e_a^m \gamma^a$), satisfying

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}, \qquad \{\gamma^m, \gamma^n\} = 2g^{mn}. \tag{2.11}$$

As mentioned above, in even dimensions, $d = 2n$, simple supersymmetry, with $\mathcal{N} = 1$, is generated by the supercharge

$$Q_1 = \mathrm{i}\not{D} = \mathrm{i}\gamma^m D_m. \tag{2.12}$$

The generally- and gauge-covariant derivative (acting on spinors)

$$D_m = \partial_m + \omega_m + A_m = \partial_m + \frac{1}{4}\omega_{mab}\gamma^{ab} + A_m^A T^A, \tag{2.13}$$

contains the connection ω and the gauge potential A, together with the generators $\gamma^{ab} = \frac{1}{2}[\gamma^a, \gamma^b]$ and T^A of spin rotations and gauge transformations. The γ-matrices are covariantly constant,

$$D_m \gamma^n = \partial_m \gamma^n + \Gamma^n_{mp}\gamma^p + [\omega_m, \gamma^n] = 0. \tag{2.14}$$

Here, Γ^n_{mp} are the Christoffel symbols,

$$\Gamma^n_{mp} = \frac{1}{2} g^{nk}\left(\partial_m g_{kp} + \partial_p g_{km} - \partial_k g_{mp}\right). \tag{2.15}$$

The involutary operator Γ in (2.2) can be identified as

$$\Gamma \equiv \gamma_* = \alpha\gamma^1 \ldots \gamma^d, \tag{2.16}$$

where the phase α is chosen such that Γ is Hermitian and squares to $\mathbb{1}$, $\alpha^2 = (-)^{d/2}$. In this way, we identify bosonic and fermionic states as states with positive and negative chirality, respectively.

The commutator of two covariant derivatives can be expressed in terms of the field strength F_{mn} and the curvature tensor R_{mn},

$$[D_m, D_n] = \mathcal{F}_{mn} = F_{mn} + R_{mn}, \tag{2.17}$$

where

$$F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n] = F^A_{mn} T^A, \tag{2.18}$$
$$R_{mn} = \partial_m \omega_n - \partial_n \omega_m + [\omega_m, \omega_n] = \frac{1}{4} R_{mnab} \gamma^{ab},$$

with Riemann curvature tensor

$$R_{mnab} = \partial_m \omega_{nab} - \partial_n \omega_{mab} + \omega_{ma}{}^c \omega_{ncb} - \omega_{na}{}^c \omega_{mcb}. \tag{2.19}$$

Thus, our supersymmetric Hamiltonian is given by

$$H = Q_1^2 = (\mathrm{i}\not{D})^2 = -g^{mn} D_m D_n - \frac{1}{2}\gamma^{ab}\mathcal{F}_{ab}. \tag{2.20}$$

Note that the two covariant derivatives in (2.20) act on different types of fields. The derivative on the right acts as defined in (2.13), whereas the derivative on the left acts on a spinor with an additional coordinate index,

$$D_m \Psi_n = \partial_m \Psi_n + \omega_m \Psi_n - \Gamma^p_{mn} \Psi_p + A_m \Psi_n. \tag{2.21}$$

Next, we want to enlarge the superalgebra to the case $\mathcal{N} \geq 2$. Motivated by the construction of *non-standard Dirac operators* and the results obtained in [31], we choose the following ansatz for the additional supercharges,

$$Q(I) = \mathrm{i} I^m{}_n \gamma^n D_m. \tag{2.22}$$

This also reflects the fact, that the free Dirac operator $\mathrm{i}\not{\partial}$ and $\mathrm{i}I^m{}_n\gamma^n\partial_m$ lead to the same square for any orthogonal matrix I.

The anticommutator of two such charges with different matrices I and J gives

$$\begin{aligned}\{Q(I), Q(J)\} = &-\frac{1}{2}(IJ^t + JI^t)^{mn}\{D_m, D_n\} - \frac{1}{2}\gamma^{mn}(I^t\mathcal{F}J + J^t\mathcal{F}I)_{mn} \\ &- \{(I\gamma)^p D_p (J\gamma)^q + (J\gamma)^p D_p (I\gamma)^q\} D_q.\end{aligned} \tag{2.23}$$

For $I = J$, we read off that $Q(I)$ squares to our Hamiltonian H in (2.20), if and only if

$$g^{mn} = (II^t)^{mn}, \qquad \mathcal{F}_{mn} = (I^t\mathcal{F}I)_{mn}, \qquad D_m I^p{}_q = 0. \tag{2.24}$$

In particular, this implies that the tensor field I must be covariantly constant. The corresponding integrability condition reads

$$0 = I^r{}_m [D_a, D_b] I_{rn} = I^r{}_m R_{rsab} I^s{}_n - R_{mnab}, \quad \text{or} \quad R_{mn} = (I^t R I)_{mn}, \tag{2.25}$$

and (2.24) implies that the same relation holds for the gauge curvature, too,

$$F_{mn} = (I^t F I)_{mn}. \tag{2.26}$$

Result: the charge

$$Q(I) = \mathrm{i} I^m_n \gamma^n D_m, \tag{2.27}$$

with real matrix I, is Hermitian and squares to H in (2.20), if and only if the following conditions hold,

$$D_m I = 0, \qquad I I^t = \mathbb{1}, \qquad [I, F] = 0. \tag{2.28}$$

The trivial solution $I = \mathbb{1}$ gives us back the original Dirac operator,

$$Q(\mathbb{1}) = \mathrm{i} \not{D} = \mathrm{i} \gamma^m D_m. \tag{2.29}$$

In view of (2.1), $Q(\mathbb{1})$ should anticommute with all other supercharges,

$$0 \stackrel{!}{=} \{Q(\mathbb{1}), Q(I)\} = -\frac{1}{2}(I + I^t)^{mn} \{D_m, D_n\}, \tag{2.30}$$

so I must be antisymmetric. Because of (2.28), it squares to $-\mathbb{1}$, hence it defines an almost complex structure [32, 33] on (the tangent bundle of) our manifold $\mathcal{M}$. Since I is covariantly constant, $\mathcal{M}$ must be a Kähler manifold.

On any Kähler manifold $\mathcal{M}$, and in a gauge field background where the gauge field strength F_{mn} commutes with the complex structure, the Hamiltonian H in (2.20) possesses two supercharges, $Q(\mathbb{1})$ and $Q(I)$. Now this can easily be generalized to higher supersymmetry.

Result: the $\mathcal{N}$ charges

$$Q(\mathbb{1}) = \mathrm{i}\not{D} \qquad \text{and} \qquad Q(I_i) = \mathrm{i}I_i{}^m{}_n\gamma^n D_m, \qquad i = 1, 2, \ldots \mathcal{N} - 1, \tag{2.31}$$

are Hermitian and generate an extended superalgebra (2.1), if and only if

$$\{I_i, I_j\} = -2\delta_{ij}\mathbb{1}, \qquad I_i^t = -I_i, \qquad D_m I_i = 0, \qquad [I_i, F] = 0. \tag{2.32}$$

Observe that, if $\{I_1, \ldots, I_k, F\}$ satisfy the conditions, then also $\{I_1, \ldots, I_{k+1}, F\}$ do, where $I_{k+1} \equiv I_1 I_2 \ldots I_k$, provided $k = 4n + 2$. It follows, for example, that the superalgebra with three supercharges can always be extended to a superalgebra with four supercharges, so $\mathcal{N} = 3$ implies $\mathcal{N} = 4$, and similarly, $\mathcal{N} = 7$ implies $\mathcal{N} = 8$.

From (2.32) we read off that the covariantly conserved complex structures form a d-dimensional *real* representation of the Euclidean Clifford algebra with $\mathcal{N} - 1$ γ-matrices. From the theory of Clifford algebras it is known that a matrix realization for the γ's – and thus for the I_i – exists only in certain dimensions. This implies that the dimension of $\mathcal{M}$ cannot be arbitrary. We call the matrix representation *irreducible*, if only $\mathbb{1}$ commutes with all γ-matrices. Taking into account that all matrices have to be real, these irreducible representations exist only in particular dimensions, which we summarize in Table 2.1.

$\mathcal{N}$	$8n + 1$	$8n + 7$	$8n + 8$
$\#\gamma$	$8n$	$8n + 6$	$8n + 7$
d	16^n	$8 \cdot 16^n$	$8 \cdot 16^n$

Table 2.1.: Supersymmetry ($\mathcal{N}$), number of γ matrices ($\#\gamma$), vs. dimension of $\mathcal{M}$ (d).

2.3. Higher Supersymmetries

Let us consider $\mathcal{N} = 4$ first. Observe that this case is not contained in Table 2.1. The reason for that is the following: the Clifford algebra with $\mathcal{N} - 1 = 3$ generators can be realized in four dimensions in two inequivalent ways, by either choosing selfdual (SD) or anti-selfdual (ASD) matrices,

$$\begin{aligned} \text{SD:} \qquad & \tilde{I}_1 = \mathrm{i}\sigma_0 \otimes \sigma_2, \qquad \tilde{I}_2 = \mathrm{i}\sigma_2 \otimes \sigma_3, \qquad \tilde{I}_3 = \mathrm{i}\sigma_2 \otimes \sigma_1 = -\tilde{I}_1 \tilde{I}_2, \\ \text{ASD:} \qquad & \tilde{I}_1 = \mathrm{i}\sigma_3 \otimes \sigma_2, \qquad \tilde{I}_2 = \mathrm{i}\sigma_2 \otimes \sigma_0, \qquad \tilde{I}_3 = \mathrm{i}\sigma_1 \otimes \sigma_2 = \tilde{I}_1 \tilde{I}_2, \end{aligned} \tag{2.33}$$

The dimension of the matrices I_i (which equals the dimension of the manifold) must be a multiple of four, $d = 4n$, so we define $I_i = \tilde{I}_i \otimes \mathbb{1}_n$. They generate two commuting $\mathfrak{so}(3)$ subalgebras of $\mathfrak{so}(4n)$. The conditions (2.32) imply that the curvature tensor (R_{ab}) and gauge field strength (F_{ab}) commute with all three I_i. For example, in four dimensions both must be *selfdual* or *anti-selfdual*. A four-dimensional manifold with (anti-)selfdual curvature is hyper-Kähler. More generally, a $4n$-dimensional manifold is hyper-Kähler if it admits three covariantly constant and anticommuting complex structures. We see that $(\mathrm{i}\not{D})^2$ admits four supersymmetries if and only if the underlying space $\mathcal{M}$ is hyper-Kähler and the gauge field strength commutes with the three complex structures. We remark that other choices for the complex structures than those obtained from (2.33) are possible.

According to Table 2.1, we can find six or seven real and antisymmetric matrices I_i, for example the eight-dimensional (irreducible) matrices

$$\begin{aligned} &\tilde{I}_1 = \mathrm{i}\sigma_1 \otimes \sigma_0 \otimes \sigma_2, \quad \tilde{I}_3 = \mathrm{i}\sigma_2 \otimes \sigma_1 \otimes \sigma_0, \quad \tilde{I}_5 = \mathrm{i}\sigma_0 \otimes \sigma_2 \otimes \sigma_1, \\ &\tilde{I}_2 = \mathrm{i}\sigma_3 \otimes \sigma_0 \otimes \sigma_2, \quad \tilde{I}_4 = \mathrm{i}\sigma_2 \otimes \sigma_3 \otimes \sigma_0, \quad \tilde{I}_6 = \mathrm{i}\sigma_0 \otimes \sigma_2 \otimes \sigma_3, \\ &\tilde{I}_7 = \tilde{I}_1 \tilde{I}_2 \tilde{I}_3 \tilde{I}_4 \tilde{I}_5 \tilde{I}_6 = -\mathrm{i}\sigma_2 \otimes \sigma_2 \otimes \sigma_2, \end{aligned} \tag{2.34}$$

tensored with $\mathbb{1}_n$. Thus we can satisfy (2.32) in $8n$ dimensions. In eight dimensions there is no nontrivial solution to

$$[I_i, \mathcal{F}] = 0, \quad i = 1, \ldots, 7, \tag{2.35}$$

since the only matrix commuting with all I_i in (2.34) is the identity matrix. Hence, the manifold must be flat, and the gauge field strength must vanish. In eight dimensions, only the free Dirac operator admits an $\mathcal{N} = 8$ supersymmetry. However, in $8n$ dimensions with $n = 2, 3, \ldots$, there are nontrivial solutions to the constraints in (2.32). For example, every field strength $(F_{ab}) = \mathbb{1}_8 \otimes \tilde{F}$ with antisymmetric $\tilde{F}$ commutes with the I_i listed in (2.34).

2.4. $\mathcal{N} = 2$ and Number Operators

Let us now focus on the case $\mathcal{N} = 2$. We have seen that, on any Kähler manifold $\mathcal{M}$, the Dirac operator admits such an extended supersymmetry, if the field strength commutes with the complex structure. In a suitably chosen orthonormal basis, I is given by $I = \mathrm{i}\sigma_2 \otimes \mathbb{1}$. In this basis, $[I, F] = 0$ is equivalent to

$$F_{ab} = \begin{pmatrix} U & V \\ -V & U \end{pmatrix}_{ab}, \qquad U^t = -U, \qquad V^t = V. \tag{2.36}$$

Now we use the complex nilpotent charges

$$Q = \frac{1}{2}\left(Q(\mathbb{1}) + \mathrm{i}Q(I)\right) = \mathrm{i}\psi^a D_a, \tag{2.37}$$

where

$$\psi^a = P^a{}_b \gamma^b, \qquad P^a{}_b = \frac{1}{2}(\mathbb{1} + \mathrm{i}I)^a{}_b. \tag{2.38}$$

P projects onto the n-dimensional I-eigenspace corresponding to the eigenvalue $-\mathrm{i}$, its complex conjugate, $\bar{P}$, onto the n-dimensional eigenspace $+\mathrm{i}$. These two eigenspaces are complementary and orthogonal, $P + \bar{P} = \mathbb{1}$ and $P\bar{P} = 0$. The ψ^a and their adjoints form a fermionic algebra,

$$\{\psi^a, \psi^b\} = \{\psi^{a\dagger}, \psi^{b\dagger}\} = 0, \qquad \{\psi^a, \psi^{b\dagger}\} = 2P^{ab}, \tag{2.39}$$

and we can define a number operator,

$$N = \frac{1}{2}\psi_a^\dagger \psi^a = \frac{1}{4}\left(d + \mathrm{i}I_{ab}\gamma^{ab}\right). \tag{2.40}$$

The factor $\frac{1}{2}$ is due to the fact that only n of the $2n$ creation operators are linearly independent. The eigenvalues of N are raised and lowered by $\psi^{a\dagger}$ and ψ^a,

$$[N, \psi^{a\dagger}] = P^a{}_b \psi^{b\dagger} = \psi^{a\dagger}, \qquad [N, \psi^a] = -P^a{}_b \psi^b = -\psi^a. \tag{2.41}$$

N commutes with the covariant derivative, because $D_m I = 0$ is equivalent to

$$[D_m, N] = \partial_m N + [\omega_m, N] = 0, \tag{2.42}$$

therefore Q decreases N by one, while $Q^\dagger$ increases it by one,

$$[N, Q] = -Q, \qquad [N, Q^\dagger] = Q^\dagger. \tag{2.43}$$

The corresponding real supercharges are given by

$$Q(\mathbb{1}) = Q + Q^\dagger = \mathrm{i}\not{D}, \qquad Q(I) = \mathrm{i}(Q^\dagger - Q) = \mathrm{i}[N, \mathrm{i}\not{D}]. \tag{2.44}$$

The Clifford vacuum $|0\rangle$, which is annihilated by the action of any of the ψ^a, has particle number $N = 0$. The raising and lowering operators $\psi^{a\dagger}$ and ψ^a are linear combinations of γ^a and therefore anticommute with Γ in (2.16). Hence, they map left- into right-handed spinors and vice versa. Since the Clifford vaccum $|0\rangle$ is unique, and since $\Gamma\,|0\rangle$ is annihilated by all ψ^a,

$$\psi^a(\Gamma\,|0\rangle) = -\Gamma\psi^a\,|0\rangle = 0, \tag{2.45}$$

we conclude that $|0\rangle$ has definite chirality. It follows that all states with even N have the same chirality as $|0\rangle$, and all states with odd N have opposite chirality,

$$\Gamma = \pm(-)^N, \tag{2.46}$$

depending on the choice of α in (2.16). Thus, bosonic and fermionic states in our theory can equivalently be described as states with even or odd particle number and positive or negative chirality.

The n linearly independent raising operators give rise to the following grading of the Clifford space,

$$\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_n, \qquad \dim \mathcal{C}_\wp = \binom{n}{\wp}. \tag{2.47}$$

Subspaces are labelled by their particle number,

$$N\big|_{\mathcal{C}_\wp} = \wp \cdot \mathbb{1}. \tag{2.48}$$

In particular, the one-dimensional subspace $\mathcal{C}_0$ is spanned by $|0\rangle$, and the n-dimensional subspace $\mathcal{C}_1$ by the linearly independent states $\psi^{a\dagger}|0\rangle$. Along with the Clifford space, the Hilbert space of all square integrable spinor fields on $\mathcal{M}$ decomposes as

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n. \tag{2.49}$$

Since the Hamiltonian commutes with N, it leaves each $\mathcal{H}_\wp$ invariant, whereas the nilpotent charge Q maps $\mathcal{H}_\wp$ into $\mathcal{H}_{\wp-1}$, and its adjoint $Q^\dagger$ maps $\mathcal{H}_\wp$ into $\mathcal{H}_{\wp+1}$.

2.5. Superpotentials on Kähler Manifolds

We have seen that the super-Hamiltonian $(\mathrm{i}\not{D})^2$ admits an extended supersymmetry if it commutes with the number operator N or, equivalently, if the complex supercharge is nilpotent and decreases the particle number by one. Then the manifold is Kähler and the complex structure commutes with the gauge field strength. Now we shall see that this in turn is the condition for the existence of a *superpotential* g from which the spin connection and gauge potential can be derived.

Let $\mathcal{M}$ be a Kähler manifold of real dimension $d = 2n$. First, we summarize some well-known facts concerning these spaces. Kähler manifolds are particular complex manifolds, and we may introduce complex coordinates $(z^\mu, \bar{z}^{\bar{\mu}})$ with $\mu, \bar{\mu} = 1, \ldots, n$ [32]. The real and complex coordinate differentials are related as follows

$$\mathrm{d}z^\mu = \frac{\partial z^\mu}{\partial x^m}\mathrm{d}x^m \equiv f^\mu_m \mathrm{d}x^m, \qquad \mathrm{d}\bar{z}^{\bar{\mu}} = \frac{\partial \bar{z}^{\bar{\mu}}}{\partial x^m}\mathrm{d}x^m \equiv f^{\bar{\mu}}_m \mathrm{d}x^m, \tag{2.50}$$

$$\partial_\mu = \frac{\partial x^m}{\partial z^\mu}\partial_m \equiv f^m_\mu \partial_m, \qquad \partial_{\bar{\mu}} = \frac{\partial x^m}{\partial \bar{z}^{\bar{\mu}}}\partial_m \equiv f^m_{\bar{\mu}} \partial_m. \tag{2.51}$$

Vanishing of the Nijenhuis tensor,

$$0 = N^i_{jk} = I^l{}_j \partial_l I^i{}_k - I^l{}_k \partial_l I^i{}_j - I^i{}_l \partial_j I^l{}_k + I^i{}_l \partial_k I^l{}_j, \tag{2.52}$$

is the integrability condition for the $\mathrm{d}z^\mu$ to be differentials of complex coordinate functions z^μ. This condition is automatically satisfied on a Kähler manifold.

The f^μ and f_μ are left and right eigenvectors of the complex structure,

$$f^\mu_m I^m{}_n = -\mathrm{i} f^\mu_n \quad \text{and} \quad I^m{}_n f^n_\mu = -\mathrm{i} f^m_\mu, \qquad \mu = 1, \ldots, n. \tag{2.53}$$

Since $I^m{}_n$ is antisymmetric with respect to the scalar product $(A, B) = g^{mn} A_m B_n$, the eigenvectors with different eigenvalues are orthogonal in the following sense,

$$g^{mn} f^\mu_m f^\nu_n = g_{mn} f^m_\mu f^n_\nu = 0. \tag{2.54}$$

Identity and complex structure possess the spectral decompositions

$$\delta^m_n = f^m_\mu f^\mu_n + f^m_{\bar\mu} f^{\bar\mu}_n, \qquad \mathrm{i} I^m{}_n = f^m_\mu f^\mu_n - f^m_{\bar\mu} f^{\bar\mu}_n, \tag{2.55}$$

and the relations $\partial z^\mu / \partial z^\nu = \delta^\mu_\nu$ and $\partial z^\mu / \partial \bar z^{\bar\nu} = 0$ translate into

$$f^\mu_m f^m_\nu = \delta^\mu_\nu \quad \text{and} \quad f^\mu_m f^m_{\bar\nu} = 0. \tag{2.56}$$

With (2.54) the line element takes the form

$$\mathrm{d}s^2 = g_{mn} \mathrm{d}x^m \mathrm{d}x^n = 2 h_{\mu\bar\nu} \mathrm{d}z^\mu \mathrm{d}\bar z^{\bar\nu}, \qquad h_{\mu\bar\nu} = h_{\bar\nu\mu} = g_{mn} f^m_\mu f^n_{\bar\nu}, \tag{2.57}$$

where the $h_{\mu\bar\nu}$ are derived from a real Kähler potential K as follows,

$$h_{\mu\bar\nu} = \frac{\partial^2 K}{\partial z^\mu \partial \bar z^{\bar\nu}} \equiv \partial_\mu \partial_{\bar\nu} K. \tag{2.58}$$

Covariant and exterior derivatives split into holomorphic and antiholomorphic pieces,

$$D = \mathrm{d}z^\mu D_\mu + \mathrm{d}\bar z^{\bar\mu} D_{\bar\mu}, \qquad \mathrm{d} = \mathrm{d}z^\mu \partial_\mu + \mathrm{d}\bar z^{\bar\mu} \partial_{\bar\mu} = \partial + \bar\partial, \tag{2.59}$$

and the only nonvanishing components of the Christoffel symbols are

$$\Gamma^\rho_{\mu\nu} = h^{\rho\bar\sigma} \partial_\mu h_{\bar\sigma\nu} = h^{\rho\bar\sigma} \partial_{\bar\sigma\mu\nu} K, \qquad \Gamma^{\bar\rho}_{\bar\mu\bar\nu} = h^{\bar\rho\sigma} \partial_{\bar\mu} h_{\sigma\bar\nu} = h^{\bar\rho\sigma} \partial_{\sigma\bar\mu\bar\nu} K. \tag{2.60}$$

Along with the derivatives the forms split into holomorphic and antiholomorphic parts. For example, the first Chern class, $c_1 = (2\pi\mathrm{i})^{-1} h_{\mu\bar\nu} \mathrm{d}z^\mu \mathrm{d}\bar z^{\bar\nu}$, is a $(1,1)$-form and the gauge potential $A = A_\mu \mathrm{d}z^\mu + A_{\bar\mu} \mathrm{d}\bar z^{\bar\mu}$ a sum of a $(1,0)$- and a $(0,1)$-form. With the help of

(2.60) the covariant derivative of a $(1,0)$-vector field can be written as

$$D_\mu(B^\nu\partial_\nu) = \big(\partial_\mu B^\rho + \Gamma^\rho_{\mu\nu}B^\nu\big)\partial_\rho = \big(\partial_\mu B^\rho + h^{\rho\bar\sigma}(\partial_\mu h_{\bar\sigma\nu})B^\nu\big)\partial_\rho = h^{\rho\bar\sigma}\partial_\mu(h_{\bar\sigma\nu}B^\nu)\partial_\rho.$$

Let us introduce complex vielbeine $e_\alpha = e^\mu_\alpha\partial_\mu$ and $e^\alpha = e^\alpha_\mu \mathrm{d}z^\mu$, such that $h_{\bar\mu\nu} = \frac{1}{2}\delta_{\bar\alpha\beta}e^{\bar\alpha}_{\bar\mu}e^\beta_\nu$. The components of the complex connection can be related to the metric $h_{\bar\mu\nu}$ and the vielbeine with the help of Leibniz' rule and (2.60),

$$\begin{aligned}\omega^\beta_{\mu\alpha}e_\beta \equiv D_\mu e_\alpha &= D_\mu(e^\nu_\alpha\partial_\nu) = (\partial_\mu e^\nu_\alpha)\partial_\nu + e^\nu_\alpha\Gamma^\rho_{\mu\nu}\partial_\rho \\ &= (\partial_\mu e^\nu_\alpha)\partial_\nu + e^\nu_\alpha h^{\rho\bar\sigma}\partial_\mu(h_{\bar\sigma\nu})\partial_\rho = h^{\rho\bar\sigma}\partial_\mu(e^\nu_\alpha h_{\bar\sigma\nu})\partial_\rho = e^\beta_\rho h^{\rho\bar\sigma}\partial_\mu(e^\nu_\alpha h_{\bar\sigma\nu})e_\beta.\end{aligned} \quad (2.61)$$

Comparing the coefficients of e_β yields the connection coefficients $\omega^\beta_{\mu\alpha}$. The remaining coefficients are obtained in the same way, and one finds

$$\omega^\beta_{\mu\alpha} = e^{\beta\bar\sigma}\partial_\mu e_{\bar\sigma\alpha}, \qquad \omega^{\bar\beta}_{\mu\bar\alpha} = e^{\bar\beta}_{\bar\sigma}\partial_\mu e^{\bar\sigma}_{\bar\alpha}, \qquad \omega^{\bar\beta}_{\bar\mu\bar\alpha} = e^{\bar\beta\sigma}\bar\partial_{\bar\mu}e_{\sigma\bar\alpha}, \qquad \omega^\beta_{\bar\mu\alpha} = e^\beta_\sigma\bar\partial_{\bar\mu}e^\sigma_\alpha, \quad (2.62)$$

where, for example, $e^{\beta\bar\sigma} = h^{\bar\sigma\rho}e^\beta_\rho$. Having recapitulated these facts, we are ready to rewrite the Dirac operator in complex coordinates. For that we insert the completeness relation (2.55) in $\mathrm{i}\not{D} = \mathrm{i}\gamma^n\delta^m_n D_m$ and obtain

$$\mathrm{i}\not{D} = Q + Q^\dagger \equiv 2\mathrm{i}\psi^\mu D_\mu + 2\mathrm{i}\psi^{\dagger\bar\mu}D_{\bar\mu}, \quad (2.63)$$

where we have introduced the independent fermionic raising and lowering operators,

$$\psi^\mu = \frac{1}{2}f^\mu_m\gamma^m, \quad \psi^{\dagger\bar\mu} = \frac{1}{2}f^{\bar\mu}_m\gamma^m, \quad (2.64)$$

and the complex covariant derivatives

$$D_\mu = f^m_\mu D_m, \quad D_{\bar\mu} = f^m_{\bar\mu}D_m. \quad (2.65)$$

Of course, the supercharge Q in (2.63) is just the charge in (2.37) rewritten in complex coordinates. Unlike the annihilation operators ψ^a, the fermionic operators ψ^μ are independent. They fulfill the anticommutation relations

$$\{\psi^\mu, \psi^\nu\} = \{\psi^{\dagger\bar\mu}, \psi^{\dagger\bar\nu}\} = 0, \qquad \{\psi^\mu, \psi^{\dagger\bar\nu}\} = \frac{1}{2}h^{\mu\bar\nu}, \quad (2.66)$$

where $h^{\mu\bar{\nu}} = f^{\mu}_{m} f^{\bar{\nu}m}$ is the inverse of $h_{\mu\bar{\nu}}$ in (2.57). This can be seen as follows,

$$h^{\bar{\mu}\sigma} h_{\sigma\bar{\nu}} = f^{\bar{\mu}m} f^{\sigma}_{m} \cdot f^{n}_{\sigma} f_{n\bar{\nu}} \stackrel{(2.54)}{=} f^{\bar{\mu}m} \left(f^{n}_{\sigma} f^{\sigma}_{m} + f^{n}_{\bar{\sigma}} f^{\bar{\sigma}}_{m} \right) f_{n\bar{\nu}} \stackrel{(2.55)}{=} f^{\bar{\mu}m} f_{m\bar{\nu}} \stackrel{(2.56)}{=} \delta^{\bar{\mu}}_{\bar{\nu}}. \tag{2.67}$$

The operators ψ^{μ} lower the value of the Hermitian number operator,

$$N = 2 h_{\bar{\mu}\nu} \psi^{\dagger\bar{\mu}} \psi^{\nu}, \tag{2.68}$$

by one, while the $\psi^{\dagger\bar{\mu}}$ raise it by one. The proof is simple,

$$[N, \psi^{\sigma}] = 2 h_{\bar{\mu}\nu} [\psi^{\dagger\bar{\mu}} \psi^{\nu}, \psi^{\sigma}] = -2 h_{\bar{\mu}\nu} \{\psi^{\dagger\bar{\mu}}, \psi^{\sigma}\} \psi^{\nu} = -h_{\bar{\mu}\nu} h^{\bar{\mu}\sigma} \psi^{\nu} = -\psi^{\sigma}. \tag{2.69}$$

With (2.55) the fermionic operators in (2.39) and (2.64) are related as follows,

$$\psi^{m} = \frac{1}{2} (\mathbb{1} + \mathrm{i} I)^{m}{}_{n} \gamma^{n} = 2 f^{m}_{\mu} \psi^{\mu}, \qquad \psi^{\dagger m} = \frac{1}{2} (\mathbb{1} - \mathrm{i} I)^{m}{}_{n} \gamma^{n} = 2 f^{m}_{\bar{\mu}} \psi^{\dagger\bar{\mu}}, \tag{2.70}$$

and we conclude that the number operators in (2.40) and (2.68) are indeed equal,

$$\frac{1}{2} \psi^{\dagger m} \psi_{m} = 2 g_{mn} f^{m}_{\bar{\mu}} f^{n}_{\nu} \psi^{\dagger\bar{\mu}} \psi^{\nu} = 2 h_{\bar{\mu}\nu} \psi^{\dagger\bar{\mu}} \psi^{\nu}. \tag{2.71}$$

Now we are ready to prove that in cases where $(\mathrm{i}\not{D})^2$ admits an extended supersymmetry there exists a superpotential for the spin and gauge connections. Indeed, if $\mathcal{M}$ is Kähler and the gauge field strength commutes with the complex structure,

$$F_{mn} = (I^{t} F I)_{mn}, \tag{2.72}$$

then the complex covariant derivatives commute,

$$[D_{\mu}, D_{\nu}] = \mathcal{F}_{\mu\nu} = f^{m}_{\mu} f^{n}_{\nu} \mathcal{F}_{mn} = 0. \tag{2.73}$$

This is just the integrability condition (cf. Yang's equation [34]) for the existence of a superpotential g such that the complex covariant derivative can be written as

$$D_{\mu} = g \partial_{\mu} g^{-1} = \partial_{\mu} + g \left(\partial_{\mu} g^{-1} \right) = \partial_{\mu} + \omega_{\mu} + A_{\mu}. \tag{2.74}$$

This useful property is true for D_{μ} acting on any (possibly charged) tensor field on a Kähler manifold, provided (2.72) holds. If the Kähler manifold admits a spin structure, as for example $\mathbb{C}P^n$ for odd values of n, then (2.74) holds true for a (possibly charged) spinor field, too.

Of course, the superpotential g depends on the representation according to which the fields transform under the gauge and Lorentz group. One of the more severe technical problems in our applications is to obtain g in the representation of interest. It consists of two factors, $g = g_A g_\omega$. The first factor g_A is the path-ordered integral of the gauge potential. According to (2.62) and (2.74) the matrix g_ω in the vector representation is just the vielbein $e^{\beta\bar{\sigma}}$. If one succeeds in rewriting this g_ω as the exponential of a matrix, then the transition to any other representation is straightforward: one contracts the matrix in the exponent with the generators in the given representation. This will be done for the complex projective spaces in Appendix B.

Now let us assume that we have found the superpotential g. Then we can rewrite the complex supercharge in (2.63) as follows,

$$Q = 2\mathrm{i}\psi^\mu D_\mu = gQ_0 g^{-1}, \quad Q_0 = 2\mathrm{i}\psi_0^\mu \partial_\mu, \quad \psi_0^\mu = g^{-1}\psi^\mu g. \tag{2.75}$$

Here, the annihilation operators ψ^μ are covariantly constant,

$$D_\mu \psi^\nu = \partial_\mu \psi^\nu + \Gamma^\nu_{\mu\rho}\psi^\rho + [\omega_\mu, \psi^\nu] = 0. \tag{2.76}$$

The relation (2.75) between the free supercharge Q_0 and the g-dependent supercharge Q is the main result of this Section. It can be used to determine zero modes of the Dirac operator as follows. With (2.44) we find

$$\mathrm{i}\not{D}\Psi = 0 \qquad \Longleftrightarrow \qquad Q\Psi = 0, \quad Q^\dagger \Psi = 0. \tag{2.77}$$

In sectors with particle number $N = 0$ or $N = n$ one can easily solve for all zero modes. For example, $Q^\dagger$ automatically annihilates all states in the sector with $N = n$, such that zero modes only need to satisfy $Q\Psi = 0$. Because of (2.75), the general solution of this equation reads

$$\Psi = \bar{f}(\bar{z}) g \psi^{\dagger\bar{1}} \cdots \psi^{\dagger\bar{n}} |0\rangle, \tag{2.78}$$

where $\bar{f}(\bar{z})$ is some antiholomorphic function. Of course, the number of *normalizable* solutions depends on the gauge and gravitational background fields encoded in the superpotential g. With the help of the novel result (2.78) we shall find the explicit form of the zero modes on $\mathbb{C}P^n$ in Appendix B.

With the construction of zero modes we conclude our investigations concerning the implications of supersymmetry on the geometry of the manifold and the possible gauge field content. We have identified $\mathrm{i}\not{D}$ as a particular supercharge of the Hamiltonian $H = (\mathrm{i}\not{D})^2$. There exists a second supercharge (square-root of H), if $\mathcal{M}$ is a Kähler manifold and if $[F, I] = 0$. We have demonstrated how $\mathcal{N} = 3$ implies $\mathcal{N} = 4$, which is equivalent to $\mathcal{M}$ being hyper-Kähler and F_{mn} being (anti-)selfdual in $d = 4$ dimensions (more generally, $[F, I_i] = 0$ for the three complex structures I_i which are defined on the tangent bundle of $\mathcal{M}$). Similarly, $\mathcal{N} = 7$ implies $\mathcal{N} = 8$, and we have seen that in $d = 8$ dimensions only the free Dirac operator in flat space admits such a high amount of symmetry.

2.6. The Supersymmetric Coulomb Problem

In the following Sections we restrict our attention to the flat space $\mathbb{R}^d$ and Hamiltonians with scalar superpotentials. For two prominent examples, the $1/r$-potential of the Coulomb problem, as well as the r^2-potential of the harmonic oscillator, we derive the supersymmetric generalizations and solve the eigenvalue problem for the associated super-Hamiltonians by purely algebraic means. The basic results have been published in [AK1].

2.6.1. The Coulomb Problem and its Symmetries in d Dimensions

We will use the notion *Coulomb problem* and *hydrogen atom* simultaneously, thereby we refer to the $1/r$-potential in any dimension d, although this potential solves the Poisson equation in three dimensions only. It is well known [35] that the three dimensional Coulomb problem exhibits a *hidden* or *dynamical* symmetry algebra $\mathfrak{so}(4) \simeq \mathfrak{su}(2)\times\mathfrak{su}(2)$ that is bigger than the obvious rotational symmetry algebra $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$. On the level of classical mechanics, not only the components of the angular momentum vector

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}, \tag{2.79}$$

are conserved quantities, but in addition, for this particular potential, there exists the conserved Laplace-Runge-Lenz[2] vector [37]

$$\boldsymbol{C} = \frac{1}{m}\boldsymbol{p} \times \boldsymbol{L} - \frac{e^2}{r}\boldsymbol{r}, \tag{2.80}$$

where m is the reduced mass of the two-body system, and e represents the electromagnetic coupling constant in the case of an electron moving in the field of a nucleus, or the gravitational coupling constant, for a planet moving under the influence of the gravitational field of the sun.

Quantum mechanically, one defines the Hermitian Laplace-Runge-Lenz vector operator,

$$\boldsymbol{C} = \frac{1}{2m}(\boldsymbol{p} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{p}) - \frac{e^2}{r}\boldsymbol{r}. \tag{2.81}$$

Pauli [35] calculated the spectrum of the hydrogen atom, by exploiting the existence of this conserved operator. He noticed, that the components of $\boldsymbol{L}$, together with

$$\boldsymbol{K} = \sqrt{\frac{m}{-2H}}\boldsymbol{C}, \tag{2.82}$$

which is well-defined and Hermitian on bound states (with negative energies), generate the $\mathfrak{so}(4)$ algebra,

$$[L_a, L_b] = \mathrm{i}\hbar\epsilon_{abc}L_c, \qquad [L_a, K_b] = \mathrm{i}\hbar\epsilon_{abc}K_c, \qquad [K_a, K_b] = \mathrm{i}\hbar\epsilon_{abc}L_c. \tag{2.83}$$

Furthermore, the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\Delta - \frac{e^2}{r}, \tag{2.84}$$

can be expressed in terms of $\boldsymbol{K}^2 + \boldsymbol{L}^2$, one of the two second-order Casimir operators of this algebra,

$$H = -\frac{me^4}{2}\frac{1}{\boldsymbol{K}^2 + \boldsymbol{L}^2 + \hbar^2}. \tag{2.85}$$

The remaining Casimir operator $\boldsymbol{K} \cdot \boldsymbol{L}$ vanishes identically on wave functions. This condition singles out the symmetric representations of $\mathfrak{so}(4)$, and the eigenvalues of H

[2] A more suitable name for this constant of motion would be Hermann-Bernoulli-Laplace vector [36].

as well as the degeneracies of the corresponding energy levels can be read off from text books on group theory, e.g. [38].

Let us generalize these facts to arbitrary dimensions d. It will turn out that, in all cases, the algebra of rotations, $\mathfrak{so}(d)$, generated by the components of the angular momentum operators L_{ab}, can be extended to the *true* symmetry algebra $\mathfrak{so}(d+1)$ by combining them with the components of an appropriate generalization of the Laplace-Runge-Lenz vector (2.81).

The Schrödinger equation in d dimensions can be simplified to

$$H\Psi = E\Psi, \qquad H = p^2 - \frac{\eta}{r}, \qquad p_a = -\mathrm{i}\partial_a, \qquad a = 1, \dots d, \tag{2.86}$$

if we introduce dimensionless coordinates: distances are measured in units of the Compton wavelength $\lambda_c = \hbar/mc$, η is twice the fine-structure constant and the dimensionless energy E is measured in units of $mc^2/2$. The central force is attractive for positive η.

The d-dimensional generalization of the angular momentum (pseudo)vector $\boldsymbol{L}$ is given by the antisymmetric matrix L_{ab},

$$L_{ab} = x_a p_b - x_b p_a, \qquad a, b = 1, 2, \dots, d, \tag{2.87}$$

and its components generate the familiar $\mathfrak{so}(d)$ commutation relations,

$$[L_{ab}, L_{cd}] = \mathrm{i}(\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{ad}L_{bc} - \delta_{bc}L_{ad}). \tag{2.88}$$

Generalizing the Laplace-Runge-Lenz vector to d dimensions,

$$C_a = L_{ab}p_b + p_b L_{ab} - \frac{\eta x_a}{r}, \tag{2.89}$$

we find that its components C_a commute with the Hamiltonian and form a vector with respect to d-dimensional rotations induced by L_{ab},

$$[L_{ab}, C_c] = \mathrm{i}(\delta_{ac}C_b - \delta_{bc}C_a). \tag{2.90}$$

The commutator of C_a and C_b is proportional to the product of L_{ab} and H,

$$[C_a, C_b] = -4\mathrm{i}L_{ab}H. \tag{2.91}$$

In analogy to the three dimensional case, we rescale the components C_a on the subspace of bound states (with negative energy),

$$K_a = \frac{1}{\sqrt{-4H}} C_a, \qquad \text{such that} \qquad [K_a, K_b] = \mathrm{i} L_{ab}. \tag{2.92}$$

The generators L_{ab} and K_a form a closed algebra of dimension $\frac{1}{2}d(d+1)$. Using the combinations

$$L_{AB} = \left(\begin{array}{c|c} L_{ab} & K_a \\ \hline -K_b & 0 \end{array} \right), \qquad A, B = 1, 2, \dots d+1, \tag{2.93}$$

one verifies that this algebra is, in fact, $\mathfrak{so}(d+1)$,

$$[L_{AB}, L_{CD}] = \mathrm{i}(\delta_{AC} L_{BD} + \delta_{BD} L_{AC} - \delta_{AD} L_{BC} - \delta_{BC} L_{AD}). \tag{2.94}$$

The square of the Laplace-Runge-Lenz vector,

$$C_a C^a = -4 K_a K^a H = \eta^2 + (2 L_{ab} L^{ab} + (d-1)^2) H, \tag{2.95}$$

can be solved for the Hamiltonian,

$$H = p^2 - \frac{\eta}{r} = -\frac{\eta^2}{(d-1)^2 + 4\mathcal{C}_{(2)}}. \tag{2.96}$$

$\mathcal{C}_{(2)}$ is the second-order Casimir operator of the dynamical symmetry algebra $\mathfrak{so}(d+1)$,

$$\mathcal{C}_{(2)} = \frac{1}{2} L_{AB} L^{AB} = \frac{1}{2} L_{ab} L^{ab} + K_a K^a. \tag{2.97}$$

It remains to characterize all those representations of this algebra that are realized in our Hilbert space of square-integrable functions, $\mathcal{H} = L_2(\mathbb{R}^d)$. In three dimensions, they are fixed by the observation that $\boldsymbol{K} \cdot \boldsymbol{L}$ vanishes, so only symmetric representations appear. In general, the question of *which representations are realized* cannot be answered by studying algebraic properties of $\mathfrak{so}(d+1)$, but we must analyze the action of the differentiation/multiplication operators on wave functions on $\mathbb{R}^n$. This will be done in Appendices A.1 and A.2 for the case of even and odd dimensional spaces. The rotation groups for odd/even dimensions belong to different series in the Cartan classification of semi-simple Lie algebras [39], therefore we have to treat these cases separately. Nevertheless, the final results are of the same form for any d.

Here we just summarize our results from the appendices. Only the symmetric representations of $\mathfrak{so}(d+1)$ appear. They correspond to the set $(\ell, 0, \ldots, 0)$ of eigenvalues for the Cartan generators, or equivalently, to the Young tableau of the form $\begin{array}{|c|c|c|c|}\hline 1 & & & \ell \\ \hline\end{array}$.

Highest weight states are given by

$$\Psi_{\text{h.w.}} = z_1^\ell \mathrm{e}^{-\gamma_\ell r}, \qquad \gamma_\ell = \frac{\eta}{d-1+2\ell}, \tag{2.98}$$

where z_1 is the complex coordinate in the $x_1 - x_2$ plane. The dimension of the corresponding representation (the whole multiplet can, of course, be obtained by acting with all lowering operators on $\Psi_{\text{h.w.}}$) is given by

$$\dim V_\ell = \binom{\ell+d}{\ell} - \binom{\ell+d-2}{\ell-2}. \tag{2.99}$$

The Casimir operator in a symmetric representation characterized by ℓ, takes values

$$\mathcal{C}_{(2)} = \ell(\ell+d-1), \qquad \ell = 0, 1, 2, \ldots \tag{2.100}$$

Therefore, the energy eigenvalues are given by

$$E_\ell = -\frac{2me^4}{\hbar^2}\frac{1}{(d-1+2\ell)^2}\,. \tag{2.101}$$

The appearance of the accidental degeneracy – phrased in the language of representation theory – corresponds to the following branching rule: the totally symmetric representations of $\mathfrak{so}(d+1)$, labelled by an index ℓ, decompose into representations of its subalgebra $\mathfrak{so}(d)$ as

$$\begin{array}{|c|c|c|c|}\hline 1 & & & \ell \\ \hline\end{array}\ \Big|_{\mathfrak{so}(d+1)} \quad \longrightarrow \quad \mathbb{1} \oplus \begin{array}{|c|}\hline \ \\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \oplus \ldots \oplus \begin{array}{|c|c|c|c|}\hline 1 & & & \ell \\ \hline\end{array}\ \Big|_{\mathfrak{so}(d)}, \tag{2.102}$$

where all representations on the right hand side possess the same energy E_ℓ. In $d=3$

dimensions we find the following well-known results,

$$\begin{aligned} \dim V_n &= (\ell+1)^2 = n^2, \\ \mathcal{C}_{(2)} &= \ell(\ell+2) = (n-1)(n+1), \\ E_n &= -\frac{me^4}{2\hbar^2}\frac{1}{n^2}, \end{aligned} \tag{2.103}$$

where we have identified the *principal quantum number* $n \equiv \ell + 1 = 1, 2, \ldots$

2.6.2. $\mathcal{N} = 2$ Supersymmetric Quantum Mechanics

In this Section we want to show how the Coulomb problem (or any other higher dimensional quantum system) can be embedded into a supersymmetric theory. For this we take the complex nilpotent supercharges from Section 2.1 and write the Hamiltonian as

$$H = \{Q, Q^\dagger\}, \qquad Q^2 = Q^{\dagger 2} = 0. \tag{2.104}$$

Let us specify details of the theory we are going to investigate. We take $\mathcal{M}$ to be $2d$-dimensional Euclidean space and consider Abelian gauge fields. In that case, the superpotential contains the gauge connection only. We may use the polar decomposition for the superpotential, $g = UR$, with unitary U and positive Hermitian R. One can show that g and R generate gauge-equivalent potentials, thus, we can always choose $g = R = \mathrm{e}^{-\chi}$. In addition we *assume* χ and all other quantities to be independent of half of the coordinates. Effectively, we remain with d coordinates x^a and $2d$ Hermitian γ-matrices γ^a and γ^{d+a}, where $a = 1, 2 \ldots, d$. For a fixed choice of the complex structure, the complex nilpotent supercharges contain the linear combinations $\psi^a = \frac{1}{2}(\gamma^a - \mathrm{i}\gamma^{d+a})$ and $\psi_a^\dagger = \frac{1}{2}(\gamma_a + \mathrm{i}\gamma_{d+a})$ of the latter ones.

We find

$$\{\psi_a, \psi_b^\dagger\} = \delta_{ab}, \qquad \{\psi_a, \psi_b\} = \{\psi_a^\dagger, \psi_b^\dagger\} = 0, \qquad a, b = 1, 2, \ldots, d. \tag{2.105}$$

The Clifford vacuum $|0\rangle$ is annihilated by ψ_a, and we can construct the finite-dimensional Clifford space by acting with $\psi_a^\dagger$ on $|0\rangle$. As before, the particle number is given by $N = \sum_{a=1}^d \psi_a^\dagger \psi^a$. The superpotential $g = \mathrm{e}^{-\chi}$, with $\chi = \chi(x_1, \ldots, x_d)$, gives now rise to the following supercharge (cf. the deformation (2.75))

$$Q = gQ_0g^{-1} = \mathrm{i}\sum_a \psi^a(\partial_a + \partial_a\chi), \qquad Q^\dagger = g^{-1}Q_0^\dagger g = \mathrm{i}\sum_a \psi^{a\dagger}(\partial_a - \partial_a\chi), \tag{2.106}$$

where the free supercharge and its adjoint are given by

$$Q_0 = \mathrm{i}\psi^a\partial_a, \qquad Q_0^\dagger = \mathrm{i}\psi^{a\dagger}\partial_a. \tag{2.107}$$

From (2.105) it follows that $Q_0^2 = 0$, and since Q and Q_0 are related by a similarity transformation, Q is nilpotent, too. Since Q ($Q^\dagger$) contains one fermionic annihilation (creation) operator, we conclude that

$$[N, Q] = -Q, \qquad [N, Q^\dagger] = Q^\dagger. \tag{2.108}$$

The Hamiltonian (2.104) is now given by the $2^d \times 2^d$-matrix Schrödinger operator,

$$H = \big(-\Delta + (\nabla\chi)^2 + \Delta\chi\big)\,\mathbb{1}_{2^d} - 2\sum_{a,b}\psi^{a\dagger}\chi_{,ab}\psi^b, \quad \text{where} \quad \chi_{,ab} \equiv \frac{\partial^2\chi}{\partial x^a\partial x^b}. \tag{2.109}$$

The nilpotent supercharge Q gives rise to the following Hodge decomposition of the Hilbert space,

$$\mathcal{H} = Q\mathcal{H} \oplus Q^\dagger\mathcal{H} \oplus \ker H, \tag{2.110}$$

where $\ker H$ is finite dimensional and spanned by the zero modes of H. Indeed, on the orthogonal complement of $\ker H$ we may invert H and write

$$\mathcal{H}_0^\perp = (QQ^\dagger + Q^\dagger Q)H^{-1}\mathcal{H}_0^\perp = Q\left(Q^\dagger H^{-1}\mathcal{H}_0^\perp\right) + Q^\dagger\left(QH^{-1}\mathcal{H}_0^\perp\right). \tag{2.111}$$

The supercharge Q maps every energy eigenstate in $Q^\dagger\mathcal{H} \cap \mathcal{H}_p$ with positive energy into an eigenstate in $Q\mathcal{H} \cap \mathcal{H}_{p-1}$ with the same energy. Its adjoint maps eigenstates in $Q\mathcal{H} \cap \mathcal{H}_p$ into those in $Q^\dagger\mathcal{H} \cap \mathcal{H}_{p+1}$ with the same energy. With the exception of zero energy states, there is an exact pairing between bosonic and fermionic eigenstates (or equivalently, between eigenstates with even and odd particle number N). Generically, one expects to find the situation depicted in Figure 2.1.

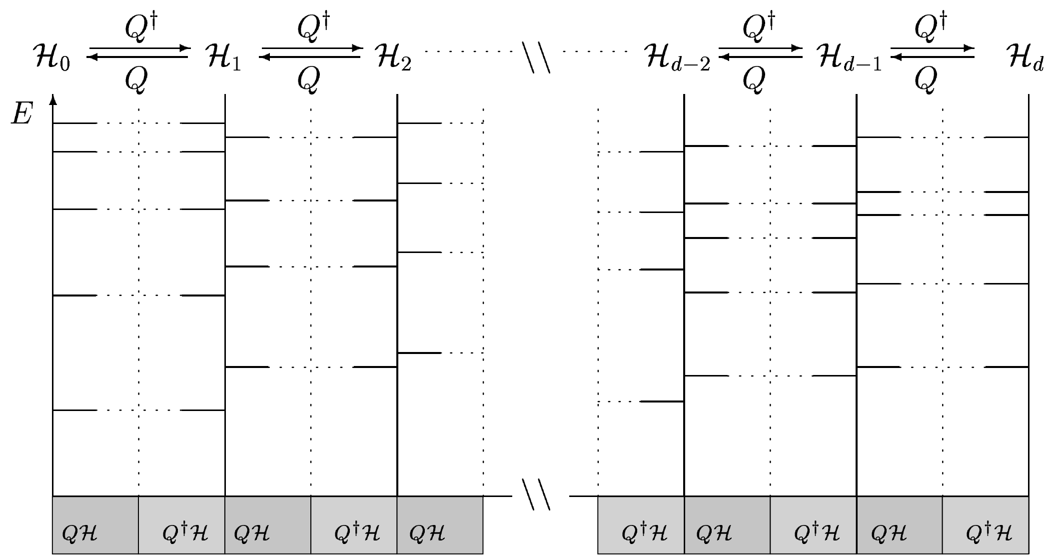

Figure 2.1.: Generic spectra of a supersymmetric Hamiltonian in d dimensions.

2.6.3. The Supersymmetric Laplace-Runge-Lenz Vector

For the particular case of a spherically symmetric superpotential, $\chi = \chi(r)$, we can define conserved total angular momenta J_{ab}. As is common in quantum mechanics, J_{ab} contains an orbital part L_{ab} and a spin part S_{ab},

$$J_{ab} = L_{ab} + S_{ab}, \qquad L_{ab} = x_a p_b - x_b p_a, \qquad S_{ab} = -\mathrm{i}(\psi_a^\dagger \psi_b - \psi_b^\dagger \psi_a). \tag{2.112}$$

The supercharge and its adjoint read

$$Q = \mathrm{i}\psi^a(\partial_a + x_a f), \qquad Q^\dagger = \mathrm{i}\psi^{a\dagger}(\partial_a - x_a f), \qquad f = \chi' r^{-1}. \tag{2.113}$$

A prime denotes differentiation with respect to the argument r. Q and $Q^\dagger$ are scalars under $\mathfrak{so}(d)$ rotations, induced by the combined action of L_{ab} (acting on the x and ∂ indices) and S_{ab} (acting on ψ and $\psi^\dagger$) indices.

Next, we define the supersymmetric generalization of the Laplace-Runge-Lenz vector. It will turn out that such a conserved vector only exists for the particular potential $\chi = -\lambda r$, which corresponds to a matrix Hamiltonian (2.109) that contains the Coulomb problem as its bosonic subsector with $N = 0$. Motivated by the structure of C_a in the ordinary case (2.89), we take the following ansatz for the supersymmetric version,

$$C_a = J_{ab}p_b + p_bJ_{ab} + x_a\tilde{f}(r)A. \tag{2.114}$$

First, for C_a to be a vector, the operator A must be a scalar under generalized rotations. Second, in the zero-particle sector $N = 0$, C_a should coincide with (2.89). Third, A should commute with the particle number N, since the J_{ab} do. A should not contain derivatives, since all derivatives are encoded in the $J_{ab}p_b$ terms. The most general ansatz for A, subject to these conditions is

$$A = \alpha\mathbb{1} - \beta N - \gamma S^\dagger S, \qquad S = r^{-1}x_a\psi^a. \tag{2.115}$$

The constants α, β and γ and the function $\tilde{f}(r)$ can be determined from the requirement

$$[C_a, Q] = 0 = [C_a, Q^\dagger], \tag{2.116}$$

which implies that C_a really generates symmetries of our Hamiltonian,

$$[C_a, H] = 0. \tag{2.117}$$

It turns out that only $\tilde{f}(r) = f(r)$ can yield a vanishing commutator. Furthermore,

$$\begin{aligned}[C_a, Q] = 2\{f\psi_b + f'Sx_b\}J_{ab} + \beta f x_a Q_0 + \mathrm{i} f x_a\{(\beta+\gamma)rf + \gamma\partial_r\}S \\ + \mathrm{i}\{f\psi_a + f'Sx_a\}(1 - d - A) + \mathrm{i}\gamma x_a r^{-1} f(d - N - 1)S.\end{aligned} \tag{2.118}$$

The terms containing derivatives,

$$f x_a(\beta - 2)Q_0 + 2\mathrm{i}r(f + rf')S\partial_a + \mathrm{i}(\gamma f - 2rf')Sx_a\partial_r, \tag{2.119}$$

cancel, provided

$$f = -\lambda r^{-1}, \qquad \chi = -\lambda r, \qquad \beta = -\gamma = 2. \tag{2.120}$$

We remain with

$$[C_a, Q] = \mathrm{i}\lambda r^{-1}(\alpha - d + 1)(\psi_a - x_a r^{-1}S), \tag{2.121}$$

which vanishes for $\alpha = d-1$. Hence, a supersymmetric extension of the Laplace-Runge-Lenz vector exists for $\chi = -\lambda r$ and is given by

$$C_a = J_{ab}p_b + p_b J_{ab} - \lambda x_a r^{-1} A, \qquad A = (d-1)\mathbb{1} - 2N + 2S^\dagger S. \tag{2.122}$$

The choice $\chi = -\lambda r$ for the superpotential yields the following supersymmetric extension of the Coulomb Hamiltonian,

$$H = -\Delta + \lambda^2 - \lambda r^{-1} A. \tag{2.123}$$

Restricted to the sector with $N = 0$, this is the Hamiltonian of the hydrogen atom[3].

In more general situations, we can have any ordinary quantum mechanical system as the $N = 0$ subsystem of a supersymmetric theory. In that case, we have the relation

$$\chi = \log \varphi_0 \tag{2.124}$$

between the superpotential χ and the ground state wave function φ_0 of the quantum system. For the hydrogen atom this implies

$$\varphi_0 \sim \exp -\lambda r \quad \longrightarrow \quad \chi = -\lambda r, \tag{2.125}$$

and for the harmonic oscillator (which we will briefly discuss in Section 2.7),

$$\varphi_0 \sim \exp -\frac{\omega}{4} r^2 \quad \longrightarrow \quad \chi = -\frac{\omega}{4} r^2. \tag{2.126}$$

2.6.4. Algebraic Determination of the Spectrum

In analogy with the bosonic case (2.91), we calculate the commutator of the components of the supersymmetrized Laplace-Runge-Lenz vector,

$$[C_a, C_b] = -4\mathrm{i} J_{ab}(H - \lambda^2). \tag{2.127}$$

This agrees with our earlier result, up to the shift in H and the replacement $L_{ab} \to J_{ab}$. In particular, on states with energy less than λ^2, the right hand side is positive, and we

[3] We have identified $\eta \equiv \lambda(d-1)$. The additional shift $+\lambda^2$ makes the lowest eigenvalue of this operator to be equal to zero, as we expect for a supersymmetric theory.

may do the following rescaling,

$$K_a = \frac{1}{\sqrt{4(\lambda^2 - H)}}\, C_a, \qquad \text{such that} \qquad [K_a, K_b] = \mathrm{i} J_{ab}. \tag{2.128}$$

As before, K_a and J_{ab} can be arranged to form generators J_{AB} of $\mathfrak{so}(d+1)$,

$$J_{AB} = \left(\begin{array}{c|c} J_{ab} & K_a \\ \hline -K_b & 0 \end{array}\right), \qquad A, B = 1, 2, \dots d+1. \tag{2.129}$$

Finally, we should calculate $C_a C^a$ and express the Hamiltonian in terms of Casimir operators. It turns out that this cannot be done so easily, but an additional piece of information is needed. The calculation gives

$$\begin{aligned} C_a C^a = {} & -2\lambda^2 J_{ab} J^{ab} + \left(2 J_{ab} J^{ab} + (d - 2N - 1)^2\right) Q Q^\dagger \\ & + \left(2 J_{ab} J^{ab} + (d - 2N + 1)^2\right) Q Q^\dagger. \end{aligned} \tag{2.130}$$

Now we need the Hodge decomposition (2.110) of our Hilbert space. The Hamiltonian leaves this decomposition invariant, and we may consider each subspace separately. Since $Q^2 = 0$ and $Q^{\dagger 2} = 0$, we find $H\big|_{Q\mathcal{H}} = QQ^\dagger$ and $H\big|_{Q^\dagger\mathcal{H}} = Q^\dagger Q$, so that we can solve (2.130) in each of these subspaces,

$$H\big|_{Q\mathcal{H}} = QQ^\dagger = \lambda^2 - \frac{(d-2N-1)^2\lambda^2}{(d-2N-1)^2 + 4\mathcal{C}_{(2)}}, \tag{2.131}$$

$$H\big|_{Q^\dagger\mathcal{H}} = Q^\dagger Q = \lambda^2 - \frac{(d-2N+1)^2\lambda^2}{(d-2N+1)^2 + 4\mathcal{C}_{(2)}}, \tag{2.132}$$

where $\mathcal{C}_{(2)}$ is the second-order Casimir of $\mathfrak{so}(d+1)$,

$$\mathcal{C}_{(2)} = \frac{1}{2} J_{AB} J^{AB} = \frac{1}{2} J_{ab} J^{ab} + K_a K^a. \tag{2.133}$$

All zero energy states are annihilated by Q and by $Q^\dagger$, and according to (2.130), the second-order Casimir must vanish on these states,

$$\mathcal{C}_{(2)}\big|_{\ker H} = 0. \tag{2.134}$$

Hence, every normalizable zero mode Ψ_0 of H transforms trivially under the dynamical symmetry group, $J_{AB}\Psi_0 = 0$.

With (2.131) and (2.132), we have obtained the algebraic solution for the supersymmetric Coulomb problem. Again, as in the bosonic case, we still have to decide which irreducible representations of $\mathfrak{so}(d+1)$ are realized. This gives us the possible eigenvalues of $\mathcal{C}_{(2)}$, hence those of H and the corresponding degeneracies.

The explicit expressions for the $\mathfrak{so}(d+1)$ Cartan generators, for the simple roots etc. in terms of coordinates, derivatives and spinors ψ^a and $\psi^\dagger_a$ are again deferred to Appendices A.1 and A.2 for even and odd values of d, respectively. The results given there can be made plausible by the following arguments. With respect to $\mathfrak{so}(d)$, all states in the sector with $N = 0$ furnish (totally) symmetric representations of the form $\begin{array}{|c|c|c|c|}\hline 1 & & & \ell \\ \hline\end{array}$. This we already know from the bosonic case. The states $\psi^{a_1\dagger}\ldots\psi^{a_\wp\dagger}\,|0\rangle$ in Clifford space form a (totally) antisymmetric representation of $\mathfrak{so}(d)$, which is denoted by $\begin{array}{|c|}\hline 1 \\ \hline \\ \hline \\ \hline \wp \\ \hline\end{array}$. Energy eigenstates are in the tensor product of symmetric (wave functions) and antisymmetric (Clifford structure) states. If we use the abbreviations

$$\mathcal{D}^\ell_\wp \sim \begin{array}{|c|c|c|c|}\hline 1 & & & \ell \\ \hline \\ \cline{1-1} \multicolumn{1}{|c|}{} \\ \cline{1-1} \multicolumn{1}{|c|}{\wp} \\ \cline{1-1}\end{array}\;, \qquad \mathcal{D}^\ell_1 \sim \begin{array}{|c|c|c|c|}\hline 1 & & & \ell \\ \hline\end{array}\;, \qquad \mathcal{D}^1_\wp \sim \begin{array}{|c|}\hline 1 \\ \hline \\ \hline \\ \hline \wp \\ \hline\end{array}\;, \tag{2.135}$$

to characterize the various representations, we conclude that all states Ψ transform according to the tensor product representation

$$\mathcal{D}^1_\wp \otimes \mathcal{D}^\ell_1 = \mathcal{D}^\ell_{\wp-1} \oplus \mathcal{D}^{\ell-1}_\wp \oplus \mathcal{D}^{\ell+1}_\wp \oplus \mathcal{D}^\ell_{\wp+1}. \tag{2.136}$$

Recall that ℓ is the order of the homogeneous polynomials in which we can expand our wave functions, and $\wp$ is the particle number of the sector in which Ψ lives[4]. Now, all representations of the algebra of rotations $\mathfrak{so}(d)$ that appear in the tensor product

[4]For special values of ℓ and $\wp$ some subtleties arise: for $\Psi \in \mathcal{H}_0$ (or $\Psi \in \mathcal{H}_d$), the first factor on the left hand side of (2.136) becomes the trivial representation and we only obtain the fully symmetric representations $\mathcal{D}^\ell_1$ on th right, in agreement with our earlier results. In the sectors $\mathcal{H}_1$ ($\mathcal{H}_d$) the first (the last) representation on the right hand side is absent. For linear functions with $\ell = 1$, the second representation on the right is missing. Finally, one should keep in mind that the representations $\mathcal{D}^\ell_\wp$ and $\mathcal{D}^\ell_{d-\wp}$ of $\mathfrak{so}(d)$ are equivalent, and that for even dimensions the representations $\mathcal{D}^\ell_{d/2}$ are reducible: they contain a selfdual and an anti-selfdual multiplet.

(2.136) should group together into multiplets of $\mathfrak{so}(d+1)$.

The branching rule in the supersymmetric case, which can be obtained from either [40] or by using the package LiE [41], reads

$$\mathcal{D}^{\ell}_{\wp}\big|_{\mathfrak{so}(d+1)} \quad\longrightarrow\quad \mathcal{D}^{\ell}_{\wp} \oplus \mathcal{D}^{\ell-1}_{\wp} \oplus \ldots \oplus \mathcal{D}^{1}_{\wp} \oplus \mathcal{D}^{\ell}_{\wp-1} \oplus \mathcal{D}^{\ell-1}_{\wp-1} \oplus \ldots \oplus \mathcal{D}^{1}_{\wp-1}\big|_{\mathfrak{so}(d)}. \tag{2.137}$$

Using the inverse of this relation, it turns out that in each sector with fixed particle number $\wp$, all $\mathfrak{so}(d)$ multiplets can be derived from just two multiplets of $\mathfrak{so}(d+1)$, according to the rule

$$\mathcal{D}^{\ell}_{\wp} \oplus \mathcal{D}^{\ell}_{\wp+1}\big|_{\mathfrak{so}(d+1)} \quad\longrightarrow\quad \mathcal{D}^{1}_{\wp} \otimes \left(\mathbb{1} \oplus \mathcal{D}^{1}_{1} \oplus \ldots \oplus \mathcal{D}^{\ell}_{1}\right) - \mathcal{D}^{\ell+1}_{\wp} + \mathcal{D}^{\ell}_{\wp-1}\big|_{\mathfrak{so}(d)}.$$

There is one notable exception to these branching rules for even d. In the middle sector $\mathcal{H}_{n=d/2}$, the correct branching rule reads

$$\mathcal{D}^{\ell}_{n} \oplus \mathcal{D}^{\ell}_{n}\big|_{\mathfrak{so}(d+1)} \quad\longrightarrow\quad \mathcal{D}^{1}_{n} \otimes \left(\mathbb{1} \oplus \mathcal{D}^{1}_{1} \oplus \ldots \oplus \mathcal{D}^{\ell}_{1}\right) - \mathcal{D}^{\ell+1}_{n} - \mathcal{D}^{\ell}_{n} - \mathcal{D}^{\ell}_{n-1}\big|_{\mathfrak{so}(d)}.$$

In odd dimensions, $d = 2n+1$, the representations of $\mathfrak{so}(d+1)$ that contain bound states are depicted in Figure 2.2.

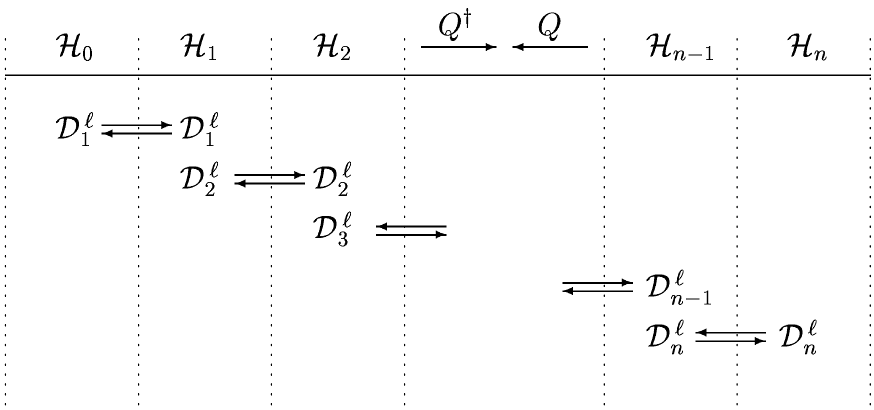

Figure 2.2.: Distribution of bound states in $d = 2n + 1$ dimensions.

In all sectors but $\mathcal{H}_0$ we have $\ell \in \mathbb{N}$. For $\mathcal{H}_0$ we have $\ell \in \mathbb{N}_0$, and $\ell = 0$ corresponds to the trivial representation. The sectors $\mathcal{H}_{\wp > n}$ turn out not to support any bound states, see the discussion in Appendix A.3. Therefore it is sufficient to consider sectors with $N \leq n$.

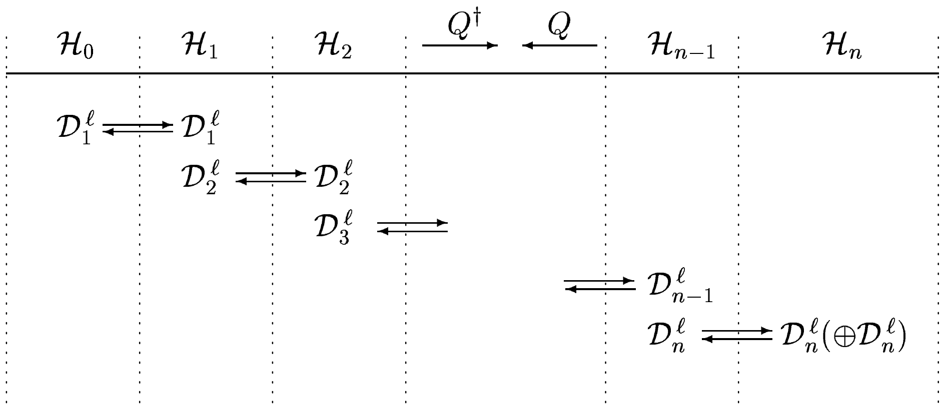

Figure 2.3.: Distribution of bound states in $d = 2n$ dimensions.

Similarly, for even dimensions, $d = 2n$, we find the situation depicted in Figure 2.3.

For any d, the value of the second-order Casimir is given by

$$\mathcal{C}_{(2)}\left(\mathcal{D}_\wp^\ell\right) = d(\ell + \wp - 1) + \ell(\ell - 1) - \wp(\wp - 1). \tag{2.138}$$

For the odd-dimensional case, the dimension of this representation is given by (A.22), where we have to set $d = 2n - 1$, and in the even dimensional case we can use (A.45) with $d = 2n$. Additionally we must set $(\ell_1, \ell_2, \ldots, \ell_n) = (\underbrace{\ell, 1, \ldots, 1}_{\wp}, 0, \ldots, 0)$.

Using (2.138), we can determine all eigenvalues of the supersymmetric Hamiltonian (2.123): in $\mathcal{H}_0$ only the symmetric representations $\mathcal{D}_1^\ell$ of $\mathfrak{so}(d+1)$ are realized. In addition, $Q\big|_{\mathcal{H}_0} = 0$, so we use (2.131),

$$\mathcal{H}_0 : \qquad E_0(\mathcal{D}_1^\ell) = \lambda^2 - \left(\frac{d-1}{d-1+2\ell}\right)^2 \lambda^2, \qquad \ell \in \mathbb{N}_0. \tag{2.139}$$

The index 0 at the energy E indicates the zero particle sector. The multiplet $\mathcal{D}_1^\ell$ of $\mathfrak{so}(d+1)$ is paired by the action of $Q^\dagger$ with a multiplet in the one-particle sector. According to the Figures 2.2 and 2.3, there is an additional multiplet $\mathcal{D}_2^\ell$ in $\mathcal{H}_1$. It is paired by the action of $Q^\dagger$ with states in the two-particle sector. Hence $H = QQ^\dagger$ on this multiplet, and we can apply (2.131),

$$E_1(\mathcal{D}_1^\ell) = \lambda^2 - \left(\frac{d-1}{d-1+2\ell}\right)^2 \lambda^2, \qquad \ell \in \mathbb{N}, \tag{2.140}$$

$$E_1(\mathcal{D}_2^\ell) = \lambda^2 - \left(\frac{d-3}{d-1+2\ell}\right)^2 \lambda^2, \qquad \ell \in \mathbb{N}. \tag{2.141}$$

Note that $\ell = 0$ does not occur in $\mathcal{H}_1$. In $\mathcal{H}_0$ the state $\ell = 0$ has vanishing energy and hence is annihilated by $Q^\dagger$ (instead of being mapped into $\mathcal{H}_1$). Now we can continue to $\mathcal{H}_2$, $\mathcal{H}_3$ etc. In each step we use

$$E_\wp(\mathcal{D}_\wp^\ell) = Q^\dagger Q\big|_{\mathcal{H}_\wp}(\mathcal{D}_\wp^\ell) = \lambda^2 - \left(\frac{d+1-2\wp}{d-1+2\ell}\right)^2, \qquad \ell \in \mathbb{N}, \tag{2.142}$$

$$E_\wp(\mathcal{D}_{\wp+1}^\ell) = QQ^\dagger\big|_{\mathcal{H}_\wp}(\mathcal{D}_{\wp+1}^\ell) = \lambda^2 - \left(\frac{d-1-2\wp}{d-1+2\ell}\right)^2, \qquad \ell \in \mathbb{N}. \tag{2.143}$$

2.6.5. Some Examples: Two, Three and Four Dimensions

As an illustration, let us apply our results from the previous Section to the particular cases $d = 2, 3$ and 4. In the two-dimensional case, we emphasise that a supersymmetric version of the Laplace-Runge-Lenz vector exists, contrary to what has been claimed in the literature [42]. The three-dimensional quantum system is, of course, the most interesting case. We have included the four-dimensional problem, since it already shows all additional structures that will be present in higher dimensions.

The Hilbert space of the two-dimensional systems splits into three sectors,

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2. \tag{2.144}$$

The Hamiltonian is a 4×4-matrix. Acting on a basis of the form $\{|0\rangle, |1\rangle, |2\rangle, |12\rangle\}$, where $|1\rangle = \psi^{1\dagger}|0\rangle$, $|2\rangle = \psi^{2\dagger}|0\rangle$ and $|12\rangle = \psi^{1\dagger}\psi^{2\dagger}|0\rangle$, it is given by

$$H = -\Delta + \lambda^2 + \lambda r^{-1} \begin{pmatrix} -1 & 0 & 0 \\ 0 & \delta_{ab} - 2x_a x_b r^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.145}$$

Obviously, for $\lambda > 0$, there are no bound states in the two-particle subspace. Only the multiplets $\mathcal{D}_1^{\ell\geq 0}$ in $\mathcal{H}_0$ and $\mathcal{D}_1^{\ell>0}$ in $\mathcal{H}_1$ give rise to bound states. There is one zero-energy state ($\ell = 0$), and the remaining eigenvalues are

$$E_\ell = \lambda^2 - \lambda^2 \frac{1}{(1+2\ell)^2}, \qquad \ell \in \mathbb{N}. \tag{2.146}$$

At $E = \lambda^2$ the continuum of scattering states starts, and we find the spectra depicted in Figure 2.4 for the supersymmetric system.

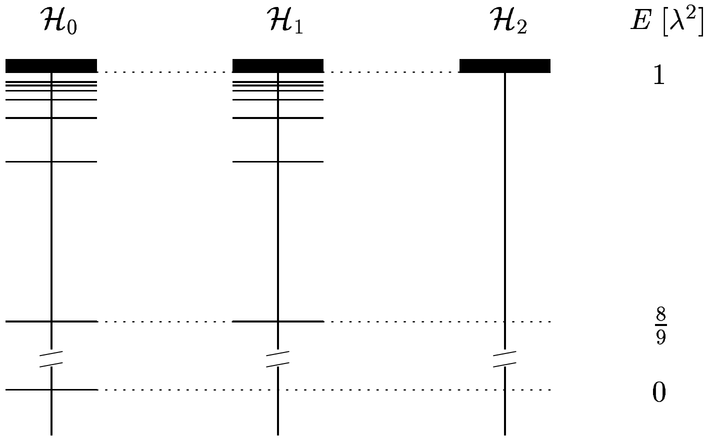

Figure 2.4.: Eigenvalues of H from (2.123) in $d = 2$ dimensions.

For the interesting three-dimensional case we have the splitting

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3, \tag{2.147}$$

and the Hamiltonian is an 8×8-matrix. Restricted to the zero-particle sector it coincides with the (shifted) Coulomb Hamiltonian,

$$H\big|_{\mathcal{H}_0} = -\Delta + \lambda^2 - 2\lambda r^{-1}. \tag{2.148}$$

In this sector, we find the ordinary hydrogen spectrum,

$$E_\ell = \lambda^2 - \lambda^2(1+\ell)^{-2}. \tag{2.149}$$

With the exception of the ground state, these states have partners in $\mathcal{H}_1$. There are no additional bound states in $\mathcal{H}_1$, since their would-be partners in $\mathcal{H}_2$ are in a sector which does not admit any bound states. The spectra for the three-dimensional system can be found in Figure 2.5.

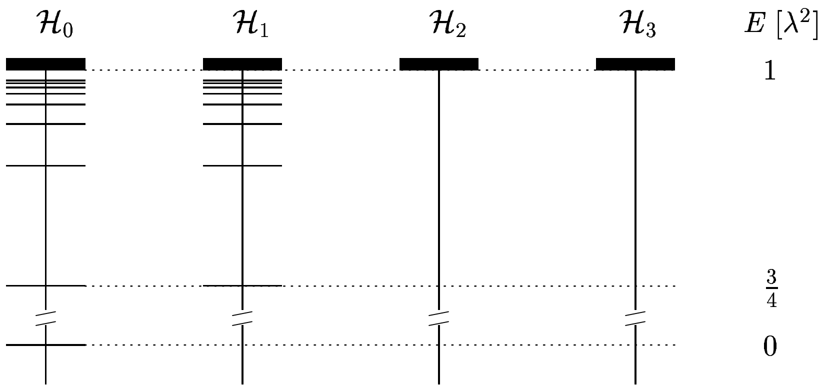

Figure 2.5.: Eigenvalues of H from (2.123) in $d = 3$ dimensions.

Finally, the four-dimensional system contains five subsectors,

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4. \tag{2.150}$$

The Hamiltonian in the zero particle sector,

$$H\big|_{\mathcal{H}_0} = -\Delta + \lambda^2 - 3\lambda r^{-1}, \tag{2.151}$$

admits a normalizable zero mode and a series $\mathcal{D}_1^\ell$ of bound states, similar to the three-dimensional case. These states are paired with states $\mathcal{D}_1^\ell$ in $\mathcal{H}_1$ with the same energies,

$$E_\ell = \lambda^2 - \lambda^2 \left(\frac{3}{2\ell + 3}\right)^2. \tag{2.152}$$

Now we do find an additional series $\mathcal{D}_2^\ell$ in $\mathcal{H}_1$, together with the partner states $\mathcal{D}_2^\ell$ in $\mathcal{H}_2$ and with energies

$$E_\ell = \lambda^2 - \lambda^2(2\ell + 3)^{-2}. \tag{2.153}$$

According to our general considerations, the pairing ceases in this sector, and there are no bound states, neither in $\mathcal{H}_3$ nor in $\mathcal{H}_4$. Figure 2.6 summarizes these results.

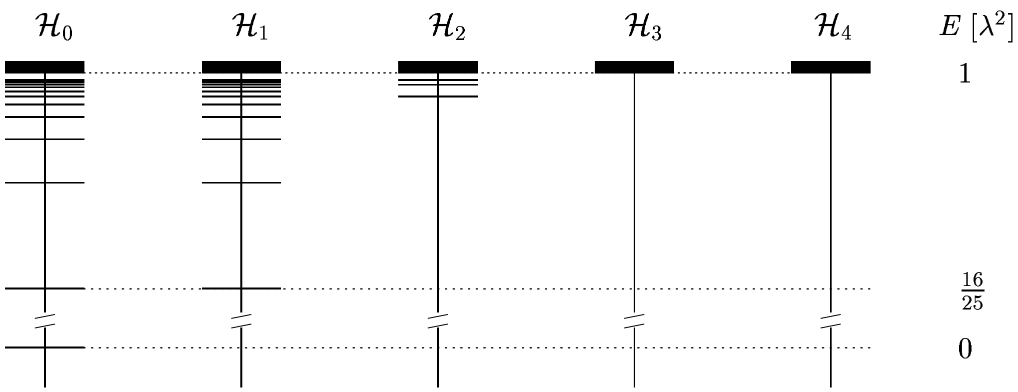

Figure 2.6.: Eigenvalues of H from (2.123) in $d = 4$ dimensions.

2.6.6. Concluding Remarks

We conclude this Section with some remarks: in the non-supersymmetric case, Itzykson and Bander [43] distinguished between the infinitesimal and the global method to solve the Coulomb problem. The former is based on the Laplace-Runge-Lenz vector and is the method used here (for both, the non-supersymmetric and the supersymmetric theory). In the global method one performs a stereographic projection of the d-dimensional momentum space to the unit sphere in $d+1$ dimensions, which in turn implies a $\mathfrak{so}(d+1)$ symmetry algebra. It would be interesting to perform a similar global construction for the supersymmetric systems, but so far this has not been accomplished.

2.7. The Supersymmetric Harmonic Oscillator

Having constructed the algebraic solution of the supersymmetric Coulomb problem, let us briefly discuss a very similar procedure for the case of the d-dimensional supersymmetric harmonic oscillator. In the Coulomb case, the whole construction is based on the fact that, apart from the generators of rotations L_{ab} (or J_{ab}), there is an additional set of operators, the components of the Laplace-Runge-Lenz vector C_a, which also commute with the Hamiltonian. Together they form generators of the *true* symmetry algebra $\mathfrak{so}(d+1)$ of this Hamiltonian. The construction can be applied in the usual bosonic theory as well as in its supersymmetric version.

In this Section, we will use the well-known fact that the real symmetry algebra of the isotropic harmonic oscillator in d dimensions is $\mathfrak{su}(d)$ rather than the algebra of rotations, $\mathfrak{so}(d)$. We will define the analogue of the (supersymmetric) Laplace-Runge-Lenz vector and use this enlarged symmetry to obtain the algebraic solution.

2.7.1. Bosonic Case

The Hamiltonian of the harmonic oscillator,

$$H = p^2 + \frac{1}{4}\omega^2 r^2, \tag{2.154}$$

for a particle of mass $m = \frac{1}{2}$ and in units where $\hbar = 1$, can be rewritten as

$$H = \omega \sum_{b=1}^{d} a_b^\dagger a^b + \frac{\omega d}{2} = \omega \left(N_B + d/2\right), \tag{2.155}$$

where we have introduced raising and lowering operators,

$$a_b^\dagger = \frac{\sqrt{\omega}}{2} x_b - \frac{\mathrm{i}}{\sqrt{\omega}} p_b, \qquad a_b = \frac{\sqrt{\omega}}{2} x_b + \frac{\mathrm{i}}{\sqrt{\omega}} p_b, \tag{2.156}$$

such that

$$[a_a, a_b] = [a_a^\dagger, a_b^\dagger] = 0, \qquad [a_a, a_b^\dagger] = \delta_{ab}\ . \tag{2.157}$$

Here N_B denotes the number of (bosonic) excitations. The invariance of H under unitary transformations is obvious in this notation. A change of basis

$$\boldsymbol{a} \to \boldsymbol{a}' = U\boldsymbol{a}, \qquad \boldsymbol{a}^\dagger \to \boldsymbol{a}'^\dagger = \boldsymbol{a}^\dagger U^\dagger, \tag{2.158}$$

is a canonical transformation [44] since the commutation relations between the primed operators are the same as (2.156)–(2.157) and the form of the Hamiltonian is unchanged, provided $U^\dagger = U^{-1}$.

H commutes with the $\frac{1}{2}d(d-1)$ components of the angular momenta

$$L_{ab} = -L_{ba} = x_a p_b - x_b p_a = -\mathrm{i}\left(a_a^\dagger a_b - a_b^\dagger a_a\right), \tag{2.159}$$

as well as with all components of

$$T'_{ab} = T'_{ba} = \frac{2}{\omega} p_a p_b + \frac{\omega}{2} x_a x_b = a_a^\dagger a_b + a_a a_b^\dagger. \tag{2.160}$$

The symmetric matrix T' is the equivalent of the Laplace-Runge-Lenz vector, and it seems that in the literature there is no name associated with it. T' itself was discovered in 1939 [45], much later than the Laplace-Runge-Lenz vector. Most likely the ease of solving the equations of motion of the harmonic oscillator, both classically and quantum mechanically, forestalled an active interest in finding the constants of motion, whereas, as we have shown, the Laplace-Runge-Lenz vector plays a very useful rôle in obtaining as well as exhibiting the solution of the hydrogen problem [23].

Observe that tr $T' = \frac{2}{\omega} H$, so T' contains only $\frac{1}{2}d(d+1) - 1$ independent components. For what follows, it is convenient to use the shifted operators

$$T_{ab} = T'_{ab} - \delta_{ab} = a_a^\dagger a_b + a_b^\dagger a_a. \tag{2.161}$$

The commutation relations between the $\frac{1}{2}d(d-1) + \frac{1}{2}d(d+1) - 1 = d^2 - 1$ components of L and T are

$$\begin{aligned}
[L_{ab}, L_{cd}] &= \mathrm{i}\left(\delta_{ac} L_{bd} + \delta_{bd} L_{ac} - \delta_{ad} L_{bc} - \delta_{bc} L_{ad}\right), \\
[L_{ab}, T_{cd}] &= \mathrm{i}\left(\delta_{ac} T_{bd} - \delta_{bd} T_{ac} + \delta_{ad} T_{bc} - \delta_{bc} T_{ad}\right), \\
[T_{ab}, T_{cd}] &= \mathrm{i}\left(\delta_{ac} L_{bd} + \delta_{bd} L_{ac} + \delta_{ad} L_{bc} + \delta_{bc} L_{ad}\right).
\end{aligned} \tag{2.162}$$

They can be combined to form generators of $\mathfrak{su}(d)$, cf. Appendix A.4.

The fact that the true symmetry algebra of the d-dimensional harmonic oscillator is $\mathfrak{su}(d)$ was published for the first time in [45]. The particular case of the two-dimensional harmonic oscillator represents an interesting curiosity because its symmetry group is definitely the unitary unimodular group and not its factor group, the three-dimensional orthogonal group. This distiction is readily demonstrated because the two-dimensional oscillator has degenerate levels of every integer multiplicity, and only odd-dimensional representations can occur for the rotation group.

Using

$$T^2 + L^2 = \sum_{a,b=1}^{d} T_{ab}^2 + \sum_{a,b=1}^{d} L_{ab}^2 = \frac{4}{\omega^2} H(H-\omega) - d(d-2), \tag{2.163}$$

the Hamiltonian (2.154) can be expressed in terms of the quadratic Casimir operator $\mathcal{C}_{(2)}$ of $\mathfrak{su}(d)$, cf. (A.78):

$$H^2 = \frac{d\omega^2}{d-1} \left(\mathcal{C}_{(2)} + \frac{1}{4} d(d-1) \right) . \tag{2.164}$$

Since the $\mathfrak{su}(d)$ algebra is realized on wave functions, only the symmetric representations appear, very similar to the Coulomb case. Let us use our abbreviation $\mathcal{D}_{\wp}^{\ell}$ for Young tableaux, cf. (2.135). From Appendix A.4, in particular formulae (A.80) and (A.83), we can read off the eigenvalues of H and the degeneracies of the eigenspaces,

$$E_\ell = \omega \left(\ell + d/2 \right), \qquad \dim(V_\ell) = \frac{(d+\ell-1)!}{l!(d-1)!}. \tag{2.165}$$

These are the well-known formulae: if ℓ is identified with the number N_B of excitations, $\dim(V_\ell)$ in (2.165) is just the number of possibilities to distribute these excitations with respect to the d independent one-dimensional oscillators.

The representations $\mathcal{D}_1^{\ell}$ of $\mathfrak{su}(d)$ can be decomposed into representations of $\mathfrak{so}(d)$ with the help of the following branching rule,

$$\mathcal{D}_1^{\ell}\Big|_{\mathfrak{su}(d)} \longrightarrow \mathcal{D}_1^{\ell} \oplus \mathcal{D}_1^{\ell-2} \oplus \mathcal{D}_1^{\ell-4} \oplus \ldots \Big|_{\mathfrak{so}(d)}, \tag{2.166}$$

cf. (A.84). In Table 2.2 the representations of $\mathfrak{su}(d)$, the energies and their degeneracies as well as the representations of $\mathfrak{so}(d)$ (for the particular case of $d=3$) are shown.

2.7.2. Supersymmetric Harmonic Oscillator

The construction of the supersymmetric Hamiltonian parallels our discussion for the Coulomb case in Section 2.6. As already mentioned there, the ground state wave function of the harmonic oscillator $\varphi_0(r) \sim \exp -\frac{\omega}{4} r^2$ implies the superpotential $\chi(r) = -\frac{\omega}{4} r^2$.

Representations of $\mathfrak{su}(3)$	$\mathbb{1}$	$\square$	$\square\square$	$\square\square\square$
Energy $[\omega]$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$
Degeneracy	1	3	6	10
Representations of $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$	$\mathbb{1}$	$\square$	$\square\square \oplus \mathbb{1}$	$\square\square\square \oplus \square$

Table 2.2.: $d = 3$ as a particular example.

With this choice, the Hamiltonian in (2.109) reads

$$H = p^2 + \frac{1}{4}\omega^2 r^2 + \omega\,(N - d/2) = \omega(N_B + N). \tag{2.167}$$

Restricted to the zero-fermion sector ($N = 0$), this is the standard bosonic oscillator, shifted by $-\frac{d}{2}\omega$ to have the lowest eigenvalue equal to zero.

In any other sector with fixed particle number N, the Hamiltonian corresponds to a shifted harmonic oscillator. Therefore, eigenvalues and degeneracies in all sectors can be read off immediately. Nevertheless, we will use the $\mathfrak{su}(d)$ structure that is also present in the supersymmetric case to derive the spectrum in an alternative way.

The components of T in (2.161) still commute with the Hamiltonian. But T is not supersymmetric in the sense that T does not commute with Q and $Q^\dagger$. However, a supersymmetric form of T can easily be obtained due to its simple structure,

$$T_{ab} = a_a^\dagger a_b + a_b^\dagger a_a + \psi_a^\dagger \psi_b + \psi_b^\dagger \psi_a. \tag{2.168}$$

Again, Q and $Q^\dagger$ are scalars with respect to rotations induced by the total angular momenta

$$J_{ab} = L_{ab} + S_{ab} = -\mathrm{i}\left(a_a^\dagger a_b - a_b^\dagger a_a + \psi_a^\dagger \psi_b - \psi_b^\dagger \psi_a\right). \tag{2.169}$$

The components of J and T (in (2.168)) obey commutation relations similiar to (2.162). As in the bosonic case, they can be combined to form generators of the $\mathfrak{su}(d)$ algebra.

The quadratic Casimir now reads (cf. (A.87))

$$\mathcal{C}_{(2)} = \frac{1}{4}(T^2 + J^2) - \frac{T_{aa}^2}{4d}. \tag{2.170}$$

It is possible to express $\mathcal{C}_{(2)}$ in terms of $QQ^\dagger$ and $Q^\dagger Q$,

$$\begin{aligned}\mathcal{C}_{(2)} = &\frac{1}{\omega}\left(d - 1 - 2N + \frac{1}{\omega}\frac{d-1}{d}QQ^\dagger\right)QQ^\dagger \\ &+ \frac{1}{\omega}\left(d + 1 - 2N + \frac{1}{\omega}\frac{d-1}{d}Q^\dagger Q\right)Q^\dagger Q. \end{aligned}\tag{2.171}$$

Again, this is sufficient to derive the whole spectrum of H, due to the Hodge decomposition (2.110) of the Hilbert space $\mathcal{H}$. On any of its eigenstates, H is either of the form $QQ^\dagger$ or $Q^\dagger Q$. We thus find for $H\big|_{Q\mathcal{H}} = QQ^\dagger$:

$$H^2 + \frac{d\omega}{d-1}\left(d - 1 - 2N\right)H = \frac{d\omega^2}{d-1}\mathcal{C}_{(2)}, \tag{2.172}$$

and similar for $H\big|_{Q^\dagger\mathcal{H}} = Q^\dagger Q$:

$$H^2 + \frac{d\omega}{d-1}\left(d + 1 - 2N\right)H = \frac{d\omega^2}{d-1}\mathcal{C}_{(2)}. \tag{2.173}$$

Since now only representations of the form $\mathcal{D}_\wp^\ell$ appear, we can read off the value of $\mathcal{C}_{(2)}$ and the dimensions of those representations from (A.87),

$$\mathcal{C}_{(2)}\left(\mathcal{D}_\wp^\ell\right) = (\wp + \ell - 1)\left(d - \frac{\wp + \ell - 1}{d}\right) + \ell^2 - \wp^2 + \wp - l, \tag{2.174}$$

$$\dim\left(\mathcal{D}_\wp^\ell\right) = \frac{(d + \ell - 1)!}{(d-\wp)!(\wp-1)!(\ell-1)!(\wp+\ell-1)}. \tag{2.175}$$

This fixes the eigenvalues of $QQ^\dagger$ and $Q^\dagger Q$ (and therefore those of H) as well as the degeneracies of the energy eigenspaces. As a particular example we show the results for $d = 3$ in Figure 2.7. There are four sectors with fermion number $N = 0, 1, 2, 3$. The energies together with their degeneracies and the pairings are indicated as well.

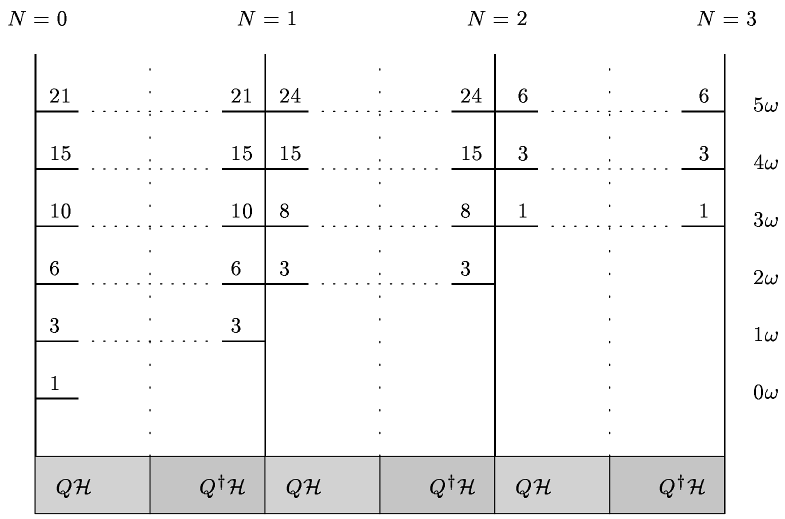

Figure 2.7.: $d = 3$ as a particular example in the supersymmetric case.

2.8. Outlook I: Kustaanheimo-Stiefel Transformations

We conclude our discussion of the supersymmetric Coulomb problem and the harmonic oscillator with the attempt to establish a connection between the two Hamiltonians. In particular, we are interested in a transformation, which relates the former problem in d dimensions to the latter one in D dimensions. In the non-supersymmetric case, such transformations are known in the framework of classical mechanics [46], where numerical studies of the $1/r$-potential can be regularized by mapping the problem to the r^2-potential (including a scaling of time in the extended phase space). For the problem of two bodies under the influence of their mutual gravitational attraction, one uses the two-dimensional, or Levi-Civita transformation. For the analogous three-body problem (a genuine three-dimensional problem), one employes the Kustaanheimo-Stiefel transformation. In the quantum mechanical setting, this relation can be applied to exactly solve the path integral for the Green function of the hydrogen atom, by rewriting it in terms of oscillator variables [47].

A construction for the supersymmetric case can be found in [48]. There, however, only the radial problem is analyzed, instead of the full problem we have in mind here.

Let us consider the non-supersymmetric case first. To find a connection between the two systems, we observe that the Schrödinger equation for the radial function $R(u)$ of the harmonic oscillator in D dimensions is given by

$$\frac{\mathrm{d}^2R}{\mathrm{d}u^2}+\frac{D-1}{u}\frac{\mathrm{d}R}{\mathrm{d}u}-\frac{L(L+D-2)}{u^2}R+\frac{2m}{\hbar^2}\left(E-\frac{1}{2}m\omega^2u^2\right)R=0. \tag{2.176}$$

$L=0,1,2,\ldots$ are the eigenvalues of the angular momentum operator. If we define now $r=u^2$, and use the relations

$$\frac{1}{u}\frac{\mathrm{d}}{\mathrm{d}u}=2\frac{\mathrm{d}}{\mathrm{d}r},\qquad \frac{\mathrm{d}^2}{\mathrm{d}u^2}=2\frac{\mathrm{d}}{\mathrm{d}r}+4r\frac{\mathrm{d}^2}{\mathrm{d}r^2}, \tag{2.177}$$

then (2.176) transforms into

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+\frac{d-1}{r}\frac{\mathrm{d}R}{\mathrm{d}r}-\frac{l(l+d-2)}{r^2}R+\frac{2m}{\hbar^2}\left(\epsilon+\frac{\eta}{r}\right)R=0, \tag{2.178}$$

which is the radial Schrödinger equation for the Coulomb problem, provided we make the following identifications

$$d=\frac{D}{2}+1,\qquad l=\frac{L}{2},\qquad \epsilon=-\frac{m\omega^2}{8},\qquad \eta=\frac{E}{4}. \tag{2.179}$$

The identification of (2.176) and (2.178) should be understood in a formal sense. In particular, since L takes values in $\mathbb{N}_0$, the angular momentum eigenvalue l of the Coulomb problem in d dimensions takes on half-integer values. In (2.179), the oscillator frequency ω is transformed into the energy eigenvalue ϵ, whereas the energy eigenvalue E determines the coupling constant η. Odd values of D lead to half-integer values of d, which cannot be given a sensible interpretation in terms of space dimensions. Thus, we restrict our attention to the case where D is even. Additionally, such a transformation exists for $D=1$ and $d=1$ [49].

The coordinate transformation, that includes also the angular dependence between coordinates x_i, $i=1,\ldots,d$, of the Coulomb problem and u_μ, $\mu=1,\ldots,D$, of the oscillator is given by $x=u^2$ for $D=1$ and $d=1$, by

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}u_1&-u_2\\u_2&u_1\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}, \tag{2.180}$$

for $D=2$ and $d=2$, by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} u_3 & -u_4 & u_1 & -u_2 \\ u_4 & u_3 & u_2 & u_1 \\ u_1 & u_2 & -u_3 & -u_4 \\ u_2 & -u_1 & -u_4 & u_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \tag{2.181}$$

for $D=4$ and $d=3$, and by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & -u_5 & -u_6 & -u_7 & -u_8 \\ u_5 & u_6 & -u_7 & -u_8 & u_1 & u_2 & -u_3 & -u_4 \\ u_6 & -u_5 & u_8 & -u_7 & -u_2 & u_1 & -u_4 & u_3 \\ u_7 & u_8 & u_5 & u_6 & u_3 & u_4 & u_1 & u_2 \\ u_8 & -u_7 & -u_6 & u_5 & u_4 & -u_3 & -u_2 & u_1 \\ u_2 & -u_1 & u_4 & -u_3 & u_6 & -u_5 & u_8 & -u_7 \\ u_3 & -u_4 & -u_1 & u_2 & -u_7 & u_8 & u_5 & -u_6 \\ u_4 & u_3 & -u_2 & -u_1 & -u_8 & -u_7 & u_6 & u_5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}, \tag{2.182}$$

for $D=8$ and $d=5$. These transformations are known as Euler [50], Levi-Civitá [51], Kustaanheimo-Stiefel [52] and Hurwitz transformations [53], respectively. A theorem by Hurwitz [53] states, that such transformations are possible only in $D=1,2,4$ and 8.

This analytic approach is difficult to generalize to the supersymmetric case. In particular, it is unknown, how the relations (2.180)–(2.182) for the coordinates and the corresponding formulae for the differentials translate into relations between the spinors ψ^a and $\psi_a^\dagger$. There is, however, a more algebraic approach for the non-supersymmetric case, outlined in [54].

We follow this approach for the particular case of the three-dimensional hydrogen atom. There, the Hamiltonian H can be expressed in terms of the (rescaled) Laplace-Runge-Lenz vector $\boldsymbol{K}$ and the angular momentum (pseudo)vector $\boldsymbol{L}$ as

$$H = -\frac{\eta^2/4}{1+\boldsymbol{K}^2+\boldsymbol{L}^2}, \qquad \boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}, \qquad \boldsymbol{K} = \sqrt{\frac{m}{-2H}}\boldsymbol{C}, \tag{2.183}$$

If we define the linear combinations

$$\boldsymbol{M} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{K}), \qquad \boldsymbol{N} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{K}), \tag{2.184}$$

then orthogonality $\boldsymbol{L} \cdot \boldsymbol{K} = \boldsymbol{K} \cdot \boldsymbol{L} = 0$ and commutation relations,

$$[L_a, L_b] = \mathrm{i}\epsilon_{abc} L_c, \qquad [L_a, K_b] = \mathrm{i}\epsilon_{abc} K_c, \qquad [K_a, K_b] = \mathrm{i}\epsilon_{abc} L_c, \tag{2.185}$$

translate into

$$\boldsymbol{M}^2 = \boldsymbol{N}^2, \qquad [M_a, M_b] = \mathrm{i}\epsilon_{abc} M_c, \qquad [N_a, N_b] = \mathrm{i}\epsilon_{abc} N_c, \qquad [M_a, N_b] = 0. \tag{2.186}$$

The algebra (2.186) can be realized in terms of four sets of bosonic creation and annihiliation operators a_1, a_2, a_3, a_4 and $a_1^\dagger, a_2^\dagger, a_3^\dagger, a_4^\dagger$ (cf. Schwinger's oscillator model of angular momentum [55]),

$$\boldsymbol{M} = \frac{1}{2} a^\dagger \boldsymbol{\sigma} a, \qquad \boldsymbol{N} = \frac{1}{2} b^\dagger \boldsymbol{\sigma} b, \qquad \text{where} \qquad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad b = \begin{pmatrix} a_3 \\ a_4 \end{pmatrix}. \tag{2.187}$$

This is the analogue of the Kustaanheimo-Stiefel transformation (2.181). From (2.186) we conclude that

$$\boldsymbol{M}^2 = N_{(a)}(N_{(a)} + 2) = \boldsymbol{N}^2 = N_{(b)}(N_{(b)} + 2), \tag{2.188}$$

so the number operators $N_{(a)}$ and $N_{(b)}$ associated with the two sets (a_1, a_2) and (a_3, a_4) of harmonic oscillators must be equal. Here

$$N_{(a)} = a_1^\dagger a_1 + a_2^\dagger a_2, \qquad N_{(b)} = a_3^\dagger a_3 + a_4^\dagger a_4. \tag{2.189}$$

Now we can use (2.183) and express H in terms of $N = N_{(a)} = N_{(b)}$,

$$H = -\frac{\eta^2/4}{(N+1)^2}, \qquad N = 0, 1, 2, \ldots \tag{2.190}$$

The degeneracy of the energy levels can be computed as follows. For a given N, there are $N+1$ possibilities to excite oscillators no. 1 and no. 2. At the same time, there are $N+1$ possibilities for the remaining oscillators, no. 3 and no. 4, so together this gives the (correct) degeneracy for the hydrogen spectrum, $(N+1)^2$.

Result: We can start with the Coulomb-Hamiltonian, and reexpress the Laplace-Runge-Lenz vector, the angular momenta etc. in terms of raising and lowering operators of four harmonic oscillators. The symmetry condition $\boldsymbol{K}\cdot\boldsymbol{L}=0$ for the Coulomb systems yields the restriction $N_{(a)}=N_{(b)}$ for the excitations of the oscillators. Together, this gives the correct eigenvalues and degeneracies for the Coulomb Hamiltonian. Note that one can carry out a similar construction for the $D=2$ and $d=2$ case and, presumably, also for $D=8$ and $d=5$.

This algebraic approach can be extended – at least partially – to the supersymmetric case. Let us focus again on $D=4$ and $d=3$. The super-extensions of $\boldsymbol{M}$ and $\boldsymbol{N}$ are given by

$$\boldsymbol{M}=\frac{1}{2}(a^\dagger\boldsymbol{\sigma}a+\chi^\dagger\boldsymbol{\sigma}\chi),\qquad \boldsymbol{N}=\frac{1}{2}(b^\dagger\boldsymbol{\sigma}b+\xi^\dagger\boldsymbol{\sigma}\xi), \tag{2.191}$$

where

$$a=\begin{pmatrix}a_1\\a_2\end{pmatrix},\qquad b=\begin{pmatrix}a_3\\a_4\end{pmatrix},\qquad \chi=\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix},\qquad \xi=\begin{pmatrix}\psi_3\\\psi_4\end{pmatrix}. \tag{2.192}$$

$\psi_1,\ldots,\psi_4$ are the fermionic annihilation operators for the super-oscillator in (2.105).

As pointed out in Section 2.6.4, we cannot express the Coulomb Hamiltonian H in terms of $\mathcal{C}_{(2)}$, but we have to utilize the Hodge decomposition (2.110). In the subspace $Q\mathcal{H}$ we find

$$H\big|_{Q\mathcal{H}}=QQ^\dagger=\eta^2-\frac{(2-2N)^2\eta^2}{(2-2N)^2+4\mathcal{C}_{(2)}}, \tag{2.193}$$

$$\begin{aligned}4\mathcal{C}_{(2)}&=2N_{(a)}^2+2N_{(b)}^2+4N_{(a)}+4N_{(b)}-6N_{(\chi)}^2-6N_{(\xi)}^2\\&\quad+12N_{(\chi)}+12N_{(\xi)}-4N_{(a)}N_{(\chi)}-4N_{(b)}N_{(\xi)}\\&\quad+8\sum_{m,n=1}^{n}a_m^\dagger a_n\psi_n^\dagger\psi_m+8\sum_{m,n=3}^{4}a_m^\dagger a_n\psi_n^\dagger\psi_m.\end{aligned} \tag{2.194}$$

So far, we have not suceeded in rewriting $QQ^\dagger$ in terms of $N_{(a)}$, $N_{(b)}$, $N_{(\chi)}$ and $N_{(\xi)}$ only, that is in terms of the number operators of the bosonic and fermionic oscillators. This is work in progress, and we hope to be able to report on this soon.

2.9. Outlook II: Supersymmetry on a Spatial Lattice

We conclude the first part of this thesis with some comments concerning supersymmetric field theories in $1+1$ dimensions on a spatial lattice and their relation to the supersymmetric Hamiltonians we have discussed so far. Our results can be found in [AK3] and will be discussed in much more detail in [56]. Unfortunately, there is not enough space here to present everything explicitly, and thus some familiarity with Wess-Zumino models in $1+1$ dimensions is assumed for the moment. Lattice regularization transforms our field theory into a problem of multi-dimensional quantum mechanics.

As an example, we consider the $(1+1)$-dimensional Wess-Zumino model with simple supersymmetry in the continuum formulation. In the simplest case, this theory describes the interaction of a real scalar ϕ with a two-component Majorana spinor ψ. The Lagrangian in the on-shell formulation,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - (W')^2 + \mathrm{i}\bar{\psi}\partial\!\!\!/\psi - \bar{\psi}W''\psi\right), \tag{2.195}$$

is invariant under supersymmetry transformations

$$\delta\phi = \bar{\zeta}\psi, \quad \delta\psi = -\mathrm{i}\partial\!\!\!/\phi\zeta - W'\zeta. \tag{2.196}$$

$W(\phi)$ is the superpotential, and a prime denotes the derivative with respect to ϕ. The Noether procedure, which we carry out in detail for the supersymmetric gauge theories in Chapter 3, yields the following Noether charge (supercharge) associated with the transformation (2.196),

$$Q = \int \mathrm{d}x \left(\pi - \partial_x\phi\gamma_* + \mathrm{i}W'\gamma^0\right)\psi. \tag{2.197}$$

Here, π is the momentum conjugate to ϕ. The superalgebra reads

$$\{Q_1, Q_1\} = 2(H+\mathcal{Z}), \quad \{Q_2, Q_2\} = 2(H-\mathcal{Z}), \quad \{Q_1, Q_2\} = 2P, \tag{2.198}$$

with Hamiltonian H, spatial momentum P and central charge $\mathcal{Z} = \int \mathrm{d}x \frac{\mathrm{d}W}{\mathrm{d}x}$. We use the representation

$$\gamma^0 = \sigma^2, \quad \gamma^1 = \mathrm{i}\sigma^3, \quad \gamma_* = \gamma^0\gamma^1 = -\sigma^1, \tag{2.199}$$

for the γ-matrices. Now we discretize space and leave time continuous, such that time translations generated by the Hamiltonian remain symmetries of our theory. On the one-dimensional lattice with N equidistant sites and periodic boundary conditions, we discretize the fields as follows,

$$\phi(x), \pi(x), \psi_\alpha(x) \quad \longrightarrow \quad \phi(n), \pi(n), \psi_\alpha(n), \qquad n = 1, 2, \ldots, N. \tag{2.200}$$

$\alpha = 1, 2$ indicates the upper/lower spinor component. Now derivatives become finite differences and in [AK3] the most prominent ones – left/right derivative, symmetric derivative, SLAC derivative – together with their various advantages and disadvantages are discussed. The ultimate choice for the lattice derivative is of no importance for what we are going to discuss here.

Lattice regularization always breaks part of the supersymmetry algebra, and we have to be careful enough as to leave the part which contains H intact. We *define* the Hamiltonian H_{lat} of the lattice theory to be square of the discretized supercharge Q_1,

$$\begin{aligned} H_{\text{lat}} &\equiv \frac{1}{2}\{Q_1, Q_1\} \qquad\qquad (2.201) \\ &= \frac{1}{2}(\pi, \pi) - \frac{1}{2}(\phi, \Delta\phi) + \frac{1}{2}(W', W') + \frac{1}{2}(\psi, h_{\text{F}}\psi) + (W', \partial_A\phi) - (W', \partial_S\phi), \end{aligned}$$

such that supersymmetry of H_{lat} is manifest. A bracket $(.,.)$ denotes the sum over all lattice points, as well as, possibly, the contraction of the spinor indices. h_{F} is the following matrix,

$$\begin{aligned} (h_{\text{F}})_{\alpha\beta} &= (h_{\text{F}}^0)_{\alpha\beta} + (\gamma^0)_{\alpha\beta} W'' = \mathrm{i} \begin{pmatrix} 0 & \partial \\ -\partial^\dagger & 0 \end{pmatrix}_{\alpha\beta} + (\gamma^0)_{\alpha\beta} W'' \\ &= -\mathrm{i}(\gamma_*)_{\alpha\beta}\partial_A - (\gamma^0)_{\alpha\beta}\partial_S + (\gamma^0)_{\alpha\beta} W''. \end{aligned} \tag{2.202}$$

Here we decomposed the (unspecified) lattice derivative ∂ into its Hermitian and anti-Hermitian parts, $\partial_S = \frac{1}{2}(\partial + \partial^\dagger)$ and $\partial_A = \frac{1}{2}(\partial - \partial^\dagger)$.

The point is now, that H_{lat} can be written as the Hamiltonian of a supersymmetric quantum system. Consider the Dirac operator $\mathrm{i}\not{D}$ on the direct product of Euclidean space and a torus, $\mathbb{R}^N \times \mathbb{T}^N$, with coordinates $x^M = (x^m, x^{\bar{m}})$, $m, \bar{m} = 1, 2, \ldots N$, in the background of the Abelian vector potential $A_M = (A_m, A_{\bar{m}})$. After a dimensional reduction, where the size R of the torus shrinks to zero, all fields become independent of the $x^{\bar{m}}$ coordinates. We impose the additional constraint, that $A_m = 0$ and find, using our very definition (2.20) of the Hamiltonian,

$$H_{\mathrm{qm}} \equiv \frac{1}{2}(\mathrm{i}\not{D})^2 = -\frac{1}{2}(\partial_m)^2 + \frac{1}{2}A_{\bar{m}}^2 - \frac{\mathrm{i}}{2}\gamma^m\gamma^{\bar{n}}(\partial_m A_{\bar{n}}). \tag{2.203}$$

We included a factor $\frac{1}{2}$ in order to adjust the prefactors. The lattice fields are identfied with the coordinates and γ-matrices of the quantum system as

$$\phi(n) \equiv x_n, \quad \pi(n) \equiv -\mathrm{i}\frac{\partial}{\partial x^n}, \quad \psi(n) \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}\gamma^n \\ \gamma^{\bar{n}}\end{pmatrix}. \tag{2.204}$$

If we take the gauge potential

$$A_{\bar{m}}(x^p) = -(\partial^\dagger\phi)(m) + W'(\phi(m)), \tag{2.205}$$

where ∂ is the lattice derivative and $W'(\phi(m))$ is the discretized version of $W'(\phi)$, the Hamiltonian of the lattice model (2.201) and the Hamiltonian of our quantum mechanical system (2.203) are identical:

$$A_{\bar{m}}^2 = (\partial\phi)^2 + (W')^2 - 2(W', \partial^\dagger\phi), \tag{2.206}$$

$$-\mathrm{i}\gamma^m\gamma^{\bar{n}}(\partial_m A_{\bar{n}}) = \psi(m)(\mathrm{i}\gamma_* + \gamma^0)\psi(n)\partial_m A_{\bar{n}}, \tag{2.207}$$

such that

$$H_{\mathrm{qm}} = \frac{1}{2}(\pi,\pi) + \frac{1}{2}(\partial\phi,\partial\phi) + \frac{1}{2}(W',W') + \frac{1}{2}(\psi, h_{\mathrm{F}}\psi) + (W',\partial_A\phi) - (W',\partial_S\phi).$$

Result: The Hamiltonian of the Wess-Zumino model in $1+1$ dimensions on a spatial lattice with N points can be mapped to the square of a Dirac operator defined on $\mathbb{R}^N \times \mathbb{T}^N$ in the background of a specific Abelian gauge field. A similar construction can be applied to Wess-Zumino models that consist of d copies of the simplest multiplet, and to models with $\mathcal{N}=2$ extended supersymmetry. The map between supersymmetric quantum mechanical systems and discretized field theories can now be used to explore properties of the latter, like the calculation of ground state wave functions or the discussion of supersymmetry breaking. In addition, we would like to extend these ideas to more realistic models in four dimensions. This is work in progress together with A. Wipf, J.D. Länge and F. Brauer, and we will publish some of the results soon [AK3].

3. Supersymmetric Field Theory

The second part of this thesis is devoted to the study of supersymmetric field theories. In particular, we analyze properties of an $\mathcal{N} = 2$ supersymmetric theory in three Euclidean dimensions. This theory describes a vectormultiplet and a hypermultiplet in interaction. We are interested in the behaviour of fermionic states in the background of bosonic field configurations. To this end we construct zero modes of the Dirac operator in various backgrounds. This operator contains the coupling of fermions to the gauge potential $\boldsymbol{A}$, as well as possible interaction terms of fermions with scalar fields. In this way we obtain some information about the number of zero modes of Dirac operators in three dimensions.

The question of how many zero modes the Euclidean Dirac operator possesses in a given background configuration is important for many physical applications, e.g. for the calculation of the fermionic determinant in the path integral [57] or for stability analyses of bound state systems [58].

In even dimensions, where a γ_* (the generalization of γ_5 in four dimensions) is definded, one can apply the Atiyah-Singer[1] index theorem [59] to count the difference of zero modes with positive and negative chirality. This index theorem, that holds for compact manifolds without boundary, has been generalized to manifolds with boundaries (the APS index theorem due to Atiyah, Patodi and Singer [60]) and to non-compact manifolds [61]. In general, the index can be expressed as an integral over certain characteristic classes and is related to topological invariants of the background configuration, like the magnetic flux or the instanton number. This gives a first hint as to whether or not zero modes are present. The number of zero modes is bounded from below by the absolute

[1] Recently, the Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2004, jointly to Sir Michael F. Atiyah, University of Edinburgh, and Isadore M. Singer, Massachusetts Institute of Technology, 'for their discovery and proof of the index theorem, bringing together topology, geometry and analysis, and their outstanding rôle in building new bridges between mathematics and theoretical physics', see `http://www.abelprisen.no/en/`.

value of the index. Often, a vanishing theorem can be applied, which states that zero modes of positive or negative chirality are absent. In that case, the index gives the total number of zero modes.

Explicit calculations of zero modes and the application of index theorems have been performed for field configurations in a variety of theories, like planar systems in magnetic fields [62], Dirac fields near magnetic flux strings [63], the fermion-vortex system in the two-dimensional Abelian Higgs model [64], the Schwinger model [65], fermions on the two-sphere [66], QCD on the two- and four-dimensional torus [67, 68], spetral flow and sphalerons [69], intersecting vortices [70], instanton and meron fields in the continuum [71] and on the lattice [72], (cosmic) strings and zero modes [73], string solitons with torsion [74], domain walls [75], fermions on $\mathbb{C}P^n$ [76, 77, 78, 79], and in knotted soliton backgrounds [80], in supersymmetric instanton-backgrounds [81], for noncommutative instantons [82] and for instantons and D-instantons in IIB supergravity [83]. These days, fermionic zero modes are used to detect instanton and meron configurations on the lattice [84].

The APS index theorem for manifolds with boundary has been applied to the cylinder, the two-dimensional ball B^2 and $B^2 \times B^2$ [85, 86, 87], as well as to the four dimensional ball B^4 [88]. We have generalized the last example to include gauge configurations with an arbitrary profile function and have explicitly constructed the zero modes in this case. Our results can be applied to instanton configurations that have undergone an Abelian projection. We summarize the calculations for this example in Appendix D.

In odd dimensions, no such index theorem is available. There is the Callias-Bott-Seeley index theorem [89] for zero modes of the Dirac Hamiltonian but not of the Dirac operator itself. In that case, a Higgs field is needed, and it is the topology of the Higgs field that determines the index in this case. This index theorem has been applied to magnetic monopoles and dyons [90, 91] and to domain walls in supersymmetric gauge theories [75].

Until 1986 it was not known whether there are zero-mode-supporting gauge field configurations $\boldsymbol{A}$ in three dimensions. First examples have been constructed in [24, 25]:

$$\mathrm{i}\not{D}\psi = (\mathrm{i}\not{\partial} - \not{A})\psi = 0, \tag{3.1}$$

where

$$\boldsymbol{A} = -\frac{3}{(1+r^2)^2}\begin{pmatrix} 2x_1x_3 - 2x_2 \\ 2x_2x_3 + 2x_1 \\ 1 - x_1^2 - x_2^2 - x_3^2 \end{pmatrix} \quad \text{and} \quad \psi = (1+r^2)^{-\frac{3}{2}}(\mathbb{1} + \mathrm{i}\not{x})\begin{pmatrix}1\\0\end{pmatrix}. \tag{3.2}$$

Here, a slash denotes the contraction with the vector of Pauli matrices, $\not{x} \equiv x_i\sigma^i$. In a series of publications, Adam, Muratori and Nash [26] constructed a whole class of examples, and in particular they showed that the dimension of the kernel of $\mathrm{i}\not{D}$ can be higher than one. It is now known that such zero modes exist and that their degeneracy can in principle be large, growing like $\int \mathrm{d}^3x\ |\boldsymbol{B}(\boldsymbol{x})|^{3/2}$ [25], where $\boldsymbol{B}$ is the magnetic field.

In even dimensional cases zero modes can be constructed as supersymmetry variations evaluated at a fixed background: for an instanton configuration [81, 83, 92] say, or for a static monopole configuration [93]. The very idea of employing these variations to generate zero modes goes back to Rossi in 1977 [94].

In this thesis we will prove that a similar construction cannot be carried out in the three-dimensional case. We demonstrate this for a particular supersymmetric field theory, which nevertheless exhibits all the general features. We show that one cannot obtain the examples of Adam, Muratori and Nash, which are defined in flat space $\mathbb{R}^3$, in this way. However, a further reduction of our theory to two dimensions admits nontrivial vortex-like configurations and we can construct zero modes in these background fields. For three-dimensional instanton fields in non-Abelian gauge theories, employing a version of Derricks theorem, we show that only non-normalizable zero modes can be obtained in this way. If we allow for a compactification to a three-torus, $\mathbb{R}^3 \to \mathbb{T}^3$, then zero modes can be constructed, and we present some examples: 't Hooft's constant-curvature solutions and the associated zero modes. In addition, we construct zero modes in the background of a 't Hooft-Polyakov monopole and relate our results to the well-known Jackiw-Rebbi modes [90] from the four-dimensional case.

We start with an $\mathcal{N}=2$ supersymmetric field theory in $(3+1)$-dimensional Minkowski space. Afterwards we reduce this theory to three-dimensional Euclidean space by compactifying the time direction. We study properties of the reduced theory, construct invariant background configurations and obtain zero modes of the Dirac equation.

Apart from the review article by Sohnius [16], from which we take our conventions, there are many other excellent reviews, lecture notes [95] and books [96] on supersymmetry and supersymmetric field theories.

3.1. A Toy Model

The very idea of how to generate fermionic zero modes can be understood with the help of the following simple example, which we borrow from [92]. Consider a quantum mechanical system for a boson ϕ and a two-component Majorana fermion ψ, described by the action

$$S[\phi,\psi] = \int \mathrm{d}\tau \; \mathcal{L} = \frac{1}{2}\int \mathrm{d}\tau \left\{\dot{\phi}^2 + U^2(\phi) + \psi^t\dot{\psi} + (\psi^t\sigma^2\psi)U'(\phi)\right\}. \tag{3.3}$$

$S[\phi,\psi]$ is invariant under the supersymmetry transformations

$$\delta\phi = \zeta^t\sigma^2\psi, \qquad \delta\psi = \sigma^2\dot{\phi}\zeta - U(x)\zeta, \qquad \text{such that} \qquad \delta\mathcal{L} = \frac{1}{2}\zeta^t\partial_\tau(\sigma^2\psi\dot{\phi} - \psi U).$$

The equations of motion associated with $S[\phi,\psi]$ are given by

$$\ddot{\phi} = UU' + \frac{1}{2}\psi^t\sigma^2\psi U'', \qquad \dot{\psi} = -\sigma^2 U'\psi. \tag{3.4}$$

For the particular function $U(\phi) = \frac{\sqrt{\lambda}}{2}(\phi^2 - \mu^2/\lambda)$, which gives rise to the well-known ϕ^4-potential for the boson, we find the following BPS solution,

$$\phi_0(\tau) = \frac{\mu}{\sqrt{\lambda}}\tanh\left(\frac{\mu}{\sqrt{2}}(\tau-\tau_0)\right), \qquad \psi_0 = 0. \tag{3.5}$$

This configuration[2] (a *kink-solution* for ϕ and vanishing ψ) is invariant under the restricted class of supersymmetry variations that satisfy $\zeta = \mathbb{P}_-\zeta$, where $\mathbb{P}_\pm = \frac{1}{2}(\mathbb{1} \pm \sigma^2)$. The remaining transformations, $\zeta = \mathbb{P}_+\zeta$, generate the following set of new solutions,

$$\phi_1 = \phi_0 + \delta\phi_0 = \phi_0, \qquad \psi_1 = \psi_0 + \delta\psi_0 = \sqrt{\frac{3\mu}{4\sqrt{2}}}\cosh^{-2}\left(\frac{\mu}{\sqrt{2}}(\tau-\tau_0)\right)\zeta. \tag{3.6}$$

[2]Due to translational invariance of our theory, the value of the *collective coordinate* τ_0 is not fixed.

One readily verifies, that ψ_1 is a zero mode of the Dirac operator, in the sense that it satisfies its equations of motion (3.4), whereas the boson field satisfies the linearized equation, without the bilinear term in ψ. So we obtain a fermionic zero mode as supersymmetry variation of a particular bosonic background configuration. This toy model already exhibits all generic features in which we are interested in. In what follows, we will extend these ideas to models that contain gauge interactions, too.

3.2. $\mathcal{N} = 2$ in $d = 4$ Dimensions

Let us start by considering a supersymmetric field theory in $3 + 1$ dimensions. These models are well-studied subjects in the literature cited above, so we will recapitulate them only briefly. The smallest spinor in this space is a Majorana spinor with four real components, or equivalently, a Weyl spinor with two complex degrees of freedom. A theory with $\mathcal{N} = 2$ extended supersymmetry contains two supercharges that can also be combined into a Dirac spinor, or a pair of symplectic Majorana spinors.

First, we restrict ourselves to the Abelian case and consider a vector multiplet coupled to a hypermultiplet. The vector multiplet $V(A_\mu, \lambda^i, M, N, \boldsymbol{D})$ consists of the Abelian vector potential A_μ, scalar fields M and N, and a triplet of real auxiliary fields $\boldsymbol{D} = (D_1, D_2, D_3)^t$, as well as a pair of symplectic Majorana spinors λ^i, $i = 1, 2$. The λ^i can be understood as two Dirac spinors, subject to the symplectic Majorana condition, $\lambda^i = \epsilon^{ij} S \lambda_j^*$.

Our conventions, together with some useful rules for calculations involving symplectic Majorana spinors can be found in Appendix C.

All fields in a supersymmetry multiplet transform according to the same representation of the gauge group. In particular, for $U(1)$ gauge theory, the whole vector multiplet is uncharged and thus transforms trivially,

$$\delta_{\text{gauge}} A_\mu = \partial_\mu \Lambda, \quad \delta_{\text{gauge}} M = \delta_{\text{gauge}} N = \delta_{\text{gauge}} \boldsymbol{D} = \delta_{\text{gauge}} \lambda^i = 0. \tag{3.7}$$

In the literature [16] it is shown, how V can be constructed out of an $\mathcal{N} = 1$ vector multiplet and an $\mathcal{N} = 1$ chiral multiplet. The degrees of freedom for the fields in V are summarized in Table 3.1. Note that fermionic and bosonic degrees of freedom match both off-shell and on-shell.

Under the $SU(2)$ R-symmetry, the automorphism group of the supercharges, the fields transform as singlets (A_μ, M, N), as a doublet (λ^i) and as a triplet $(\boldsymbol{D})$, respectively.

Field	A_μ	M	N	$\boldsymbol{D}$	λ^i
d.o.f. off-shell	3	1	1	3	8
d.o.f. on-shell	2	1	1	0	4

Table 3.1.: Degrees of freedom (d.o.f.) for the $\mathcal{N}=2$ vector multiplet.

The corresponding Lagrangian,

$$\mathcal{L}_{\text{vector}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mathrm{i}}{2}\bar{\lambda}_i \not{\partial}\lambda^i + \frac{1}{2}\partial_\mu M \partial^\mu M + \frac{1}{2}\partial_\mu N \partial^\mu N + \frac{1}{2}\boldsymbol{D}^2, \tag{3.8}$$

is invariant under the following supersymmetry transformations,

$$\begin{aligned} &\delta A_\mu = \mathrm{i}\bar{\zeta}_i \gamma_\mu \lambda^i, \quad \delta M = \mathrm{i}\bar{\zeta}_i \lambda^i, \qquad \delta N = -\bar{\zeta}_i \gamma_5 \lambda^i, \\ &\delta\lambda^i = \mathrm{i}(F,\Sigma)\zeta^i - \not{\partial}(M + \mathrm{i}\gamma_5 N)\zeta^i - \mathrm{i}\zeta^j \boldsymbol{\tau}_j{}^i \cdot \boldsymbol{D}, \\ &\delta\boldsymbol{D} = \boldsymbol{\tau}_i{}^j \bar{\zeta}_j \not{\partial}\lambda^i. \end{aligned} \tag{3.9}$$

Here, $\boldsymbol{\tau}$ is the three-component vector of Pauli matrices. Later on, $\boldsymbol{\tau}$ should not be confused with $\boldsymbol{\sigma}$, which is also the vector of Pauli matrices, but there the σ^m serve as representation of the Clifford algebra in three Euclidean dimensions.

One finds $\delta\mathcal{L}_{\text{vector}} = \partial_\mu V^\mu_{\text{vector}}$, where

$$V^\mu_{\text{vector}} = \bar{\zeta}_i \left(\frac{1}{2}(\tilde{F}^{\mu\nu}\gamma_5 - \mathrm{i}F^{\mu\nu})\gamma_\nu \lambda^i + \frac{\mathrm{i}}{2}\gamma^\mu \gamma^\nu \partial_\nu (M + \mathrm{i}\gamma_5 N)\lambda^i + \frac{1}{2}\boldsymbol{D}\cdot \boldsymbol{\tau}_j{}^i \gamma^\mu \lambda^j \right). \tag{3.10}$$

The Noether current J^μ associated with the symmetry transformations (3.9) is given by

$$J^\mu \equiv \kappa^\mu - V^\mu_{\text{vector}}, \qquad \text{where} \qquad \kappa^\mu \equiv \sum_\varphi \frac{\delta\mathcal{L}_{\text{vector}}}{\delta\partial_\mu\varphi}\delta\varphi\,. \tag{3.11}$$

Explicitly, one has

$$J^{\mu} = \bar{\zeta}_i \left[-(\tilde{F}^{\mu\nu}\gamma_5 + \mathrm{i}F^{\mu\nu})\gamma_\nu\lambda^i + \mathrm{i}\gamma^\nu\gamma^\mu\partial_\nu(M + \mathrm{i}\gamma_5 N) \right] \lambda^i. \tag{3.12}$$

The zero component of this current, when integrated over space, yields the supercharge Q,

$$\bar{\zeta}_i Q^i + \bar{Q}_i\zeta^i = \int \mathrm{d}^3x\ J^0, \tag{3.13}$$

such that

$$\begin{aligned} Q^i &= \int \mathrm{d}^3x \left(-\frac{1}{2}(\tilde{F}^{0m}\gamma_5 + \mathrm{i}F^{0m})\gamma_m + \frac{\mathrm{i}}{2}(\pi_M + \mathrm{i}\gamma_5\pi_N) - \frac{\mathrm{i}}{2}\gamma^0\gamma^m\partial_m(M + \mathrm{i}\gamma_5 N) \right) \lambda^i, \\ \bar{Q}_i &= \int \mathrm{d}^3x \lambda^\dagger\gamma^0 \left(-\frac{1}{2}(\tilde{F}^{0m}\gamma_5 - \mathrm{i}F^{0m})\gamma_m - \frac{\mathrm{i}}{2}(\pi_M + \mathrm{i}\gamma_5\pi_N) - \frac{\mathrm{i}}{2}\gamma^0\gamma^m\partial_m(M + \mathrm{i}\gamma_5 N) \right). \end{aligned}$$

Here, π_M and π_N are the momenta conjugate to M and N. Supersymmetry variations on any field φ are generated via its Poisson bracket,

$$\{\varphi, \bar{\zeta}_i Q^i + \bar{Q}_i\zeta^i\}_{\mathrm{PB}} = \delta\varphi. \tag{3.14}$$

The commutator of two supersymmetry transformations is given by

$$\begin{aligned} [\delta^{(1)}, \delta^{(2)}]A_\mu &= \delta^{(1)}(\mathrm{i}\bar{\zeta}_i^{(2)}\gamma_\mu\lambda^i) - (1 \leftrightarrow 2) \\ &= 2\mathrm{i}\bar{\zeta}_i^{(1)}\gamma^\rho\zeta^{(2)i}\partial_\rho A_\mu + 2\mathrm{i}\partial_\mu\bar{\zeta}_i^{(1)}\left(M + \mathrm{i}\gamma_5 N - \gamma^\rho A_\rho\right)\zeta^{(2)i}, \end{aligned} \tag{3.15}$$

and

$$[\delta^{(1)}, \delta^{(2)}]\varphi = 2\mathrm{i}\bar{\zeta}_i^{(1)}\gamma^\rho\zeta^{(2)i}\partial_\rho\varphi, \tag{3.16}$$

where φ is one of the fields M, N, λ^i or $\boldsymbol{D}$. This can be written as

$$[\delta^{(1)}, \delta^{(2)}] = \delta_{\text{translation}} + \delta_{\text{gauge}}, \tag{3.17}$$

where δ_{gauge} is defined in (3.7) with gauge parameter Λ given by

$$\Lambda = 2\mathrm{i}\bar{\zeta}_i^{(1)}(M + \mathrm{i}\gamma_5 N - \gamma^\rho A_\rho)\zeta^{i(2)}. \tag{3.18}$$

The equations of motion for the fields in the vector multiplet are

$$\partial_\mu F^{\mu\nu} = \Box M = \Box N = \boldsymbol{D} = \mathrm{i}\not{\partial}\lambda^i = 0. \tag{3.19}$$

As mentioned in the introduction, we would like to study the behaviour of charged matter fields in supersymmetric gauge theories. To this end, we add an $\mathcal{N} = 2$ hypermultiplet $H(\phi_i, F_i, \psi)$ to the theory. This hypermultiplet can be thought of two copies of $\mathcal{N} = 1$ chiral multiplets. It consists of two complex scalars ϕ_i, two complex auxiliary scalars F_i and a Dirac spinor ψ. As for the vector multiplet, ψ can also be considered as a pair of symplectic Majorana spinors. The degrees of freedom are summarized in Table 3.2.

Field	ϕ_i	F_i	ψ
d.o.f. off-shell	4	4	8
d.o.f. on-shell	4	0	4

Table 3.2.: Degrees of freedom (d.o.f.) for the $\mathcal{N} = 2$ hypermultiplet.

From the representation theory of the super-Poincaré algebra it is known, that any hypermultiplet for $\mathcal{N} = 2$ will contain particles of spin ≥ 1, unless there is a central charge $\mathcal{Z}$ which satisfies $\mathcal{Z} = \pm m$ [97]. Here m is the mass of the fields in H. If $\mathcal{Z} = \pm m$ holds true, the multiplet becomes a *short multiplet* and one can get rid of the higher-spin components.

For a massless hypermultiplet (the case that we are interested in) one finds the following Lagrangian,

$$\mathcal{L}_{\text{hyper}} = \frac{1}{2}\partial_\mu \phi^{i\dagger}\partial^\mu \phi_i + \frac{1}{2}F^{i\dagger}F_i + \mathrm{i}\bar{\psi}\not{\partial}\psi. \tag{3.20}$$

The supersymmetry transformations

$$\delta\phi_i = 2\bar{\zeta}_i\psi, \qquad \delta\psi = -\mathrm{i}\zeta^i F_i - \mathrm{i}\not{\partial}\zeta^i\phi_i, \qquad \delta F_i = 2\bar{\zeta}_i\not{\partial}\psi, \tag{3.21}$$

leave the action invariant, $\delta\mathcal{L}_{\text{hyper}} = \partial_\mu V^\mu_{\text{hyper}}$, where

$$V^\mu_{\text{hyper}} = \bar{\psi}\left(\partial^\mu\phi_i + \gamma^\mu F_i\right)\zeta^i + \bar{\zeta}_i\left(2\mathrm{i}\Sigma^{\mu\nu}\partial_\nu\phi^{i\dagger}\right)\psi. \tag{3.22}$$

The Noether current and the supercharges for the hypermultiplet can be calculated as for the vector multiplet and become

$$J^\mu = \bar{\psi}\gamma^\mu\gamma^\nu\zeta^i\partial_\nu\phi_i + \bar{\zeta}_i\gamma^\nu\gamma^\mu\partial_\nu\phi^{\dagger i}\psi, \tag{3.23}$$

such that ($\pi_{\phi_i} = \dot{\phi}^{\dagger i}$)

$$Q^i = \int \mathrm{d}^3x \left(\pi_{\phi_i} + \gamma^n\gamma^0\partial_n\phi^{\dagger i}\right)\psi, \qquad \bar{Q}_i = \int \mathrm{d}^3x \; \psi^\dagger\left(\gamma_0\pi_{\phi^{\dagger i}} + \gamma^n\partial_n\phi_i\right). \tag{3.24}$$

The supercharges induce the transformations (3.21) on the fields. The commutator of two such transformations yields

$$[\delta^{(1)}, \delta^{(2)}] = \delta_{\text{translation}} + \delta_{\text{central}}, \tag{3.25}$$

where the action of the central charge on the components of H is given by

$$\delta_{\text{central}}\phi_i = F_i, \qquad \delta_{\text{central}}\psi = \partial\!\!\!/\psi, \qquad \delta_{\text{central}}F_i = \Box\phi_i, \tag{3.26}$$

such that $\delta^2_{\text{central}} = \Box$. The equations of motion,

$$\Box\phi_i = F_i = \mathrm{i}\partial\!\!\!/\psi = 0, \tag{3.27}$$

imply that the central charge vanishes on-shell as required in the massless case.

In the next step, we want to couple the matter fields in the hypermultiplet to the vector multiplet. By construction all fields in V are uncharged under the $U(1)$ gauge symmetry. Nevertheless, the matter fields in H can transform in a different representation, in particular, they can be charged. It is well-known [98], that no supersymmetric renormalizable self-coupling terms for H exist. Thus, the only interactions of $\mathcal{N} = 2$ matter are gauge interactions. If we couple matter and gauge fields, the same supersymmetry algebra must be represented on all fields, i.e. we must extend the algebra of transformations for the vector multiplet components to include the central charge and the algebra on the matter fields to include a gauge transformation. This extension can be arranged as follows. We set

$$\delta_{\text{central}}A_\mu = \delta_{\text{central}}M = \delta_{\text{central}}N = \delta_{\text{central}}\lambda^i = \delta_{\text{central}}\boldsymbol{D} = 0, \tag{3.28}$$

and employ the minimal coupling prescription $\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}A_\mu$. The Lagrangian for the fully interacting theory is then given by [16]

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{hyper}} + \mathcal{L}_{\text{interaction}} \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mathrm{i}}{2}\bar{\lambda}_i \not{\partial}\lambda^i + \frac{1}{2}\partial_\mu M\partial^\mu M + \frac{1}{2}\partial_\mu N\partial^\mu N + \frac{1}{2}\boldsymbol{D}^2 \\
&\quad + \frac{1}{2}(D_\mu\phi_i)^\dagger D^\mu\phi_i + \mathrm{i}\bar{\psi}\not{D}\psi + \frac{1}{2}F^{i\dagger}F_i \\
&\quad + \mathrm{i}\phi^{\dagger i}\bar{\lambda}_i\psi - \mathrm{i}\bar{\psi}\lambda^i\phi_i - \bar{\psi}(M - \mathrm{i}\gamma_5 N)\psi \\
&\quad - \frac{1}{2}\phi^{\dagger i}(M^2+N^2)\phi_i + \frac{1}{2}\phi^{\dagger i}\boldsymbol{\tau}_i{}^j\cdot\boldsymbol{D}\phi_j.
\end{aligned} \tag{3.29}$$

Up to surface terms it is invariant under the supersymmetry transformations

$$\begin{aligned}
\delta A_\mu &= \mathrm{i}\bar{\zeta}_i\gamma_\mu\lambda^i, \qquad \delta M = \mathrm{i}\bar{\zeta}_i\lambda^i, \qquad \delta N = -\bar{\zeta}_i\gamma_5\lambda^i, \\
\delta\lambda^i &= \mathrm{i}(F,\Sigma)\zeta^i - \not{\partial}(M+\mathrm{i}\gamma_5 N)\zeta^i - \mathrm{i}\zeta^j\boldsymbol{\tau}_j{}^i\cdot\boldsymbol{D}, \\
\delta\boldsymbol{D} &= \boldsymbol{\tau}_i{}^j\bar{\zeta}_j\not{\partial}\lambda^i,
\end{aligned} \tag{3.30}$$

for the fields in V and

$$\begin{aligned}
\delta\phi_i &= 2\bar{\zeta}_i\psi, \\
\delta F_i &= 2\bar{\zeta}_i(\not{D} + \mathrm{i}M + \gamma_5 N)\psi - 2\bar{\zeta}_i\lambda^j\phi_j, \\
\delta\psi &= -\mathrm{i}\zeta^i F_i - (\mathrm{i}\not{D} + M + \mathrm{i}\gamma_5 N)\zeta^i\phi_i,
\end{aligned} \tag{3.31}$$

for the fields in H. Note that the supersymmetry transformations (3.30) are identical to (3.9), so they are not modified by the presence of the hypermultiplet. The change in $\mathcal{L}$ is given by

$$\delta\mathcal{L} = \delta\mathcal{L}_{\text{vector}} + \delta\mathcal{L}_{\text{hyper}} + \mathcal{L}_{\text{interaction}} = \partial_\mu V^\mu, \tag{3.32}$$

where

$$\begin{aligned}
V^\mu = \bar{\zeta}_i \Bigg(&\frac{1}{2}(\tilde{F}^{\mu\nu}\gamma_5 - \mathrm{i}F^{\mu\nu})\gamma_\nu\lambda^i + \frac{\mathrm{i}}{2}\gamma^\mu\gamma^\nu\partial_\nu(M+\mathrm{i}\gamma_5 N)\lambda^i \\
&+ 2\mathrm{i}\Sigma^{\mu\nu}(D_\nu\phi_i)^\dagger\psi - \mathrm{i}\phi^{\dagger\mathrm{i}}(M+\mathrm{i}\gamma_5 N)\gamma^\mu\psi + \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{\tau}_j{}^i\gamma^\mu\lambda^j + \frac{1}{2}\gamma^\mu\lambda^j\phi^{\dagger m}\phi_k\boldsymbol{\tau}_m{}^k\boldsymbol{\tau}_j{}^i\Bigg) \\
&+ \bar{\psi}(\gamma^\mu F_i + D^\mu\phi_i)\zeta^i.
\end{aligned} \tag{3.33}$$

Later on we will make us of the fact, that we can add a *Fayet-Iliopoulos (FI) term* $\boldsymbol{k}\cdot\boldsymbol{D}$ to $\mathcal{L}$ in (3.29). Here, $\boldsymbol{k}$ is a constant (spacetime independent) three-component vector. Due to the fact that the variation of $\boldsymbol{D}$ in (3.30) is a total divergence, adding $\boldsymbol{k}\cdot\boldsymbol{D}$ gives only a contribution to the current V^μ_{total}. Thus, the modified Lagrangian is still $\mathcal{N}=2$ invariant.

The Noether current and the supercharges now read

$$\begin{aligned}
J^\mu = \bar{\zeta}_i &\left[-(\tilde{F}^{\mu\nu}\gamma_5 + \mathrm{i}F^{\mu\nu})\gamma_\nu + \mathrm{i}\gamma^\nu\gamma^\mu\partial_\nu(M+\mathrm{i}\gamma_5 N)\right]\lambda^i \\
&+ \bar{\zeta}_i\left[\gamma^\nu\gamma^\mu(D_\nu\phi_i)^\dagger + \mathrm{i}(M+\mathrm{i}\gamma_5 N)\phi^{\dagger i}\gamma^\mu\right]\psi \\
&+ \bar{\psi}\left[\gamma^\mu\gamma^\nu(D_\nu\phi_i) - \mathrm{i}\gamma^\mu(M+\mathrm{i}\gamma_5 N)\phi_i\right]\zeta^i \\
&- \frac{1}{2}\bar{\zeta}_i\gamma^\mu\lambda^j\boldsymbol{\tau}_j{}^i\cdot\boldsymbol{\tau}_m{}^k\phi^{\dagger m}\phi_k,
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
Q^i = \int \mathrm{d}^3x \Bigg(&\left[-\frac{1}{2}(\tilde{F}^{0m}\gamma_5 + \mathrm{i}F^{0m})\gamma_m + \frac{\mathrm{i}}{2}(\pi_M + \mathrm{i}\gamma_5\pi_N) - \frac{\mathrm{i}}{2}\gamma^0\gamma^m\partial_m(M+\mathrm{i}\gamma_5 N)\right]\lambda^i \\
&+ \left[(D_0\phi_i)^\dagger - \gamma^0\gamma^m(D_m\phi_i)^\dagger + \mathrm{i}(M+\mathrm{i}\gamma_5 N)\phi^{\dagger i}\gamma^0\right]\psi \\
&-\frac{1}{4}\gamma^0\lambda^j\boldsymbol{\tau}_j{}^i\cdot\boldsymbol{\tau}_m{}^k\phi^{\dagger m}\phi_k\Bigg).
\end{aligned} \tag{3.35}$$

Again, they induce the supersymmetry transformations (3.30)–(3.31) on the fields. The commutator of two such transformations is given by (3.15)–(3.18) for the fields in V, and by

$$\begin{aligned}
[\delta^{(1)},\delta^{(2)}]\phi_j &= 2\mathrm{i}\bar{\zeta}_i^{(1)}\gamma^\mu\zeta^{i(2)}D_\mu\phi_j + 2\mathrm{i}\bar{\zeta}_i^{(1)}(F_j - \mathrm{i}(M+\mathrm{i}\gamma_5 N)\phi_j)\zeta^{i(2)}, \\
[\delta^{(1)},\delta^{(2)}]\,\psi &= 2\mathrm{i}\bar{\zeta}_i^{(1)}\gamma^\mu\zeta^{i(2)}D_\mu\,\psi + 2\mathrm{i}\bar{\zeta}_i^{(1)}\left((D_\mu^2 + \mathrm{i}(M-\mathrm{i}\gamma_5 N))\psi - \lambda^i\phi_i\right)\zeta^{i(2)}, \\
[\delta^{(1)},\delta^{(2)}]F_j &= 2\mathrm{i}\bar{\zeta}_i^{(1)}\gamma^\mu\zeta^{i(2)}D_\mu F_j + 2\mathrm{i}\bar{\zeta}_i^{(1)}\left((D^\mu D_\mu + M^2 + N^2)\phi_j\right. \\
&\qquad \left. -2\mathrm{i}\bar{\lambda}_j\psi - \boldsymbol{\tau}_j{}^k\cdot\boldsymbol{D}\phi_k + \mathrm{i}(M-\mathrm{i}\gamma_5 N)F_j\right)\zeta^{i(2)}.
\end{aligned} \tag{3.36}$$

for the matter fields. This can be written as

$$[\delta^{(1)},\delta^{(2)}] = 2\mathrm{i}\bar{\zeta}_i^{(1)}\gamma^\mu\zeta^{(2)i}D_\mu + 2\mathrm{i}\bar{\zeta}_i^{(1)}(\delta_{\text{central}} - \mathrm{i}M + \gamma_5 N)\zeta^{(2)i}, \tag{3.37}$$

where the action of δ_{central} is given by (3.28) and

$$\begin{aligned}
\delta_{\text{central}}\phi_i &= F_i, \\
\delta_{\text{central}}\,\psi &= \left(D_\mu^2 + \mathrm{i}M + \gamma_5 N\right)\psi - \lambda^i\phi_i, \\
\delta_{\text{central}}F_i &= \left(D_\mu^2 + M^2 + N^2\right)\phi_i - 2\mathrm{i}\bar\lambda_i\psi - \boldsymbol{\tau}_i{}^j \cdot \boldsymbol{D}\phi_j + 2\mathrm{i}MF_i,
\end{aligned} \tag{3.38}$$

respectively. The equations of motion for our fields are given by

$$\begin{aligned}
\partial_\mu F^{\mu\nu} &= \bar\psi\gamma^\nu\psi + \frac{\mathrm{i}}{2}\phi^{\dagger i}D^\nu\phi_i - \frac{\mathrm{i}}{2}(D^\nu\phi_i)^\dagger\phi_i, \\
\Box M &= -\bar\psi\psi - \phi^{\dagger i}\phi_i M, \\
\Box N &= \mathrm{i}\bar\psi\gamma_5\psi - \phi^{\dagger i}\phi_i N, \\
\mathrm{i}\partial\!\!\!/\lambda^i &= -\mathrm{i}\phi^{\dagger i}\psi - \mathrm{i}\epsilon^{ij}\phi_j S\psi^*, \\
\boldsymbol{D} &= -\frac{1}{2}\phi^{\dagger i}\phi_j\boldsymbol{\tau}_i{}^j,
\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}
D^2\phi_i &= 2\mathrm{i}\bar\lambda_i\psi - (M^2+N^2)\phi_i + \boldsymbol{\tau}_i{}^j \cdot \boldsymbol{D}\phi_j, \\
\mathrm{i}D\!\!\!\!/\,\psi &= \mathrm{i}\lambda^i\phi_i + (M - \mathrm{i}\gamma_5 N)\psi, \\
F_i &= 0.
\end{aligned} \tag{3.40}$$

Once a FI term is included, the last equation in (3.39) becomes

$$\boldsymbol{D} = -\frac{1}{2}\phi^{\dagger i}\phi_j\boldsymbol{\tau}_i{}^j - \boldsymbol{k}. \tag{3.41}$$

This concludes our recapitulation of the four-dimensional theory. Now we are ready to reduce this theory to three dimensions, where novel results concerning fermionic zero modes as supersymmetry variations will be derived.

3.3. Reduction to $d = 3$ Dimensions

In this Section, we are going to reduce our model from $3+1$ dimensions to three Euclidean dimensions. We use the simplest possible reduction and consider all fields to be independent of the time coordinate. Equivalently, we could consider a spacetime manifold $S^1 \times \mathbb{R}^3$, where the time direction t is compactified to a circle of radius R. Now, all fields can be expanded into Fourier series with respect to the parameter t. If we shrink the circle, $R \to 0$, the energy of all excited Fourier modes grows like $1/R^2$. For very small radii, all these modes decouple because of their high energies, and only the lowest Fourier mode, the one that is in fact independent of t, survives. There are other reductions that deal with (some of) the excited modes, too, but since we are interested in deriving a simple three dimensional model, our naive ansatz is sufficient.

The dimensional reduction of the individual terms in (3.29) will be discussed in the following. Let us call the zero component of the vector potential ϕ, $\phi \equiv A_0$, which is now taken to be time-independent. Accordingly, the field strength $F_{\mu\nu}$ decomposes into two parts, $F_{\mu\nu} \to (F_{mn}, F_{0m})$. Using

$$F_{0m} = \partial_0 A_m - \partial_m A_0 = -\partial_m \phi, \tag{3.42}$$

we find

$$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \to \quad -\frac{1}{4} F_{mn} F_{mn} + \frac{1}{2} \partial_m \phi \partial_m \phi. \tag{3.43}$$

All other bosonic fields are reduced similarly,

$$\begin{aligned} \frac{1}{2} \partial_\mu M \partial^\mu M \quad &\to \quad -\frac{1}{2} \partial_m M \partial_m M, \\ \frac{1}{2} \partial_\mu N \partial^\mu N \quad &\to \quad -\frac{1}{2} \partial_m N \partial_m N, \\ \frac{1}{2} \boldsymbol{D}^2 \quad &\to \quad \frac{1}{2} \boldsymbol{D}^2. \end{aligned} \tag{3.44}$$

During the reduction process, one should keep in mind that the reduced fields need to by rescaled by factors of $\sqrt{R}$, where R is the compactification radius. This must be done in order to guarantee their canonical dimensions, e.g.

$$F^{d=4}_{\mu=m,\nu=n} = \frac{1}{\sqrt{R}} F^{d=3}_{mn}. \tag{3.45}$$

The same holds for coupling constants, which we have set to unity in our original theory. Afterwards, the scale factors can be combined in each term of the Lagrangian to yield an overall factor R^{-1}, which cancels the time integration,

$$\int \mathrm{d}^4x\ \mathcal{L}_4 \quad \to \quad \int \mathrm{d}t \int \mathrm{d}^3x\ R^{-1}\mathcal{L}_3 = \int \mathrm{d}^3x\ \mathcal{L}_3. \tag{3.46}$$

Now, all fields have their canonical three dimensional mass dimensions. Observe that in the following we rescaled the dimensionful three-dimensional coupling constants and set them to unity again, in order to simplify our formulae. At each step they can be made explicit on dimensional grounds.

The reduction of fermionic terms is usually the most involved part. A spinor in the original space must be decomposed into spinors in the reduced space. Here, because of our particular choice (C.6) for the γ-matrices, this can be done rather easily. Every four-component spinor in four dimensions decomposes into two two-component spinors in three dimensions,

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \quad \to \quad \psi_-, \quad \psi_+. \tag{3.47}$$

$\psi_\pm$ are independent two-component Dirac spinors in three dimensions. Similarly, one finds for the symplectic Majorana spinors,

$$\lambda^1 = \begin{pmatrix} \lambda_- \\ \lambda_+ \end{pmatrix} \quad \to \quad \lambda_-, \quad \lambda_+. \tag{3.48}$$

Note that λ^2 (the symplectic partner of λ^1) gives no further independent fields. In particular, the kinetic term for λ^i in (3.29) reduces to

$$\frac{\mathrm{i}}{2}\bar{\lambda}_i \partial\!\!\!/ \lambda^i \quad \to \quad \mathrm{i}\bar{\lambda}_+ \partial\!\!\!/ \lambda_+ - \mathrm{i}\bar{\lambda}_- \partial\!\!\!/ \lambda_-. \tag{3.49}$$

Summarizing (3.43)–(3.49), the vector part of $\mathcal{L}$ in (3.29) reduces to

$$\begin{aligned} &-\frac{1}{4}F_{mn}F_{mn} + \frac{1}{2}\partial_m\phi\partial_m\phi - \frac{1}{2}\partial_m M\partial_m M \\ &-\frac{1}{2}\partial_m N\partial_m N + \mathrm{i}\bar{\lambda}_+ \partial\!\!\!/ \lambda_+ - \mathrm{i}\bar{\lambda}_- \partial\!\!\!/ \lambda_- + \frac{1}{2}\boldsymbol{D}^2. \end{aligned} \tag{3.50}$$

At the same time, the hypermultiplet part gives

$$
\begin{aligned}
&-\frac{1}{2}(D_m\phi_i)^\dagger(D_m\phi_i)+\frac{1}{2}\phi^{\dagger i}\phi_i\phi^2+\mathrm{i}\bar{\psi}_+\not{D}\psi_+\\
&-\mathrm{i}\bar{\psi}_-\not{D}\psi_- -\bar{\psi}_-\phi\psi_- -\bar{\psi}_+\phi\psi_+ +\frac{1}{2}F^{\dagger i}F_i,
\end{aligned}
\tag{3.51}
$$

and the interaction part reduces to

$$
\begin{aligned}
&\mathrm{i}\phi^{\dagger 1}(\bar{\lambda}_+\psi_- +\bar{\lambda}_-\psi_+)+\mathrm{i}\phi^{\dagger 2}(\lambda_-^t\sigma^2\psi_- +\lambda_+^t\sigma^2\psi_+)\\
&-\mathrm{i}(\bar{\psi}_+\lambda_- +\bar{\psi}_-\lambda_+)\phi_1-\mathrm{i}(\bar{\psi}_+\sigma^2\lambda_+^* +\bar{\psi}_-\sigma^2\lambda_-^*)\phi_2\\
&-\bar{\psi}_+(M+\mathrm{i}N)\psi_- -\bar{\psi}_-(M-\mathrm{i}N)\psi_+\\
&-\frac{1}{2}\phi^{\dagger i}(M^2+N^2)\phi_i+\frac{1}{2}\phi^{\dagger i}\boldsymbol{\tau}_i^{\ j}\cdot\boldsymbol{D}\phi_j.
\end{aligned}
\tag{3.52}
$$

Adding all terms from above, and using the definition (C.32) of the charge-conjugate spinor $\lambda_{\mathcal{C}}$, the reduced Lagrangian is given by

$$
\begin{aligned}
\mathcal{L}=&-\frac{1}{4}F_{mn}F_{mn}+\frac{1}{2}\partial_m\phi\partial_m\phi-\frac{1}{2}\partial_m M\partial_m M-\frac{1}{2}\partial_m N\partial_m N\\
&-\frac{1}{2}(D_m\phi_i)^\dagger(D_m\phi_i)+\mathrm{i}\bar{\psi}_+\not{D}\psi_+ -\mathrm{i}\bar{\psi}_-\not{D}\psi_- +\mathrm{i}\bar{\lambda}_+\not{\partial}\lambda_+ -\mathrm{i}\bar{\lambda}_-\not{\partial}\lambda_-\\
&-\bar{\psi}_-\phi\psi_- -\bar{\psi}_+\phi\psi_+ +\mathrm{i}\phi^{\dagger 1}(\bar{\lambda}_+\psi_- +\bar{\lambda}_-\psi_+)-\mathrm{i}(\bar{\psi}_+\lambda_- +\bar{\psi}_-\lambda_+)\phi_1\\
&+\mathrm{i}\phi^{\dagger 2}(\bar{\lambda}_{-\mathcal{C}}\psi_- +\bar{\lambda}_{+\mathcal{C}}\psi_+)-\mathrm{i}(\bar{\psi}_+\lambda_{+\mathcal{C}}+\bar{\psi}_-\lambda_{-\mathcal{C}})\phi_2\\
&-\bar{\psi}_+(M+\mathrm{i}N)\psi_- -\bar{\psi}_-(M-\mathrm{i}N)\psi_+\\
&-\frac{1}{2}\phi^{\dagger i}(M^2+N^2-\phi^2)\phi_i+\frac{1}{2}\phi^{\dagger i}\boldsymbol{\tau}_i^{\ j}\cdot\boldsymbol{D}\phi_j+\frac{1}{2}\boldsymbol{D}^2+\frac{1}{2}F^{\dagger i}F_i.
\end{aligned}
\tag{3.53}
$$

Note the wrong sign in the kinetic term for ϕ. This is a generic and presumably unavoidable feature of Euclidean supersymmetric theories. The sign problem has already been studied soon after the very invention of supersymmetry [99], and has attracted renewed interest recently [100].

The reduction of the supersymmetry parameters ζ^i yields two independent Dirac spinors, $\zeta_\pm$. They serve as parameters in the reduced theory, which exhibits therefore $\mathcal{N}=2$ supersymmetry, too. For the reduction of supersymmetry transformations we proceed along the same lines, e.g., using the explicit form of the γ-matrices, we find

$$
\delta\phi\equiv\delta A_0=\mathrm{i}\bar{\zeta}_i\gamma_0\lambda^i \quad\rightarrow\quad \mathrm{i}\left(\bar{\zeta}_+\lambda_+ +\bar{\zeta}_-\lambda_- -\bar{\lambda}_-\zeta_- -\bar{\lambda}_+\zeta_+\right). \tag{3.54}
$$

The supersymmetry transformations of the remaining fields in the (four-dimensional) vector multiplet are given by

$$\begin{aligned}
\delta A_m &= -\mathrm{i}\left(\bar\zeta_+\sigma^m\lambda_+ - \bar\zeta_-\sigma^m\lambda_- + \bar\lambda_-\sigma^m\zeta_- - \bar\lambda_+\sigma^m\zeta_+\right),\\
\delta M &= \mathrm{i}\left(\bar\zeta_+\lambda_- + \bar\zeta_-\lambda_+ - \bar\lambda_+\zeta_- - \bar\lambda_-\zeta_+\right),\\
\delta N &= \bar\zeta_+\lambda_- - \bar\zeta_-\lambda_+ - \bar\lambda_+\zeta_- + \bar\lambda_-\zeta_+,\\
\delta\lambda_\pm &= \mathrm{i}(F,\sigma)\zeta_\pm \mp \partial\!\!\!/\phi\zeta_\pm \pm \partial\!\!\!/(M\mp\mathrm{i}N)\zeta_\mp - \mathrm{i}D_3\zeta_\pm - \mathrm{i}(D_1+\mathrm{i}D_2)\zeta_{\mp C}.
\end{aligned} \tag{3.55}$$

Here, D_i are the components of $\boldsymbol{D}$. The remaining variations reduce to

$$\begin{aligned}
\delta\phi_1 &= 2\bar\zeta_+\psi_- + 2\bar\zeta_-\psi_+, \qquad \delta\phi_2 = 2\bar\zeta_{-C}\psi_- + 2\bar\zeta_{+C}\psi_+,\\
\delta\psi_\pm &= -\zeta_\pm(\mathrm{i}F_1 + (M\pm\mathrm{i}N)\phi_1) - \zeta_{\mp C}(\mathrm{i}F_2 + (M\pm\mathrm{i}N)\phi_2) + (\phi\pm\mathrm{i}D\!\!\!\!/)(\zeta_\mp\phi_1 + \zeta_{\pm C}\phi_2).
\end{aligned} \tag{3.56}$$

Invariance of $\mathcal{L}$ under these transformations follows from the invariance of $\mathcal{L}$ in 3+1 dimensions or can be checked explicitly. The equations of motion in the reduced theory can be derived as Euler-Lagrange equations of $\mathcal{L}$ in (3.53) or by reducing (3.39). Both ways, we obtain

$$\begin{aligned}
\Delta\phi &= \phi^{\dagger i}\phi_i\phi - \bar\psi_+\psi_+ - \bar\psi_-\psi_-,\\
\partial_m F_{mn} &= \bar\psi_+\sigma^n\psi_+ - \bar\psi_-\sigma^n\psi_- - \frac{\mathrm{i}}{2}\phi^{\dagger i}D_n\phi_i + \frac{\mathrm{i}}{2}\left(D_n\phi_i\right)^\dagger\phi_i,\\
\mathrm{i}\partial\!\!\!/\lambda_\pm &= \mp\mathrm{i}\phi^{\dagger 1}\psi_\mp \pm \mathrm{i}\phi_2\psi_{\pm C},\\
\Delta M &= \bar\psi_+\psi_- + \bar\psi_-\psi_+ + \phi^{\dagger i}\phi_i M,\\
\Delta N &= \mathrm{i}\bar\psi_+\psi_- - \mathrm{i}\bar\psi_-\psi_+ + \phi^{\dagger i}\phi_i N,\\
\boldsymbol{D} &= -\frac{1}{2}\phi^{\dagger i}\phi_j\boldsymbol{\tau}_i{}^j,
\end{aligned} \tag{3.57}$$

for the fields in the vector multiplet, and

$$\begin{aligned}
D^2\phi_1 &= -2\mathrm{i}\bar\lambda_+\psi_- - 2\mathrm{i}\bar\lambda_-\psi_+ + (M^2+N^2-\phi^2+\frac{1}{2}\phi^{\dagger k}\phi_k)\phi_1,\\
D^2\phi_2 &= -2\mathrm{i}\bar\lambda_{-C}\psi_- - 2\mathrm{i}\bar\lambda_{+C}\psi_+ + (M^2+N^2-\phi^2+\frac{1}{2}\phi^{\dagger k}\phi_k)\phi_2,\\
\mathrm{i}D\!\!\!\!/\psi_+ &= \mathrm{i}\lambda_-\phi_1 + \mathrm{i}\lambda_{+C}\phi_2 + (M+\mathrm{i}N)\psi_- + \phi\psi_+,\\
\mathrm{i}D\!\!\!\!/\psi_- &= -\mathrm{i}\lambda_+\phi_1 - \mathrm{i}\lambda_{-C}\phi_2 - (M-\mathrm{i}N)\psi_+ - \phi\psi_-,\\
F_i &= F^{\dagger i} = 0,
\end{aligned} \tag{3.58}$$

for the matter fields.

In the next Section, we will use these variations to construct bosonic background field configurations that are invariant under half of the supersymmetry transformations. Afterwards, we study the fermionic equations in these background fields. In four dimensions, this procedure is known to generate zero modes of the corresponding Dirac operator, e.g. for the case of a static monopole and for instanton fields [81, 93, 83].

3.4. Background Fields and Zero Modes

The most general ansatz for a bosonic background configuration contains all bosonic fields $(\boldsymbol{A}, \phi, M, N, \boldsymbol{D}, \phi_i, F_i)$ whereas all fermionic fields $(\psi_\pm, \lambda_\pm)$ are set to zero.

One verifies that in such a background configuration the supersymmetry variations of the spinors satisfy the linearized equations

$$\begin{aligned}
(\mathrm{i}\not{D} - \phi)(\delta\psi_+) &= \mathrm{i}\phi_1(\delta\lambda_-) + \mathrm{i}\phi_2(\delta\lambda_{+\mathcal{C}}) + (M + \mathrm{i}N)(\delta\psi_-), \\
(\mathrm{i}\not{D} + \phi)(\delta\psi_-) &= -\mathrm{i}\phi_1(\delta\lambda_+) - \mathrm{i}\phi_2(\delta\lambda_{-\mathcal{C}}) - (M - \mathrm{i}N)(\delta\psi_+), \\
\mathrm{i}\not{\partial}(\delta\lambda_+) &= -\mathrm{i}\phi^{\dagger 1}(\delta\psi_-) + \mathrm{i}\phi_2(\delta\psi_{+\mathcal{C}},) \\
\mathrm{i}\not{\partial}(\delta\lambda_-) &= \mathrm{i}\phi^{\dagger 1}(\delta\psi_+) - \mathrm{i}\phi_2(\delta\psi_{-\mathcal{C}}),
\end{aligned} \tag{3.59}$$

provided the scalars ϕ_i satisfy their equations of motion (3.58).

Now we ask: under which conditions on the bosons do the fermionic variations vanish at the background? For arbitrary variations with independent parameters ζ_+ and ζ_-, only trivial backgrounds (involving constant scalar fields) are invariant. Therefore, we look for configurations that are invariant under only *half* of the supersymmetry transformations. In particular, the parameters $\zeta_\pm$ will no longer be independent, but we relate them to each other in a Lorentz-invariant way. We consider the following three cases,

$$\zeta \equiv \zeta_+ \sim \zeta_-, \qquad \zeta \equiv \zeta_+ \sim \zeta_{-\mathcal{C}}, \zeta_- \sim -\zeta_{\mathcal{C}}, \qquad \zeta_+ = 0, \zeta_- \equiv \zeta. \tag{3.60}$$

Let us focus on the first case. Then, setting $\delta\lambda_\pm = 0$ implies the following conditions on the bosonic fields,

$$\phi = M + \text{const.}, \qquad \boldsymbol{B} = -\boldsymbol{\nabla} N, \qquad \boldsymbol{D} = 0. \tag{3.61}$$

Here, $\boldsymbol{B}$ is the magnetic field, the dual of the field-strength F_{mn} in three dimensions,

$$B_k = \frac{1}{2}\epsilon_{kmn}F_{mn}. \tag{3.62}$$

The Bianchi identity,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = -\Delta N = 0, \tag{3.63}$$

implies that N vanishes, since we are interested in square-integrable solutions only: any harmonic function on $\mathbb{R}^3$ is either zero or not normalizable. Non-normalizability of N implies that the spinor-variations in this background are non-normalizable, too, and we have to set $N = 0$. Accordingly, the field strength vanishes, and we can choose $\boldsymbol{A} = 0$ for the gauge potential. Furthermore, $\boldsymbol{D} = 0$ in (3.61) can be expressed in terms of ϕ_i and reads

$$\phi^{\dagger i}\boldsymbol{\tau}_i{}^j\phi_j = -2\boldsymbol{k}. \tag{3.64}$$

Here $\boldsymbol{k}$ is the constant vector in the FI term and can be choosen as

$$\boldsymbol{k} = k\begin{pmatrix}0 & 0 & 1\end{pmatrix}^t, \quad k \geq 0. \tag{3.65}$$

Note, that the left hand side of (3.64),

$$\phi^{\dagger i}\boldsymbol{\tau}_i{}^j\phi_j = \begin{pmatrix} \phi^{\dagger 1}\phi_2 + \phi^{\dagger 2}\phi_1 \\ -\mathrm{i}\phi^{\dagger 1}\phi_2 + \mathrm{i}\phi^{\dagger 2}\phi_1 \\ \phi^{\dagger 1}\phi_1 - \phi^{\dagger 2}\phi_2 \end{pmatrix} = \begin{pmatrix} 2(u_1u_3 + u_2u_4) \\ 2(u_1u_4 - u_2u_3) \\ u_1^2 + u_2^2 - u_3^2 - u_4^2 \end{pmatrix}, \tag{3.66}$$

is the standard Hopf map [33, 101], if we restrict the fields to

$$|\phi_1|^2 + |\phi_2|^2 = \text{const.} \tag{3.67}$$

We have used the parametrization

$$\phi_1 = u_1 + \mathrm{i}u_2, \qquad \phi_2 = u_3 + \mathrm{i}u_4, \tag{3.68}$$

to make the relation to the Hopf map more transparent. Under this map, the following equivalence class of fields is mapped to the same point,

$$\phi_i \sim \mathrm{e}^{\mathrm{i}\varphi}\phi_i, \qquad \phi^{\dagger i} \sim \mathrm{e}^{-\mathrm{i}\varphi}\phi^{\dagger i}. \tag{3.69}$$

Using this fact, (3.64) with the choice (3.65) reads

$$\phi_1 \equiv 0, \qquad |\phi_2|^2 = 2k. \tag{3.70}$$

The $\delta\psi_\pm$ variations on this restricted background read

$$\delta\psi_\pm = (\phi \pm \mathrm{i}\not{\partial} - M)\zeta_C\phi_2. \tag{3.71}$$

To find an invariant background, we set them to zero, and take the sum and the difference,

$$(\phi - M)\zeta_C\phi_2 = 0, \qquad \mathrm{i}\not{\partial}\zeta_C\phi_2 = 0, \tag{3.72}$$

with solution $\phi = M$ and $\phi_2 = \text{const.}$ Here, the fields $\phi = M$ are not restricted any further. Summarizing, if we allow for a general $\mathcal{N} = 1$ variation (no additional restrictions on ζ), the only solution for an invariant bosonic background is

$$\boldsymbol{A} = 0, \quad F_{mn} = 0, \quad N = 0, \quad \phi = M, \quad \phi_1 = 0, \quad \phi_2 = \text{const.} \tag{3.73}$$

If we apply a general $\mathcal{N} = 2$ supersymmetry variation to this configuration, it can be decomposed into a $\zeta_+ = \zeta_-$ variation, which vanishes by construction, and a remaining $\zeta_+ = -\zeta_- \equiv \zeta$ variation, which yields

$$\delta\lambda_\pm = -2\not{\partial}\phi\zeta, \qquad \delta\psi_\pm = \pm 2\phi\phi_2\zeta_C. \tag{3.74}$$

One verifies that these variations of the spinors satisfy the Dirac equations in the background, e.g.

$$(\mathrm{i}\partial\!\!\!/ - \phi)\delta\psi_+ = \mathrm{i}\phi_2(\delta\lambda_{+\mathcal{C}}) + M\delta\psi_- = -2\mathrm{i}\phi_2(\partial\!\!\!/\phi\zeta)_{\mathcal{C}} - \phi\delta\psi_+, \tag{3.75}$$

such that $\psi_\pm$ is a solution of the Dirac equation

$$\mathrm{i}\partial\!\!\!/\delta\psi_+ = +2\mathrm{i}\phi_2\partial\!\!\!/\phi\zeta_{\mathcal{C}}. \tag{3.76}$$

Let us come back to the remaining two cases in (3.60). Upon setting $\delta\lambda_\pm = 0$, they turn out to yield even more restricted configurations: vanishing gauge fields and constant scalar fields.

Result: In three Euclidean dimensions, only a very restricted class of bosonic background fields are invariant under half of the supersymmetry transformations. In particular, the gauge field must vanish in order to find normalizable configurations. Acting with the unbroken supercharge on such a background yields zero modes of the Dirac operator, but for fermions interacting with the scalar fields only. In particular, the zero mode examples of Adam, Muratori and Nash, for Dirac operators in three dimensional gauge fields, cannot be obtained in this way. In the subsequent Sections we are going to apply a similar construction to other physical situations, where we are able to derive more interesting results.

3.5. $\mathcal{N} = 2$ Vortices

Let us consider a more restricted bosonic background, where only the gauge field $\boldsymbol{A}$ and the scalars ϕ_i are nonzero. The Lagrangian (3.53) for these fields reduces to

$$-\mathcal{L} = \frac{1}{4}F_{mn}^2 + \frac{1}{2}(D_m\phi_i)^\dagger(D_m\phi_i) + \frac{1}{8}(\phi^{i\dagger}\boldsymbol{\tau}_i{}^j\phi_j)^2 + \frac{1}{2}\boldsymbol{k}(\phi^{i\dagger}\boldsymbol{\tau}_i{}^j\phi_j) + \frac{1}{2}\boldsymbol{k}^2, \tag{3.77}$$

where we included a FI term $\boldsymbol{k}\cdot\boldsymbol{D}$. For $\boldsymbol{k} = 0$, that is without FI term in the action, we can use the BPS trick [102] and write

$$-\mathcal{L} = \frac{1}{4}F_{mn}^2 + \frac{1}{2}(D_m\phi_i)^\dagger(D_m\phi_i) + \frac{1}{8}(\phi^{i\dagger}\boldsymbol{\tau}_i{}^j\phi_j)^2 \geq 0 \tag{3.78}$$

But now the BPS equations imply

$$F_{mn} = 0, \qquad \phi_i = 0, \tag{3.79}$$

and we are left with trivial solutions only. For nonvanishing $\boldsymbol{k}$, in particular with our convention (3.65), we find

$$\begin{aligned}
-\mathcal{L} &= \frac{1}{2}B_1^2 + \frac{1}{2}B_2^2 + \frac{1}{2}|\bar{D}\phi_1|^2 + \frac{1}{2}|D\phi_2|^2 + \frac{1}{2}(D_3\phi_i)^\dagger(D_3\phi_i) \\
&\quad + \frac{1}{8}(\phi^{1\dagger}\phi_2 + \phi^{2\dagger}\phi_1)^2 + \frac{1}{8}(\mathrm{i}\phi^{1\dagger}\phi_2 - \mathrm{i}\phi^{2\dagger}\phi_1)^2 \\
&\quad + \frac{1}{8}(|\phi_1|^2 - |\phi_2|^2 + 2k - 2B_3)^2 + kB_3 \\
&\geq kB_3.
\end{aligned} \tag{3.80}$$

Here we have introduced the complex covariant derivatives,

$$D = \partial + \mathrm{i}A, \qquad \bar{D} = \bar{\partial} + \mathrm{i}\bar{A}, \tag{3.81}$$

with respect to the complex coordinate

$$z = x_1 + \mathrm{i}x_2, \qquad \bar{z} = x_1 - \mathrm{i}x_2, \qquad \partial \equiv \frac{\partial}{\partial z}, \qquad \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}, \tag{3.82}$$

and B_i are the components of the magnetic field (pseudo)vector $\boldsymbol{B}$. In this case, the BPS equations

$$\begin{aligned}
B_1 = B_2 &= 0, \\
\bar{D}\phi_1 = 0, \quad D\phi_2 = 0, \quad D_3\phi_i = 0, \quad \phi^{\dagger 1}\phi_2 &= 0, \\
|\phi_1|^2 - |\phi_2|^2 + 2k - 2B_3 &= 0,
\end{aligned} \tag{3.83}$$

will allow for nontrivial solutions. We solve these equations using complex gauge fields and find, in particular,

$$F_{zz} = F_{\bar{z}\bar{z}} = 0, \qquad F_{z\bar{z}} = \frac{\mathrm{i}}{2}B_3. \tag{3.84}$$

In analogy with the discussion for supersymmetric quantum mechanics, the covariant derivatives can be deformed as

$$D = g\partial g^{-1}, \qquad g = \mathrm{e}^{-\chi}, \qquad A = -\mathrm{i}\partial\chi, \tag{3.85}$$

$$\bar{D} = g^{-1}\bar{\partial} g, \qquad \bar{g} = \mathrm{e}^{-\chi}, \qquad \bar{A} = \mathrm{i}\bar{\partial}\chi. \tag{3.86}$$

From (3.83) and in the gauge $A_3 = 0$ we conclude

$$\phi_1 = \mathrm{e}^{\chi} f(z), \qquad \phi_2 = \mathrm{e}^{-\chi}\bar{h}(\bar{z}). \tag{3.87}$$

A and $\bar{A}$ (and therefore χ) must be independent of x_3 in order to satisfy (3.83). This effectively reduces our three-dimensional theory to a two-dimensional one, where all fields only depend on z and $\bar{z}$. It remains to solve the BPS equations,

$$\phi^{\dagger 1}\phi_2 = 0, \tag{3.88}$$

$$|\phi_1|^2 - |\phi_2|^2 + 2k - 2B_3 = 0. \tag{3.89}$$

They imply that at each point one of the scalar fields vanishes. We choose $\bar{h}(\bar{z}) = 0$, so $\phi_2 = 0$. With $B_3 = F_{12} = 4\partial\bar{\partial}\chi$, (3.89) translates into the Liouville equation,

$$\mathrm{e}^{2\chi}|f(z)|^2 + 2k - 2\Delta\chi = 0. \tag{3.90}$$

A general solution (χ, f) of this equation gives rise to BPS solutions $\boldsymbol{A}$ and ϕ_1 of our theory, but is usually inacessible due to the intricacy of the differential equation. Let us construct particular solutions with radial symmetry, $\chi = \chi(r)$. Here, r is the radial coordinate in the complex z plane. If we call the angular coordinate θ, we find

$$A_r = 0 \quad \text{and} \quad A_\theta = \chi'. \tag{3.91}$$

With the ansatz

$$A_\theta = \chi' \equiv \frac{a(r)}{r}, \qquad \phi_1 = \mathrm{e}^{\chi} f(z) \equiv \hat{f}(r)\mathrm{e}^{\mathrm{i}m\theta}, \qquad m \in \mathbb{Z}, \tag{3.92}$$

the Liouville equation translates into

$$a'(r) = \frac{r}{2}(\hat{f}^2(r) + 2k), \tag{3.93}$$

and $f(z)$ in (3.92) is a holomorphic function, $\bar{\partial} f(z) = 0$, if the following holds

$$\hat{f}' = \frac{\hat{f}}{r}(a(r) + m). \tag{3.94}$$

Equations (3.93) and (3.94) are the standard BPS equations for the Nielsen-Olesen vortex [103] of magnetic flux Φ characterized by the integer m. No analytic solutions to this coupled system of differential equations are known, but numerical ones have been found, see for instance [104]. The same equations have also been derived in [105] starting with a supersymmetric gauge theory in $d = 2$ dimensions.

Let us assume now that we are given a BPS solution $(\boldsymbol{A}, \phi_1)$. Then we can check that this solution is invariant under half of the supersymmetry transformations, and the unbroken transformations generate solutions of the Dirac equation. If we split the two-component supersymmetry parameters as

$$\zeta_\pm = \begin{pmatrix} \zeta_{\pm 1} \\ \zeta_{\pm 2} \end{pmatrix}, \tag{3.95}$$

and use the BPS equations, as well as (3.55)–(3.56), we find

$$\delta\psi_\pm = \pm \mathrm{i} D\phi_1 \begin{pmatrix} \zeta_{\mp 2} \\ 0 \end{pmatrix}, \qquad \delta\lambda_\pm = 2\mathrm{i} B_3 \begin{pmatrix} 0 \\ \zeta_{\pm 2} \end{pmatrix}. \tag{3.96}$$

Obviously, the background is invariant under transformations generated by $\zeta_{\pm 1}$. The remaining variations generate zero modes which satisfy

$$\mathrm{i}\not{D}\delta\psi_\pm = \pm \mathrm{i}\phi_1 \delta\lambda_\mp \quad \longrightarrow \quad \mathrm{i}\not{D}\delta\psi_\pm \pm 2B_3\phi_1 \begin{pmatrix} 0 \\ \zeta_{\mp 2} \end{pmatrix} = 0, \tag{3.97}$$

$$\mathrm{i}\not{\partial}\delta\lambda_\pm = \mp \mathrm{i}\phi^{\dagger 1}\delta\psi_\mp \quad \longrightarrow \quad \mathrm{i}\not{\partial}\delta\lambda_\pm + \phi^{\dagger 1} D\phi_1 \begin{pmatrix} \zeta_{\pm 2} \\ 0 \end{pmatrix} = 0. \tag{3.98}$$

Result: Given a BPS solution $(\boldsymbol{A}, \phi_1)$ of our theory described by the Lagrangian $\mathcal{L}$ in (3.77), we can construct one zero mode $\delta\psi_\pm$ in the background of the gauge potential $\boldsymbol{A}$ and in interaction with the scalar ϕ_1. In addition, we can generate a zero mode $\delta\lambda_\pm$, which is uncharged with respect to $\boldsymbol{A}$. The Atiyah-Singer index theorem applied to two-dimensional vortex-configurations [62, 106] predicts the existence of m charged zero modes. One of them can be obtained as a supersymmetry variation in our theory.

3.6. Applications to Non-Abelian Theory

In this final Section we repeat our previous considerations, but now for a non-Abelian gauge group. In this case, it is sufficient for our purposes to study the vector multiplet V, since the spinors there are charged under the gauge group (in contrast to the Abelian case). In particular, all fields in V transform according to the adjoint representation of the gauge group.

The vector multiplet of $\mathcal{N} = 2$ Non-Abelian gauge theory is well-known (see e.g. [16]), and the corresponding Lagrangian is given by

$$\begin{aligned} \mathcal{L} = \operatorname{tr} \Bigg(& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mathrm{i}}{2}\bar{\lambda}_i\gamma^\mu D_\mu\lambda^i + \frac{1}{2}(D_\mu M)(D^\mu M) + \frac{1}{2}(D_\mu N)(D^\mu N) \\ & -\frac{1}{2}\bar{\lambda}_i[\lambda^i, M] - \frac{\mathrm{i}}{2}\bar{\lambda}_i\gamma_5[\lambda^i, N] + \frac{1}{2}[M,N]^2 + \frac{1}{2}\boldsymbol{D}^2 \Bigg), \end{aligned} \tag{3.99}$$

where, tr denotes the trace over gauge group indices.

The supersymmetry transformations are given by

$$\begin{aligned} &\delta A_\mu = \mathrm{i}\bar{\zeta}_i\gamma_\mu\lambda^i, \qquad \delta M = \mathrm{i}\bar{\zeta}_i\lambda^i, \qquad \delta N = -\bar{\zeta}_i\gamma_5\lambda^i, \\ &\delta\lambda^i = \mathrm{i}(F,\Sigma)\zeta^i - \not{D}(M + \mathrm{i}\gamma_5 N)\zeta^i + \gamma_5\zeta^i[M,N] - \mathrm{i}\zeta^j\boldsymbol{\tau}_j{}^i\cdot\boldsymbol{D}, \\ &\delta\boldsymbol{D} = \boldsymbol{\tau}_i{}^j\bar{\zeta}_j(\not{D}\lambda^i + \mathrm{i}[\lambda^i, M] - \gamma_5[\lambda^i, N]). \end{aligned} \tag{3.100}$$

Like in the Abelian case, one shows that the corresponding action is invariant under these transformations. The commutator of two transformations gives

$$[\delta^{(1)}, \delta^{(2)}] = \delta_{\text{translation}} + \delta_{\text{gauge}}, \tag{3.101}$$

with gauge parameter

$$\Lambda = 2\mathrm{i}\bar{\zeta}_i^{(1)}(M + \mathrm{i}\gamma_5 N - \gamma^\mu A_\mu)\zeta^{i(2)}. \tag{3.102}$$

Using the abbreviation

$$[\bar{\psi}, \Gamma\psi] \equiv \mathrm{i}f^{ABC}\bar{\psi}^B\Gamma\psi^C T^A, \tag{3.103}$$

for any matrix Γ in spinor space, the equations of motion can be written as

$$\begin{aligned} D_\mu F^{\mu\nu} &= -\frac{1}{2}[\bar{\lambda}_i, \gamma^\nu \lambda^i] - \mathrm{i}[M, D^\nu M] - \mathrm{i}[N, D^\nu N], \\ D^2 M &= -\frac{1}{2}[\bar{\lambda}_i, \lambda^i] - [N, [N, M]], \\ D^2 N &= -\frac{\mathrm{i}}{2}[\bar{\lambda}_i, \gamma_5 \lambda^i] - [M, [M, N]], \\ \mathrm{i}\not{D}\lambda^i &= -[M + \mathrm{i}\gamma_5 N, \lambda^i], \\ \boldsymbol{D} &= 0. \end{aligned} \tag{3.104}$$

The linearized equations of motion for the fermions hold on-shell, so we have

$$\mathrm{i}\not{D}(\delta\lambda^i) = -[M + \mathrm{i}\gamma_5 N, \delta\lambda^i] \tag{3.105}$$

for a bosonic background field configuration (A_μ, M, N).

3.6.1. Reduction to $d = 3$ Dimensions

The reduction of $\mathcal{L}$ to three dimensions can be carried out along the same lines as for the Abelian theory. The result is in this case

$$\begin{aligned} \mathcal{L} = \operatorname{tr} \Big(&-\frac{1}{4}F_{mn}F_{mn} + \frac{1}{2}(D_m\phi)(D_m\phi) - \frac{1}{2}(D_m M)(D_m M) \\ &- \frac{1}{2}(D_m N)(D_m N) + \mathrm{i}\bar{\lambda}_+ \not{D}\lambda_+ - \mathrm{i}\bar{\lambda}_- \not{D}\lambda_- \\ &- \bar{\lambda}_+[\phi, \lambda_+] - \bar{\lambda}_-[\phi, \lambda_-] + \bar{\lambda}_+[M, \lambda_-] + \bar{\lambda}_-[M, \lambda_+] - \mathrm{i}\bar{\lambda}_+[N, \lambda_-] \\ &+ \mathrm{i}\bar{\lambda}_-[N, \lambda_+] - \frac{1}{2}[\phi, M]^2 - \frac{1}{2}[\phi, N]^2 + \frac{1}{2}[M, N]^2 + \frac{1}{2}\boldsymbol{D}^2 \Big) . \end{aligned} \tag{3.106}$$

As before, ϕ is the zero component of the four-dimensional vector potential, $\phi \equiv A_0$, and the kinetic term for ϕ in (3.106) has the wrong sign. Supersymmetry variations and equations of motion reduce to

$$
\begin{aligned}
\delta\phi &= \mathrm{i}\left(\bar\zeta_+\lambda_+ + \bar\zeta_-\lambda_- - \bar\lambda_-\zeta_- - \bar\lambda_+\zeta_+\right), \\
\delta A_m &= -\mathrm{i}\left(\bar\zeta_+\sigma^m\lambda_+ - \bar\zeta_-\sigma^m\lambda_- + \bar\lambda_-\sigma^m\zeta_- - \bar\lambda_+\sigma^m\zeta_+\right), \\
\delta M &= \mathrm{i}\left(\bar\zeta_+\lambda_- + \bar\zeta_-\lambda_+ - \bar\lambda_+\zeta_- - \bar\lambda_-\zeta_+\right), \\
\delta N &= \bar\zeta_+\lambda_- - \bar\zeta_-\lambda_+ - \bar\lambda_+\zeta_- + \bar\lambda_-\zeta_+, \\
\delta\lambda_\pm &= \mathrm{i}(F,\sigma)\zeta_\pm \mp \not{D}\phi\zeta_\pm - \mathrm{i}[\phi, M\mp\mathrm{i}N]\zeta_\mp \pm \not{D}(M\mp\mathrm{i}N)\zeta_\mp \\
&\qquad \pm [M,N]\zeta_\pm - \mathrm{i}D_3\zeta_\pm - \mathrm{i}(D_1+\mathrm{i}D_2)\zeta_{\mp C},
\end{aligned}
\tag{3.107}
$$

and

$$
\begin{aligned}
D_m F_{mn} &= [\bar\lambda_-, \sigma^n\lambda_-] - [\bar\lambda_+, \sigma^n\lambda_+] \\
&\qquad + \mathrm{i}[M, D_n M] + \mathrm{i}[N, D_n N] - \mathrm{i}[\phi, D_n\phi], \\
D^2\,\phi &= [\bar\lambda_+, \lambda_+] + [\bar\lambda_-, \lambda_-] + [M,[M,\phi]] + [N,[N,\phi]], \\
D^2 M &= [\bar\lambda_+, \lambda_-] + [\bar\lambda_-, \lambda_+] + [N,[N,M]] - [\phi,[\phi,M]], \\
D^2 N &= -\mathrm{i}[\bar\lambda_+, \lambda_-] + \mathrm{i}[\bar\lambda_-, \lambda_+] + [M,[M,N]] - [\phi,[\phi,N]], \\
\mathrm{i}\not{D}\lambda_\pm &= \mp[M\mp\mathrm{i}N, \lambda_\mp] \pm [\phi, \lambda_\pm], \\
\boldsymbol{D} &= 0.
\end{aligned}
\tag{3.108}
$$

Like in the four-dimensional theory, one verifies that the linearized field equations hold on-shell, and that they are given by

$$
\mathrm{i}\not{D}(\delta\lambda_-) = [M+\mathrm{i}N, \delta\lambda_+] - [\phi, \delta\lambda_-], \tag{3.109}
$$

$$
\mathrm{i}\not{D}(\delta\lambda_+) = -[M-\mathrm{i}N, \delta\lambda_-] + [\phi, \delta\lambda_+]. \tag{3.110}
$$

3.6.2. Background Fields and Zero Modes

Similar to the Abelian case, we would like to study which background field configurations are invariant under half of the supersymmetry variations. Let us first analyze a background that consists of gauge fields only (a three-dimensional instanton). The bosonic variations vanish at this background, and the spinor variations give

$$
\delta\lambda_\pm = \mathrm{i}(F,\sigma)\zeta_\pm. \tag{3.111}
$$

So this background is invariant if and only if $F_{mn} = 0$. This is in contrast to the four dimensional case, where instanton configurations break half of the supersymmetry variations, say the ones with left-handed parameters, whereas right-handed supersymmetry parameters give rise to zero modes of the Dirac operator [81, 83].

Still, we can construct zero modes of the Dirac operator in these three dimensional instanton fields. Provided the equation of motion for F_{mn} and the Bianchi identity hold, we find with $\delta\lambda$ from (3.111),

$$\mathrm{i}\not{D}\delta\lambda_{\pm} = 0. \tag{3.112}$$

It turns out that these modes are not normalizable, since

$$\|\delta\lambda\|^2 = \int \mathrm{d}^3x \ \mathrm{tr}\ \delta\lambda^\dagger\delta\lambda = \frac{1}{4}\int \mathrm{d}^3x\ \zeta^\dagger(F^A_{mn}F^A_{mn}\mathbb{1})\zeta \sim S[\boldsymbol{A}]\zeta^\dagger\zeta. \tag{3.113}$$

Here, $S[\boldsymbol{A}]$ is the action associated with the field configuration $\boldsymbol{A}$. As we will show using a particular form of Derricks theorem [107], there are no finite-action solutions to the equations of motion for the gauge potential in any Euclidean dimensions except for $d = 4$. Assume $\boldsymbol{A}(\boldsymbol{x})$ is such a configuration in d dimensions. Define the action

$$S[\boldsymbol{A}] = -\frac{1}{4}\int \mathrm{d}^dx\ \mathrm{tr}\ F_{mn}(\boldsymbol{x})F_{mn}(\boldsymbol{x}), \tag{3.114}$$

and the following one-parameter family of gauge field configurations,

$$\boldsymbol{A}_\lambda(\boldsymbol{x}) = \lambda\boldsymbol{A}(\lambda\boldsymbol{x}). \tag{3.115}$$

The action for the members of this family is given by ($\boldsymbol{y} = \lambda\boldsymbol{x}$)

$$\begin{aligned} S[\boldsymbol{A}_\lambda] &= -\frac{1}{4}\int \mathrm{d}^dx\ \mathrm{tr}\ F^\lambda_{mn}F^\lambda_{mn} \\ &= -\frac{1}{4}\int \mathrm{d}^dy\ \lambda^{-d}\lambda^4 \mathrm{tr}\ F_{mn}(\boldsymbol{y})F_{mn}(\boldsymbol{y}) \\ &= \lambda^{4-d}S[\boldsymbol{A}]. \end{aligned} \tag{3.116}$$

Since $\boldsymbol{A}(\boldsymbol{x})$ is a solution, we have

$$0 \stackrel{!}{=} \left.\frac{\partial S[\boldsymbol{A}_\lambda]}{\partial \lambda}\right|_{\lambda=1} = (4-d)S[\boldsymbol{A}]. \tag{3.117}$$

So either $d = 4$, and there can be nontrivial solutions, or $d \neq 4$, but then $S[\boldsymbol{A}] = 0$ implying $F_{mn} = 0$, and there are only trivial solutions. In particular, for $d = 3$, there are no instanton configurations which are invariant under half of the supersymmetry transformations, and possible non-normalizable instanton configurations give rise to non-normalizable zero modes only.

As a way out, we can consider our theory on a three-dimensional torus $\mathbb{T}^3$. There, nontrivial gauge field configurations exist, and they will give rise to square-integrable zero modes. For our construction, we need instanton configurations on $\mathbb{T}^3$. Not much is known about such configurations, and the situation is similar to the better studied torus $\mathbb{T}^4$ [68, 108]. In general, no explicit (anti-)selfdual connections on $\mathbb{T}^4$ are known, except for constant-curvature solutions, the 't Hooft instantons [109], which exist for some choices of the torus size. Here we present some analogous solutions for the three-dimensional case. Let the size of the torus be $L \times L \times L$, and consider the gauge potential

$$A_1 = \alpha T x_2, \qquad A_2 = -\beta T x_1, \qquad A_3 = 0, \tag{3.118}$$

with nonvanishing curvature component

$$F_{12} = -T(\alpha + \beta). \tag{3.119}$$

Here, T is a fixed linear combination of the generators of the gauge group, e.g. of $SU(2)$. Obviously, $D_m F_{mn} = 0$ is satisfied, and we generate the zero mode

$$\delta\lambda = \mathrm{i}T(\alpha+\beta)\sigma^3\zeta, \qquad \mathrm{i}\not{D}\delta\lambda = 0. \tag{3.120}$$

The cocycle condition for gauge fields on the torus [68] restricts the possible values of α and β,

$$\alpha = \frac{2\pi n}{L^2}, \qquad \beta = \frac{2\pi m}{L^2}, \qquad m, n \in \mathbb{Z}. \tag{3.121}$$

Another example is given by

$$\begin{aligned} &A_1 = \alpha T x_2, \qquad A_2 = \beta T x_3, \qquad A_3 = 0, \\ &F_{12} = -\alpha T, \qquad F_{23} = -\beta T, \qquad D_m F_{mn} = 0, \\ &\delta\lambda = \mathrm{i}T(\alpha\sigma^3 + \beta\sigma^1)\zeta, \qquad \mathrm{i}\not{D}\delta\lambda = 0, \end{aligned} \tag{3.122}$$

where the cocycle condition again implies (3.121).

3.6.3. Jackiw-Rebbi Modes in $d = 3$

As we have seen in the previous Subsection, configurations that consist of gauge fields $\boldsymbol{A}$ only do not yield interesting solutions in the case of $\mathbb{R}^3$. If, in addition, we switch on one of the scalar fields, say M, invariant backgrounds turn out to be possible. We will focus on the case with nontrivial $\boldsymbol{A}$ and M, but the other scalars yield similar results. For definiteness, let us choose the gauge group $SU(2)$ with generators T^A for what follows. In this case, the variations read

$$\delta\lambda_+ = \mathrm{i}(F,\sigma)\zeta_+ + \not{D}M\zeta_-, \tag{3.123}$$

$$\delta\lambda_- = \mathrm{i}(F,\sigma)\zeta_- - \not{D}M\zeta_+. \tag{3.124}$$

For $\mathcal{N} = 1$ supersymmetry transformations, with $\zeta = \zeta_+ = -\mathrm{i}\zeta_-$, we find

$$\delta\lambda_+ = -\mathrm{i}\delta\lambda_- = \big(\mathrm{i}(F,\sigma) + \mathrm{i}\not{D}M\big)\,\zeta. \tag{3.125}$$

These variations vanish, if

$$D_k M = B_k, \tag{3.126}$$

where B_k is the chromomagnetic field. Assuming this equation to hold, the unbroken supersymmetry generator ($\zeta = \zeta_+ = \mathrm{i}\zeta_-$) yields the zero modes

$$\delta\lambda_+ = \mathrm{i}\delta\lambda_- = 2\mathrm{i}(F,\sigma)\zeta, \tag{3.127}$$

which satisfy the Dirac equation in this particular background, e.g.

$$\mathrm{i}\not{D}\delta\lambda_+ = 2[M, \not{D}M]\zeta. \tag{3.128}$$

Here we have used the Bianchi identity and the equation of motion for F_{mn} (3.108).

Now we should solve the equations of motion for the bosonic fields. The bosonic part of our theory corresponds to the time-independent four-dimensional Georgi-Glashow model (Yang-Mills-Higgs model) with vanishing coupling, $\lambda \to 0$, so we expect to find the well-known 't Hooft-Polyakov monopole solutions [110].

Rewriting the Lagrangian for $\boldsymbol{A}$ and M,

$$-\mathcal{L} = \mathrm{tr}\left(\frac{1}{4}F_{mn}F_{mn} + \frac{1}{2}(D_m M)(D_m M)\right), \tag{3.129}$$

and applying the BPS trick gives

$$-\mathcal{L} = \frac{1}{8}\left(F^A_{mn} \pm \epsilon_{mnk}(D_k M)^A\right)^2 + \frac{1}{4}\epsilon_{mnk}F^A_{mn}(D_k M)^A \geq \mathcal{Q}, \tag{3.130}$$

$$\mathcal{Q} = \frac{1}{4}\epsilon_{mnk}F^A_{mn}(D_k M)^A. \tag{3.131}$$

BPS solutions of the equations of motion,

$$D_m F_{mn} = \mathrm{i}[M, D_n M], \qquad D^2 M = 0, \tag{3.132}$$

satisfy the first-order differential equation (BPS bound)

$$F^A_{mn} = \mp\epsilon_{mnk}(D_k M)^A. \tag{3.133}$$

The ansatz

$$M^A = \delta_{Ap}x_p\mathcal{M}(r), \qquad A^A_m = \epsilon_{Amp}x_p\mathcal{A}(r), \tag{3.134}$$

for BPS states yields the set of coupled equations,

$$-\mathcal{A}^2 r + \mathcal{A}' \pm (\mathcal{M}' - r\mathcal{A}\mathcal{M}) = 0, \tag{3.135}$$

$$-\mathcal{A}' r - 2\mathcal{A} \pm (\mathcal{M} + r^2\mathcal{A}\mathcal{M}) = 0. \tag{3.136}$$

As usual, they are much easier to solve than the second-order problem,

$$\frac{4}{r}\mathcal{A}' + \mathcal{A}'' - 3\mathcal{A}^2 - r^2\mathcal{A}^3 - \mathcal{M}^2 - r^2\mathcal{M}^2\mathcal{A}^2 = 0, \tag{3.137}$$

$$\frac{4}{r}\mathcal{M}' - 4\mathcal{A}\mathcal{M} + \mathcal{M}'' - 2\mathcal{A}^2\mathcal{M}r^2 = 0. \tag{3.138}$$

The normalizable solutions are given by[3]

$$\mathcal{A} = -\frac{1}{r^2} + \frac{M_\infty}{r\sinh rM_\infty}, \qquad \mathcal{M} = \pm\frac{1}{r^2} \mp \frac{M_\infty}{r\tanh rM_\infty}. \tag{3.139}$$

For definiteness, let us take the BPS equation (3.133) with the plus sign. The results for the other case are just copies of the ones to be obtained in a moment.

With $\zeta = \zeta_+ = \mathrm{i}\zeta_-$ we find

$$\begin{aligned}\delta\lambda &= \delta\lambda_+ = \mathrm{i}\delta\lambda_- = 2\mathrm{i}(F,\sigma)\zeta \\ &= \mathrm{i}T^A\left[\delta_{Ak}(4\mathcal{A} + 2\mathcal{A}'r) + x_A x_k(2\mathcal{A}^2 - 2\mathcal{A}'r^{-1})\right]\sigma^k\zeta,\end{aligned} \tag{3.140}$$

where $\mathcal{A}$ is given by (3.139). One easily checks that $\delta\lambda$ has finite norm and solves the corresponding Dirac equation,

$$\mathrm{i}\not{D}(\delta\lambda) - \mathrm{i}[M,\delta\lambda] = 0. \tag{3.141}$$

The two independent components of $\delta\lambda$ represent two zero modes of the Dirac operator in the background of a 't Hooft-Polyakov monopole. These zero modes are the so-called Jackiw-Rebbi modes that have been determined in [90] (in the framework of static $d = 4$ Yang-Mills-Higgs theory). Two adjoint zero modes are to be expected for the monopole of charge one on general grounds, because of the Callias index theorem [89].

The construction of zero modes in the adjoint representation of the gauge group (here $SU(2)$) for the 't Hooft-Polyakov monopole concludes our considerations for the non-Abelian case.

[3]The constant M_∞ is the vacuum expectation value of M at infinity, a remnant of the Yang-Mills-Higgs theory with nonvanishing coupling λ.

4. Summary and Conclusions

In this thesis we have shed some new light on the connection between Dirac operators and supersymmetry. Dirac operators – viewed as particular realizations of supercharges – are used to define supersymmetric systems in quantum mechanics, while supersymmetry gives us the possibility to construct zero-eigenvalue solutions of the Dirac equation.

In the first part of the thesis we have defined the notion of a supersymmetric quantum system, which is characterized by the existence of a supercharge Q, that commutes with the Hamiltonian H, and an involutary operator Γ, which anticommutes with Q. A particular class of such theories can be constructed as follows: take the Dirac operator $\mathrm{i}\not{D}$ on a Riemannian manifold $\mathcal{M}$ in the background of gauge field configurations, characterized by the gauge curvature F. *Define* the Hamiltonian to be the square of this Dirac operator, $H = (\mathrm{i}\not{D})^2$. Then, $\mathrm{i}\not{D}$ serves as a supercharge of H. The existence of Γ restricts $\mathcal{M}$ to be even-dimensional, since in this case we can use $\Gamma \equiv \gamma_*$, the generalization of γ_5 in any even dimension.

Having defined systems with simple (or $\mathcal{N} = 1$) supersymmetry, we may ask the following question: *Under which conditions are there additional supercharges for this Hamiltonian?* It turns out that the existence of additional charges (i.e. $\mathcal{N} > 1$) puts restrictions on the dimensionality and the geometry of $\mathcal{M}$ as well as on the possible gauge field content.

We have shown how *number operators* N and *superpotentials* g can be defined, provided $\mathcal{N} \geq 2$. Using the latter, we can deform the supercharges into their free counterparts, which leads to an elegant solution for the zero modes of the Dirac operator. We have applied this method to Dirac operators on $\mathbb{C}P^n$.

Given the ground state wave function, any quantum mechanical system can be generalized to a theory with $\mathcal{N} = 2$. Among all examples in nonrelativistic quantum mechanics, the Coulomb problem and the harmonic oscillator potential are special, since they possess *hidden* or *dynamical* symmetry algebras that are larger than those generated by the angular momenta. In the Coulomb case, the existence of the conserved Laplace-Runge-Lenz vector implies a symmetry algebra $\mathfrak{so}(d+1)$ for bound state wave functions. For the oscillator potential, there exists a conserved symmetric tensor of second rank, which implies the algebra $\mathfrak{su}(d)$. In the non-supersymmetric case one can use the hidden symmetry algebras in order to solve the eigenvalue problems of the associated Hamiltonians in a purely algebraic way. We have generalized this procedure to the case with $\mathcal{N}=2$ supersymmetry by constructing the supersymmetric versions of the Laplace-Runge-Lenz vector and of the second-rank tensor. These conserved quantities have been used to find the algebraic solutions. In view of similar results for the non-supersymmetric case, we have speculated on a possible extension of Kustaanheimo-Stiefel transformations to the supersymmetric situation. While such an algebraic approach looks promising, its details are beyond the scope of this thesis and left to future work.

To pave the way to the field theory case, we have identified our supersymmetric Hamiltonians with those of interacting $(1+1)$-dimensional Wess-Zumino models on a spatial lattice. This identification has been established for theories with $\mathcal{N}=1$ and $\mathcal{N}=2$ supersymmetry and can be used to determine ground states, degeneracies etc. for the lattice theory. The extension of our ideas to more realistic theories in $3+1$ dimensions is currently work in progress.

The second part of this thesis is devoted to the study of supersymmetric field theories. Like in the previous examples, we have focused on theories with $\mathcal{N}=2$ extended supersymmetry. Our aim was to derive some new information about the possibility and degeneracy of zero modes of Dirac operators in three dimensions. In odd dimensional spaces, almost nothing is known about such modes, whereas in even dimensions index theorems can be used to derive lower bounds on their number.

We have considered those supersymmetric gauge theories that are appropriate for our construction and reduced them to three Euclidean dimensions: for Abelian gauge theories we have analyzed the coupling of a hypermultiplet H to the vector multiplet V. In the case of non-Abelian theories we have argued why it is sufficient to consider the vector multiplet V only. In analogy to the even-dimensional case, a method has been proposed how to generate zero modes of Dirac operators as supersymmetry variations in the background of bosonic field configurations.

For $\mathbb{R}^3$ we have proven that zero modes obtained in this way are either trivial (in the sense that they couple only to the scalar fields rather than the gauge field), or they are non-normalizable. Nontrivial solutions are found upon compactification to a three-torus, $\mathbb{R}^3 \to \mathbb{T}^3$. We have illustrated our construction with some examples: 't Hooft's constant-curvature solutions on $\mathbb{T}^3$ and the corresponding zero modes.

We have simplified our model further by ignoring the dependence on the third coordinate, effectively reducing the dimensions to two. In this way we were able to construct zero modes of Dirac operators in the background of vortex fields. Due to translational invariance in the third direction, these zero modes are non-normalizable in $\mathbb{R}^3$, albeit in a controlled manner.

In the course of our studies, zero modes of Dirac operators in the background of Yang-Mills-Higgs monopoles turned out to be the most interesting solutions in the non-Abelian setting. Here, we employed the BPS trick in order to find the monopole profile functions and generated the zero modes as supersymmetry variations around that background. The zero modes obtained in this way correspond to the Jackiw-Rebbi modes of the static four-dimensional theory.

Summarizing, the proposed method to generate zero modes of Dirac operators as supersymmetry variations around bosonic background field configurations yields interesting results for a variety of situations. However, for the particular case of a fermion in the background of an Abelian gauge field on $\mathbb{R}^3$, only limited insight can be gained this way. The question whether such a background may support zero modes remains unanswered so far and clearly deserves further study.

Own Publications

[AK1] A. Kirchberg, J.D. Länge, P.A.G. Pisani and A. Wipf, *Algebraic Solution of the Supersymmetric Hydrogen Atom in d Dimensions*, *Annals Phys.* **303** (2003) 359–388 `[hep-th/0208228]`.

[AK2] A. Kirchberg, J.D. Länge and A. Wipf, *Extended Supersymmetries and the Dirac Operator*, *submitted to Annals Phys.* `[hep-th/0401134]`.

[AK3] A. Kirchberg, J.D. Länge and A. Wipf, *From the Dirac Operator to Wess-Zumino Models on Spatial Lattices*, Preprint FSU-TPI 04/04.

in preparation:

[AK4] F. Bruckmann and A. Kirchberg, *Fermionic Zero Modes in Abelian Projections*, *in preparation.*

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A. Representation Theory

In this Appendix we collect all basic facts concerning Lie algebras, that are needed in the main body of the thesis (cf. [111, 112] for a more detailed discussion of these issues).

We discuss properties of the A_n, B_n and D_n series in the Cartan classification of simple Lie algebras, more precisely of their compact real forms, $\mathfrak{su}(n+1)$, $\mathfrak{so}(2n+1)$ and $\mathfrak{so}(2n)$. Let us consider the B_n and D_n series first. We start with D_n, since in even dimensions we can introduce complex coordinates which turn out to be very useful. Afterwards, we can add one more real coordinate and deal with B_n.

We shall construct the relevant irreducible representations of the total angular momentum operators

$$J_{ab} = x_a p_b - x_b p_a - \mathrm{i}\left(\psi_a^\dagger \psi_b - \psi_b^\dagger \psi_a\right), \qquad a, b = 1, \ldots, d, \tag{A.1}$$

satisfying the $\mathfrak{so}(d)$ commutation relations

$$[J_{ab}, J_{cd}] = \mathrm{i}(\delta_{ac} J_{bd} + \delta_{bd} J_{ac} - \delta_{ad} J_{bc} - \delta_{bc} J_{ad}), \tag{A.2}$$

on wave functions in

$$\mathcal{H}_\wp = L_2(\mathbb{R}^d) \times \mathbb{C}^{\binom{d}{\wp}} \quad \text{with} \quad \wp = 0, \ldots, d. \tag{A.3}$$

The fermionic operators ψ_a have been introduced earlier in Section 2.6.2. It is convenient to use the *Cartan-Weyl basis* consisting of generators H_i in the Cartan subalgebra and one raising and one lowering operator E_α and $E_{-\alpha}$ for every positive root α,

$$[H_i, E_\alpha] = \alpha_i E_\alpha \quad \text{and} \quad [E_\alpha, E_{-\alpha}] = \alpha \cdot H \quad \text{with} \quad E_{-\alpha} = E_\alpha^\dagger. \tag{A.4}$$

A.1. The D_n Series

As mentioned above, it is very convenient to introduce complex coordinates in our even-dimensional space $\mathbb{R}^d$, $d = 2n$. For that, we fix a particular form for the complex structure I and write

$$z_i = \frac{1}{\sqrt{2}}(x_{2i-1} + \mathrm{i}x_{2i}), \qquad \bar{z}_i = \frac{1}{\sqrt{2}}(x_{2i-1} - \mathrm{i}x_{2i}), \tag{A.5}$$

$$\partial_i = \frac{1}{\sqrt{2}}(\partial_{x_{2i-1}} - \mathrm{i}\partial_{x_{2i}}), \qquad \bar{\partial}_i = \frac{1}{\sqrt{2}}(\partial_{x_{2i-1}} + \mathrm{i}\partial_{x_{2i}}), \tag{A.6}$$

for $i = 1, \dots n$. Similarly, we define two sets of complex creation- and annihilation operators

$$\phi_i^\dagger = \frac{1}{\sqrt{2}}(\psi_{2i-1}^\dagger + \mathrm{i}\psi_{2i}^\dagger), \quad \bar{\phi}_i^\dagger = \frac{1}{\sqrt{2}}(\psi_{2i-1}^\dagger - \mathrm{i}\psi_{2i}^\dagger), \tag{A.7}$$

$$\phi_i = \frac{1}{\sqrt{2}}(\psi_{2i-1} - \mathrm{i}\psi_{2i}), \quad \bar{\phi}_i = \frac{1}{\sqrt{2}}(\psi_{2i-1} + \mathrm{i}\psi_{2i}), \tag{A.8}$$

for $i = 1, \dots n$. The only non-vanishing anticommutators are

$$\{\phi_i, \phi_j^\dagger\} = \{\bar{\phi}_i, \bar{\phi}_j^\dagger\} = \delta_{ij} \ . \tag{A.9}$$

From the algebra (A.2) we read off, that J_{12}, J_{34}, J_{56} etc. form a set of commuting operators. In fact, these n operators can be chosen as a basis of the Cartan subalgebra. Thus, the generators in the Cartan subalgebra take the simple form

$$H_i = J_{2i-1,2i} = z_i\partial_i - \bar{z}_i\bar{\partial}_i + \phi_i^\dagger\phi_i - \bar{\phi}_i^\dagger\bar{\phi}_i \ , \qquad i = 1, \dots, n. \tag{A.10}$$

There are two types of raising operators,

$$E_\alpha = \frac{1}{2}\big(J_{2i-1,2j-1} + J_{2i,2j} - \mathrm{i}J_{2i-1,2j} + \mathrm{i}J_{2i,2j-1}\big) \quad \text{with root} \quad \alpha = e_i - e_j, \tag{A.11}$$

and

$$E_\alpha = \frac{1}{2}\big(J_{2i-1,2j-1} - J_{2i,2j} + \mathrm{i}J_{2i-1,2j} + \mathrm{i}J_{2i,2j-1}\big) \quad \text{with root} \quad \alpha = e_i + e_j, \tag{A.12}$$

where $i < j$ is assumed. In terms of the complex coordinates/operators they read

$$E_\alpha = \frac{1}{\mathrm{i}}\left(z_i\partial_j - \bar{z}_j\bar{\partial}_i + \phi_i^\dagger\phi_j - \bar{\phi}_j^\dagger\bar{\phi}_i\right) \quad \text{with root} \quad \alpha = e_i - e_j, \tag{A.13}$$

$$E_\alpha = \frac{1}{\mathrm{i}}\left(z_i\bar{\partial}_j - z_j\bar{\partial}_i + \phi_i^\dagger\bar{\phi}_j - \phi_j^\dagger\bar{\phi}_i\right) \quad \text{with root} \quad \alpha = e_i + e_j. \tag{A.14}$$

The corresponding lowering operators are just the adjoints of the raising operators. The operators $(H_i, E_\alpha, E_{-\alpha})$ satisfy the commutation relations (A.4). The n simple roots are

$$\alpha_i = e_i - e_{i+1}\ , \quad 1 \leq i < n \quad \text{and} \quad \alpha_n = e_{n-1} + e_n, \tag{A.15}$$

and the corresponding raising operators have the form

$$E_i = \frac{1}{\mathrm{i}}\left(z_i\partial_{i+1} - \bar{z}_{i+1}\bar{\partial}_i + \phi_i^\dagger\phi_{i+1} - \bar{\phi}_{i+1}^\dagger\bar{\phi}_i\right), \quad \alpha = e_i - e_{i+1}, \tag{A.16}$$

for $1 \leq i < n$, and

$$E_n = \frac{1}{\mathrm{i}}\left(z_{n-1}\bar{\partial}_n - z_n\bar{\partial}_{n-1} + \phi_{n-1}^\dagger\bar{\phi}_n - \phi_n^\dagger\bar{\phi}_{n-1}\right), \quad \alpha = e_{n-1} + e_n. \tag{A.17}$$

With the help of the Weyl vector

$$\delta \equiv \frac{1}{2}\sum_{\alpha>0}\alpha = (n-1)e_1 + (n-2)e_2 + \cdots + e_{n-1}\ , \tag{A.18}$$

where the sum extends over all positive roots, we may calculate the dimension of an arbitrary faithful representation of $\mathfrak{so}(2n)$. Such a representation is determined by its Young tableau which contains at most n rows. The length ℓ_i of row i is bigger or equal to that of row $i+1$. Hence, a Young tableau is given by n non-negative ordered integers

$$\ell_1 \geq \ell_2 \geq \cdots \geq \ell_{n-1} \geq \ell_n, \tag{A.19}$$

and has the form

$$\wp\left\{\begin{array}{|c|c|c|c|c|c|} \hline 1 & & & & & \ell_1 \\ \hline 1 & & & & \ell_2 & \multicolumn{1}{c}{} \\ \cline{1-5} : & : & : & : & \multicolumn{2}{c}{} \\ \cline{1-4} 1 & & \ell_\wp & \multicolumn{3}{c}{} \\ \cline{1-3} \end{array}\right. \ , \quad \wp \leq n. \tag{A.20}$$

We use Weyls dimension formula [111],

$$\dim_\Lambda = \prod_{\alpha>0} \frac{\langle \alpha, \Lambda + \delta \rangle}{\langle \alpha, \delta \rangle}, \tag{A.21}$$

for a representation characterized by the highest weight Λ. Correspondingly, the representation $\mathcal{D}^{\,\ell_1,\ldots,\ell_n}$ has the dimension

$$\dim\left(\mathcal{D}^{\,\ell_1,\ldots,\ell_n}\right) = \prod_{1\le r<s\le n} \frac{\ell_r + \ell_s + 2n - r - s}{2n - r - s}\, \frac{\ell_r - \ell_s + s - r}{s - r}\,. \tag{A.22}$$

For the second-order Casimir invariant of these representations one obtains the formula

$$\mathcal{C}_{(2)}(\mathcal{D}^{\,\ell_1,\ldots,\ell_n}) = \sum_r \ell_r(\ell_r + 2n - 2r). \tag{A.23}$$

In particular, for the completely symmetric representations $\mathcal{D}^{\,\ell,0,\ldots,0} \equiv \mathcal{D}^{\ell}_1 \sim \begin{array}{|c|c|c|c|}\hline 1 & & & \ell \\ \hline\end{array}$, these formulae simplify to

$$\mathcal{C}_{(2)}(\mathcal{D}^{\ell}_1) = \ell(\ell + d - 2), \quad \text{and} \quad \dim\left(\mathcal{D}^{\ell}_1\right) = \binom{\ell + d - 1}{\ell} - \binom{\ell + d - 3}{\ell - 2}. \tag{A.24}$$

For the completely antisymmetric representations $\mathcal{D}^{\,1,1,\ldots,1} \equiv \mathcal{D}^{1}_{\wp} \sim \begin{array}{|c|}\hline 1 \\ \hline \\ \hline \\ \hline \wp \\ \hline\end{array}$, they yield

$$\mathcal{C}_{(2)}(\mathcal{D}^{1}_{\wp}) = \wp(d - \wp), \quad \text{and} \quad \dim\left(\mathcal{D}^{1}_{\wp}\right) = \binom{d}{\wp}. \tag{A.25}$$

Simultaneous eigenstates of all n generators H_i in the Cartan subalgebra have the form

$$\prod_{i=1}^{n} z_i^{m_i} \bar{z}_i^{\bar{m}_i} \left|\boldsymbol{p}, \bar{\boldsymbol{p}}\right\rangle, \qquad \left|\boldsymbol{p}, \bar{\boldsymbol{p}}\right\rangle \equiv \phi_1^{\dagger p_1} \ldots \phi_n^{\dagger p_n} \bar{\phi}_1^{\dagger \bar{p}_1} \ldots \bar{\phi}_n^{\dagger \bar{p}_n} \left|0\right\rangle, \tag{A.26}$$

where $m_i, \bar{m}_i \in \mathbb{N}_0$ and $p_i, \bar{p}_i \in \{0, 1\}$. The vacuum $|0\rangle$ is annihilated by all particle lowering operators ψ_a or equivalently by all ϕ_i and $\bar{\phi}_i$. The H_i-eigenvalues of these states are $m_i - \bar{m}_i + p_i - \bar{p}_i$.

Next we must construct the highest weight states which are annihilated by all raising operators. Every such state determines an irreducible representation. The eigenvalues of

H_i on a highest weight state is equal to the length ℓ_i of the Young tableau corresponding to the irreducible representation which is determined by this weight. The $d+1$ space-independent highest weight states are

$$|\wp\rangle = |\boldsymbol{p}, \bar{\boldsymbol{p}}\rangle \quad \text{with} \quad p_1 \geq \ldots \geq p_n \geq \bar{p}_n \geq \ldots \geq \bar{p}_1 \quad \text{and} \quad \sum (p_i + \bar{p}_i) = \wp. \tag{A.27}$$

There is an additional highest weight state in the $\wp = n$ particle sector, that arises since in this sector we have selfdual and anti-selfdual configurations. It is given by

$$p_1 = \ldots = p_{n-1} = \bar{p}_n = 1, \qquad p_n = \bar{p}_1 = \ldots = \bar{p}_{n-1} = 0. \tag{A.28}$$

Clearly, the particle number $\wp$ uniquely specifies these state since the p_i and $\bar{p}_j$ are ordered. These states define the completely antisymmetric representations

$$\mathcal{D}^1_\wp \quad \text{for} \quad \wp \leq n \quad \text{and} \quad \mathcal{D}^1_\wp \sim \mathcal{D}^1_{2n-\wp} \quad \text{for} \quad \wp \geq n\ . \tag{A.29}$$

We used the following fact: a Young tableau, the first column of which has length $n \leq \wp \leq 2n$, gives rise to the same multiplet as the tableau with first column of length $2n - \wp \leq n$. In the following one should replace $\mathcal{D}^\ell_\wp$ by $\mathcal{D}^\ell_{2n-\wp}$ if $\wp$ exceeds n. Also note that $\mathcal{D}^1_0 \sim \mathcal{D}^1_d$ is the one-dimensional trivial representation $\mathbb{1}$.

The highest weight states in the zero-particle sector are

$$\Psi_{\text{h.w.}} \sim z_1^\ell\, |0\rangle\,, \tag{A.30}$$

and they give rise to the completely symmetric representations $\mathcal{D}^\ell_1$ spanned by the harmonic polynomials of order ℓ. The relevant irreducible representation of $\mathfrak{so}(2n)$ in the $\wp$-particle sector is gotten by tensoring the antisymmetric representation $\mathcal{D}^1_\wp$ with a symmetric representation $\mathcal{D}^\ell_1$. We use

$$\mathcal{D}^1_\wp \otimes \mathcal{D}^\ell_1 = \mathcal{D}^\ell_{\wp-1} \oplus \mathcal{D}^{\ell-1}_\wp \oplus \mathcal{D}^{\ell+1}_\wp \oplus \mathcal{D}^\ell_{\wp+1}. \tag{A.31}$$

Note that for $\wp = 1$ and/or $\ell = 1$ there appear only three representations in this decomposition. For $\wp = 1$ the first representation and for $\ell = 1$ the second representation on the right hand side in (A.31) are absent. Also note that for $\wp = n$ the first and last representations are equivalent. The second to last representation $\mathcal{D}^{\ell+1}_\wp$ on the right hand side has highest weight state

$$\mathcal{Y}_s(\ell+1,\wp) = z_1^\ell \; |\wp\rangle\,, \tag{A.32}$$

as it is the product of the highest weight states of $\mathcal{D}^1_\wp$ and $\mathcal{D}^\ell_1$. To find the highest weight state of the other representations we observe that the operators

$$rS = x_a\psi_a = z_i\phi_i + \bar{z}_i\bar{\phi}_i, \tag{A.33}$$

$$rS^\dagger = x_a\psi_a^\dagger = \bar{z}_i\phi_i^\dagger + z_i\bar{\phi}_i^\dagger, \tag{A.34}$$

which have been introduced in (2.115), commute with the total angular momentum and hence map highest weight states into highest weight states. Since S decreases and $S^\dagger$ increases the particle number by one, we find the state

$$\mathcal{Y}_a(\ell,\wp+1) = rS\mathcal{Y}_s(\ell,\wp-1) = \sum_{i=1}^{\wp+1}(-)^{i+1}z_i\phi_1^\dagger\dots\check{\phi}_i^\dagger\dots\phi_{\wp+1}^\dagger z_1^{\ell-1}\,|0\rangle\,, \tag{A.35}$$

which is highest weight state of the last representation, $\mathcal{D}^\ell_{\wp+1}$, in the decomposition (A.31). Here $\check{\phi}_i^\dagger$ indicates, that $\phi_i^\dagger$ is omitted in the product.

The missing two highest weight states correspond to those representations (in the tensor product of a symmetric and an antisymmetric representation) that one obtains by taking the trace over two suitable indices. Note, that this operation is equivalent to acting with $S^\dagger$. Thus

$$\mathcal{T}_s(\ell,\wp-1) = S^\dagger\mathcal{Y}_s(\ell,\wp-1), \tag{A.36}$$

is the highest weight state of $\mathcal{D}^\ell_{\wp-1}$ in the decomposition (A.31). For the remaining highest weight state we make the ansatz

$$\mathcal{T}_a(\ell-1,\wp) = \big(SS^\dagger + \alpha S^\dagger S\big)\mathcal{Y}_s(\ell-1,\wp)\,. \tag{A.37}$$

As $\{S,S^\dagger\} = 1$, this state may have a component in the direction of $\mathcal{Y}_s(\ell-1,\wp)$. However, for the choice $\alpha = -1$, the highest weight state

$$\mathcal{T}_a(\ell-1,\wp) = [S,S^\dagger]\,\mathcal{Y}_s(\ell-1,\wp)\,, \tag{A.38}$$

is orthogonal to $\mathcal{Y}_s(\ell - 1, \wp)$.

In this way, we obtained the four highest weight states $\mathcal{Y}_s$, $\mathcal{Y}_a$, $\mathcal{T}_s$ and $\mathcal{T}_a$ corresponding to the four Young tableaux on the right hand side of (A.31).

A.2. The B_n Series

The algebra $\mathfrak{so}(2n+1)$ of rotations has the same rank as its subalgebra $\mathfrak{so}(2n)$. Hence, we may still use the Cartan generators (A.10), that is

$$H_i = J_{2i-1,2i} = z_i\partial_i - \bar{z}_i\bar{\partial}_i + \phi_i^\dagger\phi_i - \bar{\phi}_i^\dagger\bar{\phi}_i\ , \qquad i = 1, \ldots, n. \tag{A.39}$$

We use the complex coordinates (A.5) and the complex creation- and annihilation operators (A.7), supplemented by the last coordinate x_d and the last creation and annihilation operator $\psi_d^\dagger$ and ψ_d. Clearly, the raising operators (A.14) are still raising operators of $\mathfrak{so}(2n+1)$ with the same positive roots. But since

$$\dim \mathfrak{so}(2n+1) = \dim \mathfrak{so}(2n) + 2n, \tag{A.40}$$

$$\operatorname{rank} \mathfrak{so}(2n+1) = \operatorname{rank} \mathfrak{so}(2n), \tag{A.41}$$

there are n positive (and n negative) roots missing. The positive roots are

$$E_\alpha = \frac{1}{\sqrt{2}}\left(J_{2i-1,d} + \mathrm{i}J_{2i,d}\right) = \frac{1}{\mathrm{i}}\left(z_i\partial_{x_d} - x_d\bar{\partial}_i + \phi_i^\dagger\psi_d - \psi_d^\dagger\bar{\phi}_i\right), \quad \alpha = e_i, \tag{A.42}$$

where $1 \leq i \leq n$. The first $n-1$ simple roots are the same as in (A.16), but the the last one is replaced by e_n. Hence, the raising operators corresponding to the simple roots read

$$E_i = \frac{1}{\mathrm{i}}\left(z_i\partial_{i+1} - \bar{z}_{i+1}\bar{\partial}_i + \phi_i^\dagger\phi_{i+1} - \bar{\phi}_{i+1}^\dagger\bar{\phi}_i\right), \tag{A.43}$$

with root $\alpha = e_i - e_{i+1}$ $(1 \leq i < n)$, and

$$E_n = \frac{1}{\mathrm{i}}\left(z_n\partial_{x_d} - x_d\bar{\partial}_n + \phi_n^\dagger\psi_d - \psi_d^\dagger\bar{\phi}_n\right), \tag{A.44}$$

with root $\alpha = e_n$. The Young tableaux are identical to those of $\mathfrak{so}(2n)$ and hence are characterized by n ordered non-negative integers $\ell_1, \ldots, \ell_n$. The dimensions of the corresponding representations read

$$\dim\left(\mathcal{D}^{\ell_1,\ldots,\ell_n}\right) = \prod_{t=1}^{n} \frac{2\ell_t + d - 2t}{d - 2t} \prod_{1\leq r<s\leq n} \frac{\ell_r + \ell_s + d - r - s}{d - r - s} \frac{\ell_r - \ell_s + s - r}{s - r}, \quad (A.45)$$

and the formula for the second-order Casimir is the same as for the $\mathfrak{so}(2n)$ algebra,

$$\mathcal{C}_{(2)}(\mathcal{D}^{\ell_1,\ldots,\ell_n}) = \sum_r \ell_r(\ell_r + d - 2r). \quad (A.46)$$

Also the rules for tensor products are identical to those of $\mathfrak{so}(2n)$.

Since the simple roots are different, the highest weight states have a slightly different form, too. The simultaneous eigenstates of the n generators in the Cartan subalgebra read

$$f(x_d) \prod_i z_i^{m_i} \bar{z}_i^{\bar{m}_i} \; |\boldsymbol{p}, q, \bar{\boldsymbol{p}}\rangle, \qquad |\boldsymbol{p}, q, \bar{\boldsymbol{p}}\rangle = \phi_1^{\dagger\, p_1} \ldots \phi_n^{\dagger\, p_n} \psi_d^{\dagger\, q} \bar{\phi}_1^{\dagger\, \bar{p}_1} \ldots \bar{\phi}_n^{\dagger\, \bar{p}_n} \, |0\rangle, \quad (A.47)$$

where $m_i, \bar{m}_i \in \mathbb{Z}$ and $p_i, q, \bar{p}_i \in \{0, 1\}$. The $d+1$ constant highest weight states are

$$|\wp\rangle = |\boldsymbol{p}, q, \bar{\boldsymbol{p}}\rangle \quad \text{with} \quad p_1 \geq \ldots \geq p_n \geq q \geq \bar{p}_n \geq \ldots \geq \bar{p}_1, \quad (A.48)$$

where $\wp = \sum(p_i + \bar{p}_i) + q$ denotes the particle number. The highest weight of $\mathcal{D}_\wp^{\ell+1}$ in the decomposition

$$\mathcal{D}_\wp^1 \otimes \mathcal{D}_1^\ell = \mathcal{D}_{\wp-1}^\ell \oplus \mathcal{D}_\wp^{\ell-1} \oplus \mathcal{D}_\wp^{\ell+1} \oplus \mathcal{D}_{\wp+1}^\ell, \quad (A.49)$$

is again determined by the highest weight state

$$\mathcal{Y}_s(\ell + 1, \wp) = z_1^l \; |\wp\rangle. \quad (A.50)$$

As in even dimensions, one may use the scalar operators

$$rS = x_a \psi_a = z_i \phi_i + \bar{z}_i \bar{\phi}_i + x_d \psi_d, \quad (A.51)$$

$$rS^\dagger = x_a \psi_a^\dagger = \bar{z}_i \phi_i^\dagger + z_i \bar{\phi}_i^\dagger + x_d \psi_d^\dagger, \quad (A.52)$$

to obtain the highest weight states

$$\begin{aligned}
\mathcal{Y}_a(\ell,\wp+1) &= rS\mathcal{Y}_s(\ell,\wp+1) &\longrightarrow&\quad \mathcal{D}^{\ell}_{\wp+1},\\
\mathcal{T}_s(\ell,\wp-1) &= S^\dagger\mathcal{Y}_s(\ell,\wp-1) &\longrightarrow&\quad \mathcal{D}^{\ell}_{\wp-1},\\
\mathcal{T}_a(\ell-1,\wp) &= [S,S^\dagger]\mathcal{Y}_s(\ell-1,\wp) &\longrightarrow&\quad \mathcal{D}^{\ell-1}_{\wp},
\end{aligned} \tag{A.53}$$

of the remaining irreducible representations in (A.49).

A.3. Eigenstates of the Supersymmetric Hydrogen Atom

In the main body of the thesis we have seen, that the total angular momenta J_{ab} in (2.112) together with the generalized Laplace-Runge-Lenz vector K_a in (2.128) combine to generators of the algebra $\mathfrak{so}(d+1)$,

$$J_{AB} = \left(\begin{array}{c|c} J_{ab} & K_a \\ \hline -K_b & 0 \end{array}\right). \tag{A.54}$$

The algebra $\mathfrak{so}(d)$ with generators J_{ab} discussed in the previous two parts of the Appendix, must be embedded into the dynamical algebra, for $d=2n$, $\mathfrak{so}(2n)\subset\mathfrak{so}(2n+1)$, and for $d=2n+1$, $\mathfrak{so}(2n+1)\subset\mathfrak{so}(2n+2)$.

So far we have not considered which highest weight states of the dynamical symmetry group are normalizable. This is necessary as only energies corresponding to normalizable states belong to the spectrum of the Hamiltonian. Now we explicitly construct these states in all subspaces $\mathcal{H}_\wp\subset\mathcal{H}$. In Section 2.6.4 we have seen that for any $\ell\geq 1$ there are only one or two irreducible representations of $\mathfrak{so}(d+1)$, namely

$$\mathcal{D}^{\ell}_{\wp}\subset\left(\mathcal{H}_\wp\cap Q^\dagger\mathcal{H}\right) \quad\text{and}\quad \mathcal{D}^{\ell}_{\wp+1}\subset\left(\mathcal{H}_\wp\cap Q\mathcal{H}\right), \tag{A.55}$$

the multiplet $\mathcal{D}^{\ell}_{\wp}$ belonging to $Q^\dagger\mathcal{H}_{\wp-1}$ and the multiplet $\mathcal{D}^{\ell}_{\wp+1}$ belonging to the subspace $Q\mathcal{H}_{\wp+1}$. It suffices to construct the highest weight states $\Psi^{\wp}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp+1})$ of the latter multiplets. The highest weight states $\Psi^{\wp}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp})$ of the first set of multiplets in (A.55) are then just their superpartners,

$$\Psi^{\wp}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp}) = Q^{\dagger}\Psi^{\wp-1}_{\text{h.w}}(\mathcal{D}^{\ell}_{\wp}). \tag{A.56}$$

Actually, we only need to calculate the highest weight states $\Psi^{\wp}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp+1})$ for $\wp < d/2$ because of the *duality transformation* $(\wp, \lambda) \leftrightarrow (d-\wp, -\lambda)$, which leaves the Hamiltonian (2.123) invariant. Observe that for any normalizable H-eigenstate $\Psi \in Q\mathcal{H}$ the transformed state $Q^{\dagger}\Psi$ is normalizable, as can be seen from

$$(Q^{\dagger}\Psi, Q^{\dagger}\Psi) = (\Psi, QQ^{\dagger}\Psi) = (\Psi, H\Psi) = E(\Psi, \Psi). \tag{A.57}$$

Without calculating the highest weight states we can argue in which sectors bound states cannot exist. For that purpose we consider the Hamiltonian (2.123). It is easy to see that the Hermitian operator $S^{\dagger}S$, where S has been defined in (2.115), is an orthogonal projector, and hence has eigenvalues 0 and 1. It follows at once that for $\wp > d/2$ the operator A in (2.122) is negative and hence $H > \lambda^2$. We conclude that H has no bound states in the sectors $\mathcal{H}_{\wp > d/2}$. On the particular sector $\mathcal{H}_n$ the operator A has both positive and negative eigenvalues. We expect that in this sector only one of the two representations (for each ℓ) of the dynamical symmetry algebra contains bound states.

After these general considerations we proceed with computing the highest weight states $\Psi^{\wp}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp+1})$ in the subspace $\mathcal{H}_{\wp} \cap Q\mathcal{H}$. Again we treat even- and odd-dimensional spaces separately.

Even dimensions: $\mathfrak{so}(2n+1)$ has the same rank as the algebra of rotations, $\mathfrak{so}(2n)$, and we can repeat our construction in Appendix A.2, where we extended $\mathfrak{so}(2n)$ to $\mathfrak{so}(2n+1)$. Of course, we should take into account that the entries in the last column and last row of J_{AB} are the components of K_a. The Cartan generators are those in (A.39) and the first $n-1$ raising operators are given in (A.43). But the last raising operator (A.44) is replaced by

$$\begin{aligned} E_n &= \frac{1}{\sqrt{2}}(K_{d-1} + \mathrm{i}K_d) \sim \frac{1}{\sqrt{2}}(C_{d-1} + \mathrm{i}C_d) \\ &\sim -2z_n\Delta + (2r\partial_r + d - 1)\bar{\partial}_n - 2\phi^{\dagger}_n\big(\phi_i\bar{\partial}_i + \bar{\phi}_i\partial_i\big) \\ &\quad + 2\big(\phi^{\dagger}_i\partial_i + \bar{\phi}^{\dagger}_i\bar{\partial}_i\big)\bar{\phi}_n - \lambda r^{-1}z_n A. \end{aligned} \tag{A.58}$$

Since the simple roots of $\mathfrak{so}(d)$ are positive roots of the dynamical symmetry algebra, a highest weight state of $\mathfrak{so}(d+1)$ is automatically a highest weight state of $\mathfrak{so}(d)$, similar as in the purely bosonic case. Since the two groups share the same Cartan generators, the highest weight state $\Psi^{\wp}_{\text{h.w}}(\mathcal{D}^{\,\ell}_{\wp+1})$ of $\mathfrak{so}(d+1)$ must also be a highest weight state of the multiplet $\mathcal{D}^{\,\ell}_{\wp+1}$ of $\mathfrak{so}(d)$. From the branching rule (2.137) and the tensor products (2.136) it follows, that this highest weight state must be the state $\mathcal{Y}_a(\ell,\wp+1)$ given in (A.35). Hence we are lead to the ansatz

$$\Psi^{\wp}_{\text{h.w.}}(\mathcal{D}^{\,\ell}_{\wp+1}) = f(r)\mathcal{Y}_a(\ell,\wp+1). \tag{A.59}$$

It remains to determine the radial function $f(r)$ such that $\Psi^{\wp}_{\text{h.w.}}$ is annihilated by E_n. With E_n from (A.58) one finds the following equation for f,

$$(d-1+2\ell)f' + \lambda(d-1-2\wp)f = 0, \tag{A.60}$$

such that the relevant highest weight states in the $\wp$-particle take the form

$$\Psi^{\wp}_{\text{h.w.}}(\mathcal{D}^{\,\ell}_{\wp+1}) = \mathcal{Y}_a(\ell,\wp+1)\mathrm{e}^{-\gamma_{\ell\wp}r}, \quad \text{with} \quad \gamma_{\ell\wp} = \frac{d-1-2\wp}{d-1+2\ell}\lambda. \tag{A.61}$$

As λ is assumed positive, these are bound states for $\wp < n$.

Odd dimensions: The rank of the dynamical symmetry algebra $\mathfrak{so}(2n+2)$ exceeds the rank of the algebra of rotations, $\mathfrak{so}(2n+1)$, by one. The Cartan generators are given by the n operators H_i in (A.10), supplemented by

$$\begin{aligned} H_{n+1} &= K_d \sim C_d \\ &\sim -2x_d\Delta + (2r\partial_r + d - 1)\partial_d \\ &\quad - 2\psi_d^\dagger(\phi_i\bar\partial_i + \bar\phi_i\partial_i) + 2(\phi_i^\dagger\partial_i + \bar\phi_i^\dagger\bar\partial_i)\psi_d - \lambda r^{-1}x_d A. \end{aligned} \tag{A.62}$$

The raising operators are the $n-1$ operators E_i in (A.43) plus the two operators

$$E_\alpha = \frac{1}{2}\big(J_{d-2,d} + K_{d-1} - \mathrm{i}K_{d-2} + \mathrm{i}J_{d-1,d}\big), \quad \alpha = e_n - e_{n+1}, \tag{A.63}$$

$$E'_\alpha = \frac{1}{2}\big(J_{d-2,d} - K_{d-1} + \mathrm{i}K_{d-2} + \mathrm{i}J_{d-1,d}\big), \quad \alpha = e_n + e_{n+1}. \tag{A.64}$$

All highest weight states are annihilated by these two raising operators. It is convenient to use two linear combinations of these operators, in particular

$$E_n = \frac{1}{\sqrt{2}}\,(E_\alpha + E'_\alpha) = \frac{1}{\mathrm{i}}(z_n\partial_{x_d} - x_d\bar{\partial}_{z_n} + \phi_n^\dagger\psi_d - \psi_d^\dagger\bar{\phi}_n), \tag{A.65}$$

$$E_{n+1} = \frac{\mathrm{i}}{\sqrt{2}}\,(E_\alpha - E'_\alpha) \sim \frac{1}{\sqrt{2}}(C_{d-2} + \mathrm{i}C_{d-1}). \tag{A.66}$$

Invoking similar arguments as in even dimensions we are lead to the following ansatz

$$\Psi^{\wp}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp+1}) = f(r)\mathcal{Y}_a(\ell, \wp+1), \tag{A.67}$$

for the highest weight state of $\mathcal{D}^{\ell}_{\wp+1} \subset \mathcal{H}_{\wp}$. This function is annihilated by all $E_{i\leq n}$. The condition $E_{n+1}\Psi^{\wp}_{\text{h.w.}} = 0$ yields the same differential equation for the radial function $f(r)$ as before, and we obtain

$$\Psi^{\wp}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp+1}) = \mathcal{Y}_a(\ell, \wp+1)\mathrm{e}^{-\gamma_{\ell\wp}r}, \quad \text{with} \quad \gamma_{\ell\wp} = \frac{d-1-2\wp}{d-1+2\ell}\lambda. \tag{A.68}$$

For positive λ these states are normalizable for all $\wp < n$. The last Cartan generator, $H_{n+1} \sim C_d$, annihilates this state and thus it has the correct weight.

The remaining highest weight states: We have argued that the highest weight state $\Psi^{\wp+1}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp+1}) \subset \mathcal{H}_{\wp+1}$ is the superpartner of $\Psi^{\wp}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp+1})$ in (A.68). A simple calculation yields

$$\begin{aligned}\Psi^{\wp+1}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp+1}) &= Q^\dagger\Psi^{\wp}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp+1}) \\ &= \mathrm{i}\Big((\lambda - \gamma_{\ell\wp})S^\dagger\mathcal{Y}_a(\ell, \wp+1) + (\ell+\wp)\mathcal{Y}_s(\ell, \wp+1)\Big)\mathrm{e}^{-\gamma_{\ell\wp}r}\end{aligned} \tag{A.69}$$

for this state. This shows that $\Psi^{\wp+1}_{\text{h.w.}}(\mathcal{D}^{\ell}_{\wp+1})$ is a linear combination of the two highest weight states $\mathcal{Y}_s$ and $\mathcal{Y}_a$ of $\mathfrak{so}(d)$ given in formulae (A.32) and (A.35). These states lead to the second series of bound-state multiplets in the sectors $\mathcal{H}_{\wp}$ with $\wp = 1, \ldots, n$.

A.4. The A_n Series

Finally, we summarize some basic facts concerning the Lie algebra $\mathfrak{su}(d)$. They are needed for the discussion of the harmonic oscillator example. More details can be found in [38, 39, 112, 113].

Let $r = d - 1$ be the rank of $\mathfrak{su}(d)$. Accordingly, r Cartan generators H_i can be constructed, that are related to the components of the conserved tensor T as follows

$$H_i = \frac{1}{2} \left(T_{ii} - T_{i+1\, i+1} \right), \qquad i = 1, 2, \ldots r. \tag{A.70}$$

With the definition

$$e_p(H_j) = \mathrm{i} \left(\delta_{jp} - \delta_{j+1p} \right), \tag{A.71}$$

the set of all possible roots is given by $\alpha_{pq} = e_p - e_q$, $p, q = 1, 2, \ldots, r$, $p \neq q$, positive roots are α_{pq}, where $p < q$, and the simple roots are given by $\alpha_j = e_j - e_{j+1}$, $j = 1, 2, \ldots r$.

The raising and lowering operators corresponding to the root characterized by p and q are given by $E_\pm^{pq} = \frac{1}{2} T_{pq} \pm \mathrm{i} L_{pq}$, the ones corresponding to simple roots are

$$E_\pm^j = \frac{1}{2} \left(T_{j\, j+1} \pm \mathrm{i} L_{j\, j+1} \right). \tag{A.72}$$

One easily verifies

$$[H_i, H_j] = 0, \qquad [E_+^i, E_-^j] = \delta_{ij} H_j, \qquad [H_i, E_\pm^j] = \pm A^{ji} E_\pm^j, \tag{A.73}$$

with Cartan matrix

$$A^{ij} = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \cdots & \cdots & \cdots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}^{ij}, \qquad i, j = 1, 2, \ldots r. \tag{A.74}$$

The quadratic Casimir operator of $\mathfrak{su}(d)$

$$\mathcal{C}_{(2)} = \sum_{A,B=1}^{d^2-1} \kappa_{AB} T^A T^B, \tag{A.75}$$

can be written in the Chevalley-Serre basis as

$$\mathcal{C}_{(2)} = \sum_{i,j=1}^{r} G_{ij} H^i H^j + \sum_{\alpha} E^{\alpha} E^{-\alpha}. \tag{A.76}$$

Here T^A are the generators, κ_{AB} denotes the Killing metric and

$$G_{ij} = \frac{1}{d} \min(i,j) \cdot (d - \max(i,j)), \tag{A.77}$$

is the so-called quadratic form matrix (the inverse of A^{ij}). The second sum in (A.76) extends over all roots α. In terms of T_{ab} and L_{ab} the Casimir reads

$$\mathcal{C}_{(2)} = \frac{1}{4}\left(T^2 + L^2\right) - \frac{(T_{aa})^2}{4d}. \tag{A.78}$$

The eigenfunctions of the bosonic harmonic oscillator are given by

$$(a_1^{\dagger})^{n_1} (a_2^{\dagger})^{n_2} \dots (a_d^{\dagger})^{n_d} \left|0\right\rangle, \qquad N_B = \sum_{i=1}^{d} n_i. \tag{A.79}$$

These wave functions furnish totally symmetric representations of $\mathfrak{su}(d)$, corresponding to Young tableaux of the form $\boxed{1}\boxed{}\boxed{}\boxed{\ell} \equiv \mathcal{D}_1^{\ell}$, where ℓ is equal to the number N_B of excitations.

The value of the quadratic Casimir in those representations is given by

$$\mathcal{C}_{(2)}\left(\mathcal{D}_1^{\ell}\right) = \ell(\ell + d - 1) - \frac{\ell^2}{d}. \tag{A.80}$$

This can be obtained as follows [39]: A Young tableau with n_{row} rows of length $b_1, \dots b_{n_{\text{row}}}$ and with n_{col} columns labeled by $a_1, \dots a_{n_{\text{col}}}$ and a total number B of boxes, corresponds to a representation of $\mathfrak{su}(d)$ with value of the quadratic Casimir

$$\mathcal{C}_{(2)} = B(d - \frac{B}{d}) + \sum_{i=1}^{n_{\text{row}}} b_i^2 - \sum_{i=1}^{n_{\text{col}}} a_i^2. \tag{A.81}$$

The dimension of this representation can be obtained via the *Hook formula*: let the pairs (i,j) label the boxes of a given Young tableau, with i numbering the rows from top to bottom, and j numbering the column from left to right. h_{ij}, the hook length, is the number of boxes which belong to the hook that has (i,j) as its upper left hand corner. Then the dimension of the representation ist

$$\dim = \prod_{(i,j)} \frac{d-i+j}{h_{ij}}, \tag{A.82}$$

in particular

$$\dim\left(\mathcal{D}_1^\ell\right) = \frac{(d+\ell-1)!}{\ell!(d-1)!}. \tag{A.83}$$

A given representation of $\mathfrak{su}(d)$ can be decomposed with respect to its subalgebra $\mathfrak{so}(d)$ via the following branching rule [41, 112]

$$\mathcal{D}_1^\ell\big|_{\mathfrak{su}(d)} \longrightarrow \mathcal{D}_1^\ell \oplus \mathcal{D}_1^{\ell-2} \oplus \mathcal{D}_1^{\ell-4} \oplus \dots \big|_{\mathfrak{so}(d)}. \tag{A.84}$$

The symmetric state $\mathcal{D}_1^\ell$ provides an irreducible representation of $\mathfrak{su}(d)$. Taking traces is not a $\mathfrak{su}(d)$ invariant operation. At the same time this state forms a representation of $\mathfrak{so}(d)$, but now traces can be taken and this eliminates two boxes from the tableau. This procedure can be iterated and yields the right hand side of (A.84).

In the supersymmetric case we replace L_{ab} by J_{ab} and T_{ab} by its supersymmetric generalization. Now representations of the form $\mathcal{D}_\wp^\ell$ appear, since any state

$$|n_1, \dots n_d; p_1, \dots p_d\rangle \equiv (a_1^\dagger)^{n_1} \dots (a_d^\dagger)^{n_d} (\psi_1^\dagger)^{p_1} \dots (\psi_d^\dagger)^{p_d} |0\rangle\,, \tag{A.85}$$

lies in the tensor product of completely symmetric states (generated by the bosonic raising operators $a_b^\dagger$) and completely antisymmetric states (generated by the fermionic raising operators $\psi_a^\dagger$)

$$\mathcal{D}_1^\ell \otimes \mathcal{D}_\wp^1 = \mathcal{D}_{\wp+1}^\ell \oplus \mathcal{D}_\wp^{\ell+1}. \tag{A.86}$$

Observe that in any sector with fixed particle number N two representations of $\mathfrak{su}(d)$ appear: $\mathcal{D}_{N+1}^\ell$ and $\mathcal{D}_N^\ell$ (cf. Figure A.1).

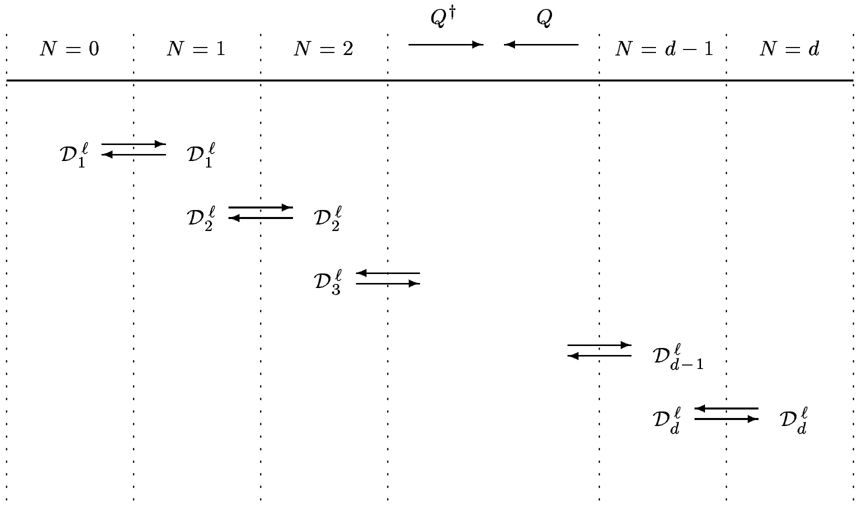

Figure A.1.: Representations of $\mathfrak{su}(d)$ distributed with respect to the different sectors.

One finds (via (A.81) and the Hook formula (A.82))

$$\begin{aligned} \mathcal{C}_{(2)} &= \frac{1}{4}(T^2 + J^2) - \frac{T_{aa}^2}{4d}, \\ \mathcal{C}_{(2)}\left(\mathcal{D}_\wp^\ell\right) &= (\wp + \ell - 1)\left(d - \frac{\wp + \ell - 1}{d}\right) + \ell^2 - \wp^2 + \wp - \ell, \\ \dim\left(\mathcal{D}_\wp^\ell\right) &= \frac{(d + \ell - 1)!}{(d - \wp)!(\wp - 1)!(\ell - 1)!(\wp + \ell - 1)}. \end{aligned} \tag{A.87}$$

The branching rules for these representations read as follows [112]:

$$\mathcal{D}_\wp^\ell\big|_{\mathfrak{su}(d)} \longrightarrow \mathcal{D}_\wp^\ell \oplus \mathcal{D}_\wp^{\ell-2} \oplus \mathcal{D}_\wp^{\ell-4} \oplus \ldots \oplus \mathcal{D}_{\wp-1}^{\ell-1} \oplus \mathcal{D}_{\wp-1}^{\ell-3} \oplus \mathcal{D}_{\wp-1}^{\ell-5} \oplus \ldots \ \Big|_{\mathfrak{so}(d)}. \tag{A.88}$$

On the right hand side the $\wp = \frac{d}{2}$ series in even dimensions – in case it appears – has to be taken twice, again, corresponding to the selfdual and the anti-selfdual representations of $\mathfrak{so}(d)$.

B. Zero Modes on $\mathbb{C}P^n$

The ubiquitous two-dimensional $\mathbb{C}P^n$ models possess remarkable similarities with non-Abelian gauge theories in 3+1 dimensions [114]. They are frequently used as toy models displaying interesting physics like fermion-number violation analogous to the electroweak theory [115] or spin excitations in quantum Hall systems [116]. Their instanton solutions have been studied in [117], and their $\mathcal{N}=2$ supersymmetric extensions describe integrable systems with known scattering matrices.

As a particular application of our considerations on theories with $\mathcal{N}=2$ supersymmetry, we consider the Dirac operator on the Kähler manifolds $\mathbb{C}P^n$. As outlined in Section 2.5 for the general situation, we shall calculate the superpotential g in (2.74) and the explicit zero modes of the Dirac operator.

B.1. Complex Projective Spaces

First we briefly recall those properties of the complex projective spaces $\mathbb{C}P^n$ that are relevant for our purposes. The space $\mathbb{C}P^n$ consists of complex lines in $\mathbb{C}^{n+1}$ intersecting the origin. Its elements are identified with the equivalence classes of points $u=(u^0,\dots,u^n)\in\mathbb{C}^{n+1}\backslash\{0\}$,

$$[u]=\{v=\lambda u|\lambda\in\mathbb{C}^*\},\tag{B.1}$$

such that $\mathbb{C}P^n$ is identified with $(\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^*$. In each class there are elements with unit norm, $\bar{u}\cdot u=\sum\bar{u}^j u^j=1$, and thus there is an equivalent characterization as a coset space of spheres, $\mathbb{C}P^n=S^{n+1}/S^1$. The u's are homogeneous coordinates of $\mathbb{C}P^n$. We define the $n+1$ open sets

$$U_k=\left\{u\in\mathbb{C}^{n+1}|u^k\neq 0\right\}\subset\mathbb{C}^{n+1}\backslash\{0\},\tag{B.2}$$

the classes of which cover the projective space. The $n+1$ maps

$$\varphi_k : \quad \mathbb{C}^n \longrightarrow [U_k], \quad z \mapsto [z^1, \dots, 1, \dots, z^n], \tag{B.3}$$

where the k^{th} coordinate is 1, define a complex analytic structure. The line element on $\mathbb{C}^{n+1}$,

$$\mathrm{d}s^2 = \sum_{j=0}^{n} \mathrm{d}u^j \mathrm{d}\bar{u}^j = \mathrm{d}u \cdot \mathrm{d}\bar{u}, \tag{B.4}$$

can be restricted to S^{2n+1}/S^1 and has the following representation on the k^{th} chart,

$$\mathrm{d}s^2 = \left(\frac{\partial u}{\partial z^\mu} \mathrm{d}z^\mu + \frac{\partial u}{\partial \bar{z}^{\bar{\mu}}} \mathrm{d}\bar{z}^{\bar{\mu}} \right) \cdot \left(\frac{\partial \bar{u}}{\partial z^\mu} \mathrm{d}z^\mu + \frac{\partial \bar{u}}{\partial \bar{z}^{\bar{\mu}}} \mathrm{d}\bar{z}^{\bar{\mu}} \right). \tag{B.5}$$

We shall use the (local) coordinates

$$u = \varphi_0(z) = \frac{1}{\rho}(1, z) \in U_0, \quad \text{where} \quad \rho^2 = 1 + \bar{z} \cdot z = 1 + r^2, \tag{B.6}$$

for representatives with non-vanishing u^0. With these coordinates the line element takes the form

$$\mathrm{d}s^2 = \frac{1}{\rho^2} \mathrm{d}z \cdot \mathrm{d}\bar{z} - \frac{1}{\rho^4} (\bar{z} \cdot \mathrm{d}z)(z \cdot \mathrm{d}\bar{z}), \tag{B.7}$$

and is derived from a Kähler potential $K = \ln \rho^2$. This concludes our summary of $\mathbb{C}P^n$.

To couple fermions to the gravitational background field we must find a complex orthonormal vielbein, $\mathrm{d}s^2 = e^\alpha \delta_{\alpha\bar{\alpha}} e^{\bar{\alpha}}$, and write it as the exponential of a matrix. We obtained the following representation for the vielbein of the Fubini-Study metric (B.7),

$$e^\alpha = e^\alpha_\mu \mathrm{d}z^\mu = \rho^{-1} \left(\mathbb{P}^\alpha{}_\mu + \rho^{-1} \mathbb{Q}^\alpha{}_\mu \right) \mathrm{d}z^\mu, \tag{B.8}$$

$$e_\alpha = e^\mu_\alpha \partial_\mu = \rho \left(\mathbb{P}^\mu{}_\alpha + \rho \mathbb{Q}^\mu{}_\alpha \right) \partial_\mu. \tag{B.9}$$

Here, we have introduced the matrices

$$\mathbb{P} = \mathbb{1} - \frac{\boldsymbol{z}\boldsymbol{z}^\dagger}{r^2} \quad \text{and} \quad \mathbb{Q} = \frac{\boldsymbol{z}\boldsymbol{z}^\dagger}{r^2}, \qquad \boldsymbol{z} = \left(z^1 \dots z^n \right)^t. \tag{B.10}$$

They satisfy

$$\mathbb{P}^2 = \mathbb{P}, \qquad \mathbb{Q}^2 = \mathbb{Q}, \qquad \mathbb{P}\mathbb{Q} = \mathbb{Q}\mathbb{P} = 0, \qquad \mathbb{P}^\dagger = \mathbb{P}, \qquad \mathbb{Q}^\dagger = \mathbb{Q}, \tag{B.11}$$

and hence are orthogonal projection operators. For the particular space $\mathbb{C}P^2$, the vielbeine are known, and can be found in [77]. These known ones are related to those in (B.9) by a local Lorentz transformation. We have not seen explicit formulae for the vielbeine for $n > 2$ in the literature. Expressing the vielbeine in terms of projection operators as in (B.9) allows us to relate the superpotentials in different representations. From (2.62) and (B.9) we obtain the connection (1,0)-form

$$\omega^\alpha_{\mu\beta} = -\frac{\bar{z}_\mu}{\rho^2}\left(\frac{1}{2}\mathbb{P}^\alpha{}_\beta + \mathbb{Q}^\alpha{}_\beta\right) + \frac{1-\rho}{\rho r^2}\mathbb{P}^\alpha{}_\mu \bar{z}_\beta. \tag{B.12}$$

B.2. Zero Modes of the Dirac Operator

In this Subsection we want to determine the zero modes of the Dirac operator $\mathrm{i}\not{D}$ on $\mathbb{C}P^n$. We use the method proposed at the end of Section 2.5. Actually, only for odd values of n a spin bundle S exists on $\mathbb{C}P^n$. We can tensor S with $L^{k/2}$, where L is the generating line bundle, and k takes on even values. In the language of field theory this means that we couple fermions to a $U(1)$ gauge potential A. For even values of n, there is no spin structure, so S does not exist globally. Similarly, for odd values of k, $L^{k/2}$ is not defined globally. There is, however, the possibility to define a generalized spin bundle S_c which is the formal tensor product of S and $L^{k/2}$, k odd [76]. Again, in the language of field theory, we couple fermions to a suitably chosen $U(1)$ gauge potential with half-integer instanton number. In both cases, the gauge potential reads

$$A = \frac{k}{2}\bar{u}\cdot \mathrm{d}u = \frac{k}{4}(\partial - \bar{\partial})K = g_A \partial g_A^{-1} + g_A^{\dagger -1}\bar{\partial} g_A^\dagger, \tag{B.13}$$

$$g_A = \mathrm{e}^{-kK/4} = (1+r^2)^{-\frac{k}{4}}, \tag{B.14}$$

with corresponding field strength

$$F = \mathrm{d}A = (\partial + \bar{\partial})A = \frac{k}{2}\bar{\partial}\partial K. \tag{B.15}$$

g_A is the part of the superpontential $g \equiv g_\omega g_A$ that gives rise to the gauge connection A. It remains to determine the spin connection part g_ω.

Using (B.9) and (B.10), equation (2.62) can be written in matrix notation as $(\omega_\mu)^\alpha{}_\beta = (S\partial_\mu S^{-1})^\alpha{}_\beta$, where

$$S = \rho(\mathbb{P} + \rho\mathbb{Q}) \stackrel{(B.11)}{=} \exp\left(\mathbb{P}\ln\rho + \mathbb{Q}\ln\rho^2\right) = \exp\Big((\mathbb{1} + \mathbb{Q})\ln\rho\Big). \tag{B.16}$$

From the matrix form of S in (B.16) we read of the superpotential g_ω in *spinor representation*,

$$g_\omega = \exp\left(\frac{1}{4}(\delta_{\bar\alpha\beta} + \mathbb{Q}_{\bar\alpha\beta})\gamma^{\bar\alpha\beta}\ln\rho\right), \tag{B.17}$$

where we have introduced

$$\gamma^{\bar\alpha\beta} \equiv \frac{1}{2}[\gamma^{\bar\alpha}, \gamma^\beta] = 2[\psi^{\dagger\bar\alpha}, \psi^\beta], \qquad \gamma^{\bar\alpha} = 2\psi^{\dagger\bar\alpha}, \qquad \gamma^\beta = 2\psi^\beta. \tag{B.18}$$

Next, we study zero modes of Q and $Q^\dagger$ in the gauge field background (B.13). In the sector of interest with $N = n$, the superpotential g_ω in spinor representation simplifies as

$$g_\omega\big|_{N=n} = (1 + r^2)^{\frac{n+1}{4}}, \quad \text{since} \quad \gamma^{\bar\alpha\beta}\big|_{N=n} = 2\delta^{\bar\alpha\beta}. \tag{B.19}$$

All states in the $N = n$ sector are annihilated by $Q^\dagger$. Zero modes Ψ satisfy in addition

$$0 = Q\Psi = 2\mathrm{i}\psi^\mu D_\mu\Psi = 2\mathrm{i}\psi^\mu g\partial_\mu g^{-1}\Psi, \qquad g = g_A g_\omega. \tag{B.20}$$

Using (B.13) and (B.19) we conclude that

$$\Psi = g\bar f(\bar z)\psi^{\dagger\bar 1}\cdots\psi^{\dagger\bar n}\left|0\right\rangle = (1 + r^2)^{\frac{n+1-k}{4}}\bar f(\bar z)\psi^{\dagger\bar 1}\cdots\psi^{\dagger\bar n}\left|0\right\rangle, \tag{B.21}$$

with some antiholomorphic function $\bar f$. Normalizability of Ψ will put restrictions on the admissible functions $\bar f$. Since the operators $\bar z^{\bar\mu}\partial_{\bar\mu}$ (no sum) commute with ∂_μ and with each other, we can diagonalize them simultaneously on the kernel of ∂_μ. Thus, we are let to the following most general ansatz

$$\bar{f}_m = (\bar{z}^{\bar{1}})^{m_1} \cdots (\bar{z}^{\bar{n}})^{m_n}, \qquad \sum_{i=1}^{n} m_i = m. \tag{B.22}$$

There are $\binom{m+n-1}{n-1}$ independent functions of this form. The solution Ψ in (B.21) is square-integrable if and only if

$$\begin{aligned} \|\Psi\|^2 &= \int \mathrm{dvol}\ (\det h)\ \Psi^\dagger \Psi \\ &\overset{(B.21)}{\propto} \int \mathrm{d}\Omega \int \mathrm{d}r\ r^{2m+2n-1}(1+r^2)^{-\frac{n+k+1}{2}} < \infty, \end{aligned} \tag{B.23}$$

so normalizable zero modes in the $N = n$ sector exist for

$$m = 0, 1, 2, \ldots, q \equiv \frac{1}{2}(k - n - 1). \tag{B.24}$$

Note, that q is always integer-valued, since k is odd (even) if n is even (odd). Also note, that there are no zero modes in this sector for $k < n + 1$ or equivalently $q < 0$. In particular, for odd n and vanishing gauge potential there are no zero modes, in agreement with [78].

For $q \geq 0$, the total number of zero modes in the $N = n$ sector is

$$\sum_{m=0}^{q} \binom{m+n-1}{n-1} = \frac{1}{n!}(q+1)(q+2)\ldots(q+n). \tag{B.25}$$

Similar considerations show that there are no normalizable zero modes in the $N = 0$ sector for $q' < 0$, where $q' = -\frac{1}{2}(k + n + 1)$. For $q' \geq 0$ there are zero modes in the $N = 0$ sector, and their number is given by (B.25) with q replaced by q'.

Observe, that the states in the $N = 0$ sector are of the same (opposite) chirality as the states in the $N = n$ sector for even (odd) n. The contribution of the zero modes in those sectors to the index of $\mathrm{i}\not{D}$ is given by

$$\frac{1}{n!}(q+1)(q+2)\ldots(q+n), \qquad q = \frac{1}{2}(k - n - 1), \tag{B.26}$$

for all $q \in \mathbb{Z}$.

On the other hand, the index theorem on $\mathbb{C}P^n$ reads [79]

$$\mathrm{ind}\ \mathrm{i}\not{D} = \int_{\mathbb{C}P^n} \mathrm{ch}(L^{-k/2})\hat{A}(\mathbb{C}P^n) = \frac{1}{n!}(q+1)(q+2)\ldots(q+n), \tag{B.27}$$

where ch and $\hat{A}$ are the Chern character and the $\hat{A}$-genus, respectively. Note, that this index coincides with (B.26). This leads us to conjecture, that for positive (negative) k *all normalizable zero modes* of the Dirac operator on the complex projective spaces $\mathbb{C}P^n$ with Abelian gauge potential (B.13) reside in the sector with $N = n$ ($N = 0$) and have the form (B.21).

We can prove this conjecture in the particular cases $n = 1$ and $n = 2$. For $\mathbb{C}P^1$ we have constructed all zero modes. The same holds true for $\mathbb{C}P^2$ for the following reason: Let us assume that there are zero modes in the $N = 1$ sector. According to (2.46) they have opposite chirality as compared to the states in the $N = 0$ and $N = 2$ sectors. Hence, the index would be less than the number of zero modes in the extreme sectors. On the other hand, according to the index theorem, the index (B.27) is equal to this number. We conclude that there can be no zero modes in the $N = 1$ sector.

Finally, note the possibility to carry out a similar construction for other interesting manifolds, like Taub-NUT space, where the explict zero modes can be compared with the corresponding index theorem [77].

C. Conventions: Field Theory

The conventions we use in the field theory part of the thesis are summarized in this Appendix. In particular, we present the conventions for the chiral representation of γ-matrices in four dimensions, our gauge theory conventions and some rules for spinor-calculations in three and four dimensions.

With these conventions we closely follow those of Sohnius [16].

We use four dimensional Minkowski space $\mathbb{M}^{1,3}$ with metric $\eta_{\mu\nu} = (+,-,-,-)$. Greek indices $\mu, \nu, \ldots$ run from 0 to 3, latin indices $m, n, \ldots$ run from 1 to 3.

C.1. Clifford Algebra

The Clifford algebra in four dimensions is generated by γ^μ,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{C.1}$$

$\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with all γ_μ. We define $\Sigma^{\mu\nu} = \frac{1}{4\mathrm{i}}[\gamma^\mu, \gamma^\nu]$, such that

$$\gamma^\rho \Sigma^{\mu\nu} = -\frac{\mathrm{i}}{2}\eta^{\mu\rho}\gamma^\nu + \frac{\mathrm{i}}{2}\eta^{\nu\rho}\gamma^\mu - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5, \tag{C.2}$$

$$\Sigma^{\mu\nu}\gamma^\rho = +\frac{\mathrm{i}}{2}\eta^{\mu\rho}\gamma^\nu - \frac{\mathrm{i}}{2}\eta^{\nu\rho}\gamma^\mu - \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5, \tag{C.3}$$

holds. The intertwiners which relate different representations of the Clifford algebra satisfy

$$\mathcal{A}\gamma^\mu\mathcal{A}^{-1} = \gamma^{\mu\dagger}, \qquad \mathcal{C}^{-1}\gamma^\mu\mathcal{C} = -\gamma^{\mu t}, \qquad \mathcal{B} \equiv \mathcal{C}\mathcal{A}^t, \tag{C.4}$$

and the Dirac-conjugated and the charge-conjugated spinors are given by

$$\bar{\psi} = \psi^\dagger \mathcal{A}, \qquad \psi_{\mathcal{C}} = \mathcal{C}\bar{\psi}^t = \mathcal{C}\mathcal{A}^t\psi^* = \mathcal{B}\psi^*. \tag{C.5}$$

Here, $*$ denotes complex conjugation.

C.2. Chiral Representation

The chiral representation for γ matrices is very convenient for our dimensional reduction. It is given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{C.6}$$

where

$$\sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma}), \qquad \tilde{\sigma}^\mu = (\mathbb{1}, -\boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3). \tag{C.7}$$

$\boldsymbol{\sigma}$ is the three-vector of Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{C.8}$$

that satisfy

$$\sigma^m \sigma^n = \delta^{mn} + \mathrm{i}\epsilon^{mnk}\sigma^k. \tag{C.9}$$

The intertwiners are given by

$$\mathcal{A} = \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} \mathrm{i}\sigma^2 & 0 \\ 0 & -\mathrm{i}\sigma^2 \end{pmatrix}, \qquad \mathcal{B} = \mathcal{C}\mathcal{A}^t = \begin{pmatrix} 0 & \mathrm{i}\sigma^2 \\ -\mathrm{i}\sigma^2 & 0 \end{pmatrix}. \tag{C.10}$$

To check the properties of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, we need

$$\sigma^2\sigma^m\sigma^2 = -(\sigma^m)^t, \quad \sigma^2\sigma^{m*}\sigma^2 = -\sigma^m, \quad \sigma^2\sigma^\mu\sigma^2 = (\tilde{\sigma}^\mu)^t, \quad \sigma^2\tilde{\sigma}^\mu\sigma^2 = (\sigma^\mu)^t. \tag{C.11}$$

Furthermore,

$$\Sigma^{\mu\nu} = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \tilde{\sigma}^{\mu\nu} \end{pmatrix}, \quad \sigma^{\mu\nu} = \frac{1}{4\mathrm{i}}(\sigma^{\mu}\tilde{\sigma}^{\nu} - \sigma^{\nu}\tilde{\sigma}^{\mu}), \qquad \tilde{\sigma}^{\mu\nu} = \frac{1}{4\mathrm{i}}(\tilde{\sigma}^{\mu}\sigma^{\nu} - \tilde{\sigma}^{\nu}\sigma^{\mu}). \tag{C.12}$$

A four-component spinor ψ decomposes into its positive and negative chirality parts $\psi_{\pm}$,

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}. \tag{C.13}$$

We define a symplectic Majorana condition by $(i, j = 1, 2)$

$$\lambda^i = \mathrm{i}\epsilon^{ij}\gamma_5\mathcal{C}\lambda_j^* = \epsilon^{ij}S\lambda_j^*, \qquad S = -\sigma^1 \otimes \sigma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}. \tag{C.14}$$

Here, symplectic refers to the metric ϵ^{ij}, which can be used to raise and lower indices. Some useful identities involving this matrix S:

$$S\gamma^{\mu *}S = \gamma^{\mu}, \quad S\gamma^0 S = \gamma^0, \quad S\gamma^0\gamma^{\mu}\ldots\gamma^{\rho}S = \gamma^{\mu t}\ldots\gamma^{\rho t}\gamma^0, \quad S\gamma_5 S = -\gamma_5. \tag{C.15}$$

Some identities for symplectic Majorana spinors:

$$\begin{aligned} \bar{\zeta}_i\chi^j &= -\bar{\chi}_m\zeta^l\epsilon_{il}\epsilon^{jm}, & \bar{\zeta}_i\gamma^{\mu}\chi^j &= -\bar{\chi}_m\gamma^{\mu}\zeta^l\epsilon_{il}\epsilon^{jm}, \\ \bar{\zeta}_i\gamma_5\chi^j &= -\bar{\chi}_m\gamma_5\zeta^l\epsilon_{il}\epsilon^{jm}, & \bar{\zeta}_i\gamma^{\mu}\gamma_5\chi^j &= +\bar{\chi}_m\gamma^{\mu}\gamma_5\zeta^l\epsilon_{il}\epsilon^{jm}, \\ \bar{\zeta}_i\gamma^{\mu}\ldots\gamma^{\rho}\chi^j &= -\bar{\chi}_m\gamma^{\rho}\ldots\gamma^{\mu}\zeta^l\epsilon_{il}\epsilon^{jm}, & \bar{\zeta}_i\Sigma^{\mu\nu}\chi^j &= \bar{\chi}_m\Sigma^{\mu\nu}\zeta^l\epsilon_{il}\epsilon^{jm}. \end{aligned} \tag{C.16}$$

In particular $(i = j)$, they imply

$$\begin{gathered} \bar{\zeta}_i\chi^i = -\bar{\chi}_i\zeta^i, \quad \bar{\zeta}_i\gamma^{\mu}\chi^i = -\bar{\chi}_i\gamma^{\mu}\zeta^i, \quad \bar{\zeta}_i\gamma_5\chi^i = -\bar{\chi}_i\gamma_5\zeta^i, \\ \bar{\zeta}_i\gamma^{\mu}\gamma_5\chi^i = \bar{\chi}_i\gamma^{\mu}\gamma_5\zeta^i, \quad \bar{\zeta}_i\gamma^{\mu}\gamma^{\nu}\chi^i = -\bar{\chi}_i\gamma^{\nu}\gamma^{\mu}\zeta^i, \quad \bar{\zeta}_i\Sigma^{\mu\nu}\chi^i = \bar{\chi}_i\Sigma^{\mu\nu}\zeta^i, \\ \bar{\zeta}_i\Sigma^{\mu\nu}\gamma^{\rho}\chi^i = \bar{\chi}_i\gamma^{\rho}\Sigma^{\mu\nu}\zeta^i, \quad \bar{\zeta}_i\gamma^{\mu}\gamma^{\nu}\gamma_5\chi^i = -\bar{\chi}_i\gamma^{\nu}\gamma^{\mu}\gamma_5\zeta^i. \end{gathered} \tag{C.17}$$

Furthermore (for any Pauli matrix τ^a and the vector $\boldsymbol{\tau}$ of Pauli matrices),

$$
\begin{aligned}
0 &= \left(\epsilon_{ik}\epsilon^{jm} + \delta_i^m \delta_k^j\right)(\tau^a)_j{}^i, \\
0 &= \delta_p^q \delta_n^m - \delta_p^m \delta_n^q - \epsilon_{pn}\epsilon^{qm}, \\
0 &= \frac{1}{2}\delta_i^l \delta_j^k - \delta_i^k \delta_j^l + \frac{1}{2}\boldsymbol{\tau}_i{}^l \boldsymbol{\tau}_j{}^k.
\end{aligned}
\tag{C.18}
$$

The Fierz identity we need for spinors in four dimensions reads

$$
4\zeta\bar{\chi} = -\bar{\chi}\zeta - \gamma_\rho(\bar{\chi}\gamma^\rho\zeta) - 2\Sigma_{\rho\sigma}(\bar{\chi}\Sigma^{\rho\sigma}\zeta) + \gamma_5\gamma_\rho(\bar{\chi}\gamma_5\gamma^\rho\zeta) - \gamma_5(\bar{\chi}\gamma_5\zeta).
\tag{C.19}
$$

C.3. Gauge Theory

In the Abelian case we use the conventions

$$
\begin{aligned}
D_\mu &= \partial_\mu + \mathrm{i}A_\mu, \\
F_{\mu\nu} &= -\mathrm{i}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu, \\
(\mathrm{i}\not{D})^2 &= -D_\mu D^\mu + (F,\Sigma), \qquad (F,\Sigma) = F_{\mu\nu}\Sigma^{\mu\nu}.
\end{aligned}
\tag{C.20}
$$

Under a gauge transformation

$$
A_\mu \quad \to \quad A'_\mu = \mathrm{e}^{-\mathrm{i}\Lambda}(A_\mu - \mathrm{i}\partial_\mu)\mathrm{e}^{\mathrm{i}\Lambda} = A_\mu + \partial_\mu\Lambda,
\tag{C.21}
$$

charged fields transform as

$$
\varphi \quad \to \quad \varphi' = \mathrm{e}^{\mathrm{i}\Lambda}\varphi,
\tag{C.22}
$$

such that

$$
\delta_{\text{gauge}} A_\mu = \partial_\mu\Lambda, \qquad \delta_{\text{gauge}}\varphi = \mathrm{i}\Lambda\varphi.
\tag{C.23}
$$

The magnetic field is given by

$$
B_k = \frac{1}{2}\epsilon_{kmn}F^{mn}, \qquad F_{mn} = \epsilon_{mnk}B^k, \qquad (F,\sigma) = F_{mn}\sigma^{mn} = -\boldsymbol{B}\cdot\boldsymbol{\sigma}.
\tag{C.24}
$$

In the non-Abelian theory we consider all fields φ in the adjoint representation,

$$\begin{aligned} D_\mu\varphi &= \partial_\mu\varphi + \mathrm{i}[A_\mu, \varphi], \\ F_{\mu\nu} &= -\mathrm{i}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}[A_\mu, A_\nu], \\ (\mathrm{i}\not{D})^2\varphi &= -D_\mu D^\mu\varphi + [(F, \Sigma), \varphi]. \end{aligned} \tag{C.25}$$

We use Hermitian generators T^A of the gauge group and real structure constants f^{ABC},

$$[T^A, T^B] = \mathrm{i}f^{ABC}T^C, \qquad \mathrm{tr}\, T^A T^B = \frac{1}{2}\delta^{AB}. \tag{C.26}$$

C.4. Three Dimensional Theory

We dimensionally reduce our theories from $\mathbb{M}^{1,3}$ to three dimensional Euclidean space $\mathbb{M}^{3,0}$. Observe, that after the dimensional reduction in time direction is carried out, we change the metric from $-\delta_{mn}$ to the more common $+\delta_{mn}$. Now we need no longer distinguish between upper and lower vector and tensor indices.

The Dirac conjugate reduces to

$$\bar{\psi}_\pm = \psi_\pm^\dagger \mathbb{1}_2 = \psi_\pm^\dagger. \tag{C.27}$$

We write a pair of symplectic Majorana spinors $\lambda^{1,2}$ and a Dirac spinor ψ as

$$\lambda^1 = \begin{pmatrix} \lambda_- \\ \lambda_+ \end{pmatrix}, \qquad \lambda^2 = \begin{pmatrix} \sigma^2\lambda_+^* \\ \sigma^2\lambda_-^* \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}, \tag{C.28}$$

in particular,

$$\frac{\mathrm{i}}{2}\bar{\lambda}_i\not{\partial}\lambda^i = \mathrm{i}\bar{\lambda}_+\not{\partial}\lambda_+ - \mathrm{i}\bar{\lambda}_-\not{\partial}\lambda_-. \tag{C.29}$$

For the intertwiners in three dimensions we choose

$$\mathcal{A} = \mathbb{1}, \qquad \mathcal{C} = \sigma^2, \qquad \mathcal{B} = \mathcal{C}\mathcal{A}^t = \sigma^2, \tag{C.30}$$

such that

$$\mathcal{A}\sigma^m\mathcal{A}^{-1} = \sigma^m = \sigma^{m\dagger}, \qquad \mathcal{C}^{-1}\sigma^m\mathcal{C} = -(\sigma^m)^t. \tag{C.31}$$

Charge conjugated spinors are defined as

$$\lambda_{\pm\mathcal{C}} = \mathcal{B}\lambda_\pm^* = \sigma^2\lambda_\pm^*, \qquad \overline{\lambda}_{\pm\mathcal{C}} = \lambda_\pm^t\sigma^2. \tag{C.32}$$

The charge conjugation matrix satisfies

$$\mathcal{B}\mathcal{B}^* = -\mathbb{1}, \qquad \text{so} \qquad (\psi_\mathcal{C})_\mathcal{C} = \mathcal{B}\psi_\mathcal{C}^* = \mathcal{B}\mathcal{B}^*\psi^{**} = -\psi. \tag{C.33}$$

D. APS Example in $d = 4$ Dimensions

In this Appendix we consider zero modes of the Dirac operator $\mathrm{i}\not{D}$ in a particular gauge field background A_μ on the four dimensional ball with radius R, $\mathcal{M} = B^4$. The boundary of $\mathcal{M}$ is a three-sphere, $\partial\mathcal{M} = S^3$.

Since there is a boundary, we cannot apply the Atiyah-Singer index theorem as mentioned in the main part of the thesis. Instead, we have to impose boundary conditions for the eigenfunctions of the Dirac operator. Bag boundary conditions [118] are local boundary conditions, which do not allow for any zero modes. We prefer the non-local APS boundary conditions which admit zero modes. In that case, the difference of the number of left- and right-handed zero modes can be counted using the Atiyah-Patodi-Singer (APS) index theorem for manifolds with boundary.

We apply the APS index theorem to the very particular case at hand. For a detailed introduction we refer to one of various books on index theorems [119], to the very clear review article [77], and, of course, to the original mathematical papers [60].

The calculations that we present in this Appendix have been carried out in [88] for the case of constant selfdual gauge fields. We generalize these results to arbitrary profile functions and we calculate the zero modes explicitly. The number of zero modes we find agrees – as it should – with the expression for the index. In fact, only zero modes of one chirality are present, so the absolute value of the index is equal to the number of zero modes.

We finally show, how our results can be applied to background configurations, that result from Abelian projected instanton fields.

D.1. The APS Index Theorem

In four Euclidean dimensions we use the chiral representation,

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}, \quad \sigma_\mu = (\boldsymbol{\sigma}, \mathrm{i}\mathbb{1}), \quad \tilde{\sigma}_\mu = (\boldsymbol{\sigma}, -\mathrm{i}\mathbb{1}), \tag{D.1}$$

such that

$$\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{D.2}$$

The Euclidean Dirac operator reads

$$\mathrm{i}\not{D} = \mathrm{i}\gamma_\mu(\partial_\mu + \mathrm{i}A_\mu) = \begin{pmatrix} 0 & L^\dagger \\ L & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} D_4 + \mathrm{i}D_3 & D_2 + \mathrm{i}D_1 \\ -D_2 + \mathrm{i}D_1 & D_4 - \mathrm{i}D_3 \end{pmatrix}. \tag{D.3}$$

Due to the spherical symmetry of $\mathcal{M}$, it is appropriate to use polar coordinates,

$$x_1 = r\sin\theta\sin\phi_1, \qquad x_2 = r\sin\theta\cos\phi_1, \tag{D.4}$$

$$x_3 = r\cos\theta\sin\phi_2, \qquad x_4 = r\cos\theta\cos\phi_2, \tag{D.5}$$

where

$$0 \le r \le R, \qquad 0 \le \theta \le \frac{\pi}{2}, \qquad 0 \le \phi_1, \phi_2 < 2\pi. \tag{D.6}$$

The covariant derivatives can be written as

$$D_2 \pm \mathrm{i}D_1 = \mathrm{e}^{\pm\mathrm{i}\phi_1}\left(\sin\theta D_r + \frac{\cos\theta}{r}D_\theta \pm \frac{\mathrm{i}}{r\sin\theta}D_{\phi_1}\right), \tag{D.7}$$

$$D_4 \pm \mathrm{i}D_3 = \mathrm{e}^{\pm\mathrm{i}\phi_2}\left(\cos\theta D_r - \frac{\sin\theta}{r}D_\theta \pm \frac{\mathrm{i}}{r\cos\theta}D_{\phi_2}\right). \tag{D.8}$$

Applicability of the APS index theorem requires L to be of the following form,

$$\tilde{L} = \mathbb{1}\partial_r + B, \qquad \tilde{L}^\dagger = -\mathbb{1}\partial_r + B. \tag{D.9}$$

This form of L must hold at least locally, in a collar neighborhood of the boundary. L in (D.3) is not of this form, so the APS formula does not hold yet. As has been pointed out in [86, 120], a phase transformation is needed to transform L into (D.9). This can be accomplished by the following change of basis, together with the gauge choice $A_r = 0$ for the vector potential,

$$\psi \to \tilde{\psi} = \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \psi, \qquad L \to \tilde{L} = MLN^{-1}, \qquad L^\dagger \to \tilde{L}^\dagger = NL^\dagger M^{-1}, \tag{D.10}$$

where

$$\begin{aligned} N &= \sqrt{r^3 \sin\theta \cos\theta} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi_-} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi_-} \end{pmatrix}, \\ M &= \sqrt{r^3 \sin\theta \cos\theta} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi_+} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi_+} \end{pmatrix}, \\ \phi_\pm &= \frac{1}{2}(\phi_1 \pm \phi_2). \end{aligned} \tag{D.11}$$

The result is called the APS standard form of $\mathrm{i}\not{D}$ [77]. In the new basis, $\tilde{L}$ is given by (D.9), with operator

$$B(r) = \frac{1}{r} \begin{pmatrix} \frac{1}{2} + \mathrm{i}D_{\phi_1} + \mathrm{i}D_{\phi_2} & D_\theta + \mathrm{i}\cot\theta D_{\phi_1} - \mathrm{i}\tan\theta D_{\phi_2} \\ -D_\theta + \mathrm{i}\cot\theta D_{\phi_1} - \mathrm{i}\tan\theta D_{\phi_2} & \frac{1}{2} - \mathrm{i}D_{\phi_1} - \mathrm{i}D_{\phi_2} \end{pmatrix}. \tag{D.12}$$

Note, that $\tilde{L}$ is of the form (D.9) not only close to the boundary $r \sim R$, but for all values of r. Later, this fact will be used for the construction of zero modes.

The APS index theorem reads[1]

$$\mathrm{ind}\, \mathrm{i}\not{D} = \dim\ker L^\dagger - \dim\ker L = \frac{1}{8\pi^2} \int_{\mathcal{M}} F \wedge F + \frac{1}{2}(h - \eta(0)). \tag{D.13}$$

For manifolds with boundary, there is a correction term to $\int F \wedge F$, given by the above combination of

$$h \equiv \dim\ker B, \qquad \eta(s) \equiv \sum_{\lambda \neq 0} \frac{\mathrm{sgn}\, \lambda}{|\lambda|^s}. \tag{D.14}$$

Here, B is the boundary operator, $B = B(R)$, whose eigenvalues are denoted by λ, and

[1] Note, that our convention for the normal derivative pointing *outward*, exchanges the rôle of L and $L^\dagger$, compared to the *inward* normal convention used in the mathematical literature.

$\eta(s)$ is the η-function associated with B. $\eta(0)$ is called the *spectral asymmetry*.

In [88] the selfdual Abelian field

$$A_\mu = \frac{1}{2} M \eta^3_{\mu\nu} x_\nu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -\eta^3_{\mu\nu} M, \tag{D.15}$$

has been analyzed. Here, M is a constant and $\eta^a_{\mu\nu}$ are the selfdual 't Hooft symbols [121],

$$\eta_{a\mu\nu} = \epsilon_{a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{\mu 4}, \qquad \eta_{a\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\eta_{a\rho\sigma}. \tag{D.16}$$

In polar coordinates, this field configuration is given by

$$A_r = 0, \qquad A_\theta = 0, \qquad A_{\phi_1} = \frac{1}{2} M r^2 \sin^2\theta, \qquad A_{\phi_2} = \frac{1}{2} M r^2 \cos^2\theta, \tag{D.17}$$

and the operator $B(r)$ in (D.12) reads

$$B(r) = \frac{1}{r}\begin{pmatrix} \frac{1}{2} + \mathrm{i}\partial_{\phi_1} + \mathrm{i}\partial_{\phi_2} - \mu(r) & \partial_\theta + \mathrm{i}\cot\theta\partial_{\phi_1} - \mathrm{i}\tan\theta\partial_{\phi_2} \\ -\partial_\theta + \mathrm{i}\cot\theta\partial_{\phi_1} - \mathrm{i}\tan\theta\partial_{\phi_2} & \frac{1}{2} - \mathrm{i}\partial_{\phi_1} - \mathrm{i}\partial_{\phi_2} + \mu(r) \end{pmatrix}, \tag{D.18}$$

where we have introduced

$$\mu(r) = \frac{1}{2} M r^2. \tag{D.19}$$

The eigenvalue problem for the boundary operator,

$$B(R)\omega_\lambda(\Omega) = \lambda\omega_\lambda(\Omega), \tag{D.20}$$

where Ω indicates the dependence on θ, ϕ_1 and ϕ_2, has been solved in [88]. Details of the calculation can be found there. Here we just state the main results: there are three different series, given by

$$\begin{aligned} \mathrm{I}: \quad & \lambda R = \frac{1}{2} + p - \mu(R), \quad p = 1, 2, \ldots, \\ \mathrm{II}: \quad & \lambda R = \frac{1}{2} + p + \mu(R), \quad p = 1, 2, \ldots, \\ \mathrm{III}: \quad & \lambda R = \frac{1}{2} \pm \sqrt{(p + \mu(R))^2 - 4\mu(R)N}, \quad p = 2, 3, \ldots, \quad 1 \leq N \leq p - 1. \end{aligned} \tag{D.21}$$

In all cases, the degeneracy of the eigenvalue λ is given by $\deg(\lambda) = p$.

Accordingly, h and $\eta(s)$ in (D.14) get contributions from all three series,

$$h = h_{\mathrm{I}} + h_{\mathrm{II}} + h_{\mathrm{III}}, \qquad \eta(s) = \eta_{\mathrm{I}}(s) + \eta_{\mathrm{II}}(s) + \eta_{\mathrm{III}}(s), \tag{D.22}$$

where

$$\begin{aligned} \eta_{\mathrm{I}}(s) &= \sum_{p=1}^{\infty} p \frac{\operatorname{sgn}\,(p + \frac{1}{2} - \mu(R))}{|p + \frac{1}{2} - \mu(R)|^s}, \qquad p \neq \mu(R) - \frac{1}{2}, \\ \eta_{\mathrm{II}}(s) &= \sum_{p=1}^{\infty} p \frac{\operatorname{sgn}\,(p + \frac{1}{2} + \mu(R))}{|p + \frac{1}{2} + \mu(R)|^s}, \qquad p \neq -\mu(R) - \frac{1}{2}, \\ \eta_{\mathrm{III}}(s) &= \sum_{p=2}^{\infty} p \sum_{N=1}^{p-1} \left(|\alpha + \frac{1}{2}|^{-s} - |\alpha - \frac{1}{2}|^{-s} \right), \qquad \alpha \equiv \sqrt{(p + \mu(R))^2 - 4\mu(R)N}. \end{aligned} \tag{D.23}$$

Regularization of these sums via the Hurwitz ζ-function is understood. Whether or not any of the series gives a contribution to h, depends on the value of $|\mu(R)| + \frac{1}{2}$. Therefore we decompose this expression into its integer and fractional part,

$$|\mu(R)| + \frac{1}{2} = l + \rho, \qquad l = 0, 1, 2, \ldots, \qquad 0 \leq \rho < 1. \tag{D.24}$$

Suppose $\rho \neq 0$ first, so $|\mu(R)| + \frac{1}{2}$ is not integer, and

$$h_{\mathrm{I}}^{\rho\neq 0} + h_{\mathrm{II}}^{\rho \neq 0} = 0. \tag{D.25}$$

Furthermore, for $\rho \neq 0$, the conditions in (D.23) are empty, and we find for the contributions to the η-function

$$\begin{aligned} &\eta_{\mathrm{I}}^{\rho\neq 0}(0) + \eta_{\mathrm{II}}^{\rho\neq 0}(0) \\ &= \lim_{s\to 0} \left(\sum_{p=1}^{\infty} \frac{p}{|p + l + \rho|^s} - \sum_{p=1}^{l-1} \frac{p}{|p + 1 - l - \rho|^s} + \sum_{p=l}^{\infty} \frac{p}{|p + 1 - l - \rho|^s} \right). \end{aligned} \tag{D.26}$$

This can be rewritten as

$$\begin{aligned}
&\eta_{\mathrm{I}}^{\rho\neq 0}(0) + \eta_{\mathrm{II}}^{\rho\neq 0}(0)\\
&= -\frac{1}{2}l(l-1) + \lim_{s\to 0}\Bigg(\sum_{p=1}^{\infty}\frac{1}{|p+l+\rho|^{s-1}} - \\
&\quad -\sum_{p=1}^{\infty}\frac{l+\rho}{|p+l+\rho|^{s}} + \sum_{p=l}^{\infty}\frac{1}{|p+1-l-\rho|^{s-1}} - \sum_{p=l}^{\infty}\frac{1-l-\rho}{|p+1-l-\rho|^{s}}\Bigg)\\
&= -\frac{1}{2}l(l-1) + \lim_{s\to 0}\Bigg(\sum_{q=0}^{\infty}\frac{1}{|q+1+l+\rho|^{s-1}} - \\
&\quad -\sum_{q=0}^{\infty}\frac{l+\rho}{|q+1+l+\rho|^{s}} + \sum_{q=0}^{\infty}\frac{1}{|q+1-\rho|^{s-1}} - \sum_{q=0}^{\infty}\frac{1-l-\rho}{|q+1-\rho|^{s}}\Bigg)\\
&= -\frac{1}{2}l(l-1) + \zeta(-1,1+l+\rho) - (l+\rho)\zeta(0,1+l+\rho) + \\
&\quad +\zeta(-1,1-\rho) - (1-l-\rho)\zeta(0,1-\rho)\\
&= \rho^2 + 2l\rho - \rho + \frac{1}{3}\\
&= -l(l-1) + \mu^2 + \frac{1}{12},
\end{aligned} \tag{D.27}$$

where we used the properties

$$\zeta(0,x) = \frac{1}{2} - x, \qquad \zeta(-1,x) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{1}{12}, \tag{D.28}$$

of the Hurwitz ζ-function. If $\rho = 0$, so that $|\mu(R)| + \frac{1}{2} = l = 1, 2, \ldots$, vanishing eigenvalues are present for $p = l-1$, and their multiplicity is $l-1$, thus

$$h_{\mathrm{I}}^{\rho=0} + h_{\mathrm{II}}^{\rho=0} = l - 1. \tag{D.29}$$

A similar calculation as (D.27) gives

$$\eta_{\mathrm{I}}^{\rho=0} + \eta_{\mathrm{II}}^{\rho=0} = l - \frac{2}{3}. \tag{D.30}$$

Independent of $\rho = 0$ or $\rho \neq 0$, the series I and II give the contribution

$$h_{\mathrm{I}} + h_{\mathrm{II}} - \eta_{\mathrm{I}}(0) - \eta_{\mathrm{II}}(0) = l(l-1) - \mu(R)^2 - \frac{1}{12} \tag{D.31}$$

to the boundary term in the APS index formula (D.13). Finally, one easily shows that $\lambda = 0$ cannot be obtained in series III, so $h_{\mathrm{III}} = 0$. The η-invariant in this case is [88]

$$\eta_{\mathrm{III}} = -\frac{1}{12}. \tag{D.32}$$

Thus, the total contribution to the index is given by

$$h - \eta(0) = l(l-1) - \mu(R)^2. \tag{D.33}$$

The bulk term gives

$$\frac{1}{8\pi^2}\int_{\mathcal{M}} F \wedge F = \frac{2M^2}{8\pi^2}\int_{\mathcal{M}} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4 = \frac{1}{2}\mu(R)^2, \tag{D.34}$$

and the index reduces to

$$\operatorname{ind} \mathrm{i}\not{D} = \frac{1}{2}l(l-1). \tag{D.35}$$

Here, l is the integer part of $|\mu(R)| + \frac{1}{2}$.

D.2. Zero Modes of i$\not{D}$

In the next step, we will calculate the corresponding zero modes of i$\not{D}$ explicitly. After the transformation (D.10), the zero modes $\tilde{\psi} = (\tilde{\varphi} \;\; \tilde{\chi})^t$, satisfy

$$\mathrm{i}\tilde{\not{D}}\tilde{\psi} = \begin{pmatrix} 0 & \tilde{L}^\dagger \\ \tilde{L} & 0 \end{pmatrix}\begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix} = 0. \tag{D.36}$$

Let us formulate the APS boundary conditions at $r = R$ for this problem. These boundary conditions guarantee, that $\tilde{L}^\dagger$ is the adjoint of $\tilde{L}$, so i$\not{D}$ is a self-adjoint operator. For more details we refer to [77, 85, 86]. Close to the boundary we expand our functions in terms of the eigenfunctions ω_λ of the boundary operator B as

$$\tilde{\varphi} = \sum_\lambda f_\lambda(r)\omega_\lambda(\Omega), \tag{D.37}$$

$$\tilde{\chi} = \sum_\lambda g_\lambda(r)\omega_\lambda(\Omega). \tag{D.38}$$

APS boundary conditions require, that the sum in (D.37) contains only contributions for $\lambda > 0$, and the sum in (D.38) contains only $\lambda \leq 0$ contributions[2]. In this case, all zero modes satisfy

$$\left.\frac{f'_\lambda}{f_\lambda}\right|_{r=R} = -\lambda, \quad \text{for} \quad \lambda > 0, \tag{D.39}$$

$$\left.\frac{g'_\lambda}{g_\lambda}\right|_{r=R} = +\lambda, \quad \text{for} \quad \lambda \leq 0, \tag{D.40}$$

and this mimics the square-integrability property for zero modes in the limit $R \to \infty$, the profile functions have negative slope at the boundary.

Zero modes $\tilde{\psi}$ are normalizable, if they satisfy

$$\int_0^R \mathrm{d}r\mathrm{d}\tilde{\Omega}\,\|\tilde{\psi}\|^2 = \int_0^R r^3\mathrm{d}r\mathrm{d}\Omega\,\|\psi\|^2 < \infty, \tag{D.41}$$

where $\mathrm{d}\tilde{\Omega} = \mathrm{d}\theta\mathrm{d}\phi_1\mathrm{d}\phi_2$, due to the transformation (D.10).

Now we are ready to determine all normalizable zero modes that satisfy APS boundary conditions. For a fixed value of λ (in one of the three series (D.21)), we use the ansatz

$$\tilde{\varphi}_\lambda(r,\Omega) = f_\lambda(r)\omega_\lambda(\Omega). \tag{D.42}$$

Since $\tilde{L} = \mathbb{1}\partial_r + B$ on all of $\mathcal{M}$ (and not only close to $\partial\mathcal{M}$), we can use this product ansatz globally[3]. Now, $\tilde{L}\tilde{\varphi}_\lambda = 0$ implies

$$f_\lambda(r) = \exp - \int^r \frac{R}{r'}\lambda(\mu(r'))\mathrm{d}r'. \tag{D.43}$$

The dependence $\lambda(\mu(r))$ can be read off from (D.21). For series I we find

$$\lambda(r)R = \frac{1}{2} + p - \frac{1}{2}Mr^2, \qquad f_\lambda(r) = r^{-\frac{1}{2}-p}\mathrm{e}^{\frac{M}{4}r^2}. \tag{D.44}$$

Normalizability requires $p < 0$, but this is not contained in (D.21), so there are no

[2] These are right handed boundary conditions. They give rise to the APS index formula (D.13). One can also define left handed boundary conditions, where the sums in (D.37) and (D.38) contain contributions from $\lambda \geq 0$ and $\lambda < 0$ only. In this case, the sign of h in (D.13) is reversed [85].

[3] Observe, that $\omega_\lambda(\Omega, r)$ is not a eigenfunction of $B(r)$ in the strict sense, but gives back a r-dependent eigenvalue.

solutions in this case. Similarly one shows, that there are no APS zero modes in case II and case III. Altogether, there are no APS zero modes of $\tilde{L}$. It remains, to analyze

$$\tilde{L}^\dagger \chi_\lambda = 0, \qquad \chi_\lambda = g_\lambda(r)\omega_\lambda(\Omega). \tag{D.45}$$

We find

$$g_\lambda(r) = \exp + \int^r \frac{R}{r'}\lambda(\mu(r'))\mathrm{d}r'. \tag{D.46}$$

In case I this can be integrated and gives

$$\lambda(r)R = \frac{1}{2} + p - \frac{1}{2}Mr^2, \qquad g_\lambda(r) = r^{p+\frac{1}{2}}\mathrm{e}^{-\frac{M}{4}r^2}. \tag{D.47}$$

Normalizability requires $p > -1$, whereas the APS condition reads $\lambda R \leq 0$ or, equivalently, $p \leq \mu(R) - \frac{1}{2}$. All values $p = 1, 2, \ldots, \left[\mu(R) - \frac{1}{2}\right]$ are allowed, where $[x]$ is the biggest integer less than (or equal to) x. Observe, that in this case only $\mu(R) > 0$ gives rise to zero modes. Since the degeneracy of λ is given by p, we find

$$\#_\chi = 1 + 2 + \ldots + \left[\mu - \frac{1}{2}\right] = \frac{1}{2}\left[\mu - \frac{1}{2}\right]\left(\left[\mu - \frac{1}{2}\right] + 1\right) = \frac{1}{2}l(l-1) \tag{D.48}$$

zero modes of $\tilde{L}^\dagger$, where l is given in (D.24).

In cases II and III no zero modes are present for $\mu > 0$. For $\mu < 0$, case I and case II are interchanged, and one obtains the same number of zero modes (we have to replace μ by $|\mu|$ in (D.48)). Thus, the total number of normalizable APS zero modes of $\tilde{L}^\dagger$ is given by

$$\#_\chi = \frac{1}{2}l(l-1). \tag{D.49}$$

Result: for the selfdual constant gauge field (D.15), we have evaluated the APS index formula and find

$$\mathrm{ind}\ \mathrm{i}\not{D} = \frac{1}{2}l(l-1). \tag{D.50}$$

In addition, we have calculated the zero modes explicitly. It turned out, that there are no zero modes of $\tilde{L}$, and the number of zero modes of $\tilde{L}^\dagger$ is given by

$$\#_\chi = \frac{1}{2} l(l-1). \tag{D.51}$$

We observe, that all zero modes have the same chirality, and the index (which counts the difference of zero modes with positive and negative chirality) equals the total number of zero modes in this case.

If we replace the selfdual gauge field by its anti-selfdual counterpart ($\bar{\eta}^3_{\mu\nu}$ instead of $\eta^3_{\mu\nu}$ in (D.15)), the index changes sign, and the rôles of $\tilde{L}$ and $\tilde{L}^\dagger$ are interchanged.

D.3. Arbitrary Profile Function

Let us generalize the above results to arbitrary profile functions. That means, we consider the gauge potential

$$\begin{aligned} A_\mu &= \frac{1}{2} M(r) \eta^3_{\mu\nu} x_\nu, \\ F_{\mu\nu} &= -\eta^3_{\mu\nu} M - \frac{M'}{2r} (\eta^3_{\mu\sigma x_\nu x_\sigma} + \eta^3_{\sigma\nu} x_\mu x_\sigma), \\ F \wedge F &= (2M^2 + MM'r) \text{ dvol}. \end{aligned} \tag{D.52}$$

Observe, that the boundary operator B and its eigenvalue problem are not affected by this generalization, provided we define now

$$\mu(r) = \frac{1}{2} M(r) r^2, \qquad |\mu(R)| + \frac{1}{2} = l + \rho. \tag{D.53}$$

In particular, the boundary contribution to the index formula is given by

$$h - \eta(0) = l(l-1) - \mu(R)^2. \tag{D.54}$$

The bulk contribution can be written as a surface integral an yields

$$
\begin{aligned}
\frac{1}{8\pi^2}\int_{\mathcal{M}} F \wedge F &= \frac{1}{8\pi^2}\int_{\mathcal{M}} r^3(2M^2 + MM'r)\mathrm{d}r\mathrm{d}\Omega \\
&= \int_0^R \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{8}M^2(r)r^4\right) \\
&= \frac{1}{8}M^2(R)R^4 = \frac{1}{2}\mu(R)^2.
\end{aligned} \tag{D.55}
$$

The index formula reads

$$
\text{ind } \mathrm{i}\not{D} = \frac{1}{8\pi^2}\int_{\mathcal{M}} F \wedge F + \frac{1}{2}(h - \eta(0)) = \frac{1}{2}l(l-1), \tag{D.56}
$$

where l is defined in (D.53). Given any profile function, the corresponding zero modes can be calculated as above, if one takes into account that $M = M(r)$.

Let us apply these results to the Abelian projected instanton field in $SU(2)$ gauge theory. Instantons in four dimensional gauge theories are known to possess zero modes. These finite-action configurations allow for a compactification $\mathbb{R}^4 \to S^4$. On that compact space, the Atiyah-Singer index theorem can be applied and yields

$$
\text{ind }[A] = \frac{1}{8\pi^2}\int \text{tr } F \wedge F = \nu[A], \tag{D.57}
$$

so there are at least $\nu[A]$ zero modes of the Dirac operator, where $\nu[A]$ is the instanton number (or Pontryagin index) of the field configuration. On the other hand, instantons and monopoles are intimately related in the Polyakov gauge and in a more complicated way in any Abelian gauge [122]. This correlation can be established after an Abelian projection of the instanton configuration. Using our method described above, we can now answer the quesion, *whether there are fermionic zero modes in Abelian projections.* Generically, the action for such a projected configurations is no longer finite. Therefore we cannot compactify the Euclidean space to a sphere as before, and the Atiyah-Singer index theorem – which holds for compact spaces only – cannot be applied. As a way out, we consider the instanton configuration on a four-dimensional ball of radius R and impose APS boundary conditions for the fermions on the boundary. In the end, we will study the limit of large radii, which should be comparable to the non-compact case, once R is much larger than the instanton size ρ.

We answer the question from above for the one instanton configuration in regular gauge, which is given by

$$A = A^A_\mu T^A \mathrm{d}x_\mu, \qquad A^A_\mu = \frac{2\eta^A_{\mu\nu} x_\nu}{r^2 + \rho^2}. \tag{D.58}$$

Here T^A are the generators of the gauge group and ρ is the size of the instanton. The action density and the Atiyah-Singer index formula for this background are given by

$$S[A] = \int \mathrm{tr}\ F \wedge \tilde{F} = 8\pi^2, \tag{D.59}$$

$$\mathrm{ind}\ [A] = \nu[A] = \frac{1}{8\pi^2} \int \mathrm{tr}\ F \wedge F = 1. \tag{D.60}$$

After an Abelian projection [123], the gauge field is reduced to its third component,

$$\alpha_\mu = A^3_\mu = \frac{2\eta^3_{\mu\nu} x_\nu}{r^2 + \rho^2}. \tag{D.61}$$

For the projected configuration we find

$$S[\alpha] = \int \mathrm{tr}\ F \wedge \tilde{F} \sim \log R, \tag{D.62}$$

$$\nu[\alpha] = \frac{1}{8\pi^2} \int \mathrm{tr}\ F \wedge F = 1. \tag{D.63}$$

So after the Abelian projection, the topological charge is unchanged, whereas the action diverges logarithmically with the radius $R \to \infty$. Compactification of such divergent configurations is not possible (cf. Uhlenbecks theorem). As mentioned before, we can consider the gauge field (D.61) on a four-dimensional ball of finite radius R. Comparing (D.52) with (D.61), we identify the profile function[4]

$$M(r) = \pm \frac{4}{r^2 + \rho^2}, \tag{D.64}$$

thus the index theorem predicts

$$\# = \frac{1}{2} l(l-1) = \frac{1}{2} \left[\frac{5R^2 + \rho^2}{2(R^2 + \rho^2)} \right] \left[\frac{3R^2 - \rho^2}{2(R^2 + \rho^2)} \right]. \tag{D.65}$$

[4] The isospin up/down components of the spinor decouple after the projection, and we can consider each problem separately. They only differ by the sign in (D.64).

This number is zero for $R < \sqrt{3}\rho$ and jumps to one for $R \geq \sqrt{3}\rho$. In particular, for $R \to \infty$ exactly one zero mode survives for each color orientation.

Finally, we calculate this zero mode explicitly, using the method proposed earlier. We have

$$\lambda(r)R = \frac{1}{2} + p + \frac{1}{2}M(r)r^2, \tag{D.66}$$

$$g_\lambda(r) = r^{p+\frac{1}{2}}(r^2+\rho^2)^{-1}, \qquad p = 1, 2, \dots \tag{D.67}$$

Normalizability requires $p > -1$, which is automatically satisfied, and APS boundary conditions imply

$$0 \geq \lambda(R)R = \frac{1}{2} + p - \frac{2R^2}{R^2+\rho^2} \quad \longrightarrow \quad p \leq \left[\frac{3R^2-\rho^2}{2(R^2+\rho^2)}\right], \tag{D.68}$$

so we find exactly one zero mode (with $p = 1$) for both, the upper and the lower isospin component, if $R \geq \sqrt{3}\rho$. Otherwise there is no zero mode, in agreement with the index theorem.

For $R \geq \sqrt{3}\rho$ the APS zero mode reads

$$\tilde{\psi} = \begin{pmatrix} 0 \\ \tilde{\chi} \end{pmatrix}, \quad \tilde{\chi} = g_\lambda(r)\omega_\lambda(\Omega) = r^{\frac{3}{2}}(r^2+\rho^2)^{-1}\begin{pmatrix} 0 \\ \sqrt{\sin\theta\cos\theta} \end{pmatrix}, \tag{D.69}$$

$$\psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \quad \chi = M^{-1}\tilde{\chi} = (r^2+\rho^2)^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{D.70}$$

In the last step we have revoked the transformation (D.10). In fact, we also used the particular form of $\omega_\lambda(\Omega)$, that we have not stated explicitly before, but which can be found in [88]. $\|\psi\|^2$ diverges like $\log R$ for $R \to \infty$.

This zero mode should be compared with the zero mode of the standard charge one instanton, which is normalizable on $\mathbb{R}^4$ and given by [124]

$$\psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix}, \qquad \chi \sim (r^2+\rho^2)^{-\frac{3}{2}} \times \text{color orientation}. \tag{D.71}$$

We will publish the results of this Appendix soon [AK4].

Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quelle gekennzeichnet.

Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen. Insbesondere habe ich hierfür nicht die entgeltliche Hilfe von Vermittlungs- bzw. Beratungsdiensten in Anspruch genommen.

Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt. Die geltende Promotionsordnung der Physikalisch-Astronomischen Fakultät ist mir bekannt.

Ich versichere ehrenwörtlich, dass ich nach bestem Wissen und Gewissen die reine Wahrheit gesagt und nichts verschwiegen habe.

Jena, den 9. Juli 2004

Andreas Kirchberg

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