Nullmoden des Dirac-Operators im Feld von Solitonen und magnetischen Monopolen

Diplomarbeit

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1. Introduction

Philosophy is written in this great book of the Universe which is continually open before our eyes, but we cannot read it without having first learnt the language and the characters in which it is written.
It is written in the language of mathematics and the characters are triangles, circles and other geometrical shapes without the means of which it is humanly impossible to decipher a single word; without which we are wandering in vain through a dark labyrinth.

Galileo Galilei, "The Assayer"

To understand nature in all its details and to describe the surrounding world with the help of some fundamental principles has been a dream of mankind since the very beginning. In order to achieve such an understanding, people are still investigating how nature works and how it is designed at its deepest level.

Is nature made up from some elementary building blocks? The ancient Greek were the first who tried to answer this question. They intended to solve the problem simply by thinking about it, without making any experiment at all. DEMOCRIT claimed that there were such building blocks, tiny, indivisible, immortal, and he introduced the notion atom (greek *atomos* = indivisible). But the majority agreed with ARISTOTLE, who assumed that the structure of matter is continuous. During the middle ages the european alchemists also took this point of view and people forgot about the ideas of DEMOCRIT.

It was not until the beginning of modern science in the seventeenth century, when first experiments were performed in order to prove or vitiate these hypotheses. DALTON, a british teacher and chemist, often called father of modern atomic theory and chemistry, published a famous book *A New System of Chemical Philosophy* in 1808, in which he explained his theories: all matter is made up from atoms and all atoms of a certain chemical element are identical, whereas atoms of different elements have different masses and properties. Soon scientist were able to classify those atoms and to determine their intrinsic properties, and in 1870 the Russian chemist MENDELEJEV published his periodic table of the elements. Ingenious experiments revealed the sizes of such atoms: they are as small as 10^{-8} centimeters.

Between 1894 and 1897 THOMSON analysed the cathode rays that had been discovered in 1858: it turned out that atoms are not indivisible, but all of them contain negatively charged electrons, which can be emitted under the influence of an electric field. In 1902 LORD KELVIN proposed a model of the atom that was later called THOMSON's Model: a positively charge ball with imbedded electrons.

In 1910 RUTHERFORD and his cooperators discovered (with the help of alpha-particle scattering experiments) that every atom contains a very tiny, positively charged and massive core. These results were published in 1911 and the RUTHERFORD Model was born. The nucleus of the lightest element hydrogen is called proton (greek *protos* = the first one). In 1932 CHADWICK identified the second component of the nucleus and called it neutron becaused it is electrically neutral and its mass is close to the mass of the proton.

At that time the set of all known fundamental building blocks consisted of photons, electrons, protons and neutrons. This was enough to explain all observed phenomena. Almost. Already in 1931 PAULI postulated the existence of an additional neutral particle, nowadays known as the neutrino, for the purpose of explaining the beta-decay consistently, which causes for instance the transmutation from tritium into helium. Without the neutrino the spin and energy conservation laws would have been violated. COWAN and REINES verified the existence of the neutrino in 1956.

Furthermore the Dirac equation, already deduced in 1928, predicted the existence of so-called antimatter: all particles have mirror images of the same mass but opposite charge. ANDERSON observed the first antiparticle, the positron, in 1932. The antiproton has been discovered in 1955.

Still this is not the end of the story: owing to the careful investigation of the high energy cosmic radiation and the use of capable accelerators, more and more fundamental particles came into play: muons, tauons with their associated neutrinos, pions, kaons, B-mesons, sigmas, chis..., all of them together with their antiparticles.

So their number became larger and larger, smashing the hope of the scientists that nature can be described in a simple and elegant fashion at its deepest level. All those particles are characterized by their quantum numbers, such as mass, charge, spin and baryon number.

Table 1.1.: Nucleons and Pions.

Particle	**Mass**[MeV]	**Spin**	**Charge**
Proton p	938.3	$\frac{1}{2}$	+1
Neutron n	939.6	$\frac{1}{2}$	0
Pion π^+	139.6	0	+1
Pion π^0	135.0	0	0
Pion π^-	139.6	0	-1

But while tabulating the system of all elementary particles people discovered some symmetries among them. Particles with very similar properties can be arranged in so-called multiplets, indicating that some more fundamental theory should relate them to each other. As illustrative examples we list the nucleon doublet and the pion triplet in Table 1.1.

In 1961 Gell-Mann [1] and Ne'eman [2] showed the possibility of arranging those multiplets into larger families, called supermultiplets[1], cf. Figures 1.1, 1.2 and 1.3. Here S and I_3 denote strangess and the third component of isospin as quantum numbers of the particles. Gell-Mann called this model *The Eightfold Way* and he was able to predict the existence of the Ω^- particle, which was dected right at that time.

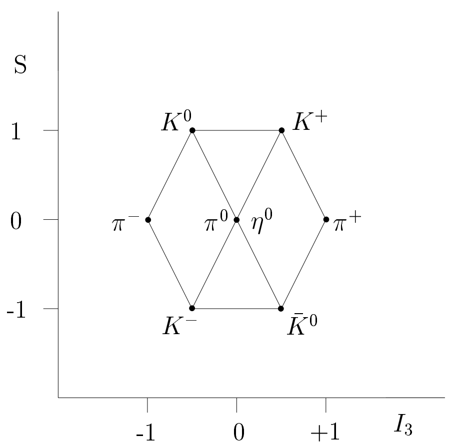

Figure 1.1.: Octet of spin-0 mesons.

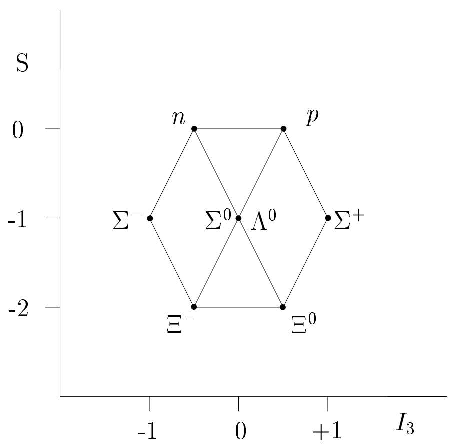

Figure 1.2.: Octet of spin-$\frac{1}{2}$ baryons.

One year leater, Ne'eman and Goldberg-Ophir made the suggestion that each baryon is made up from three more fundamental building blocks: each carrying baryon-number $\frac{1}{3}$ as well as fractional electric charge. Gell-Mann [3] and Zweig [4] improved this model in 1964, independent of each other they published a more precise formulated theory: all known hadrons are made up from some fundamental building blocks, called quarks[2]. They labeled them by a new intrinsic property, the quark flavour: there are u (up), d (down) and s (strange) quarks. The previously (except for the context of crystallography) unfamiliar mathematical notions of group theory were used here. The underlying symmetry group turned out to be the group of special unitary 3×3 matrices, $SU(3)_f$, where f indicates that this symmetry refers to the flavour of the quarks. This is an approximate symmetry, broken by the different quark masses. Today three more quarks are known: c (charm), b (bottom) and t (top).

Soon physicists realized that within some hadrons two or three of the quarks should be in the same quantum mechanical state (for instance the Ω^- consists of the combination $\{sss\}$), but since quarks are fermions this would violate Pauli's exclusion principle. Han

[1] These should not be confused with the multiplets of supersymmetric theories.

[2] Adopted from the book *Finnegans Wake* by James Joyce.

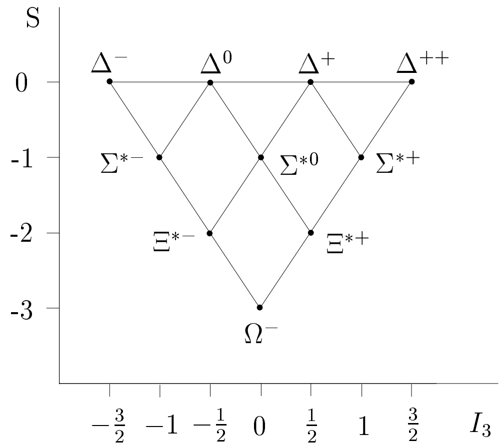

Figure 1.3.: Decuplet of heavy baryons with spin $\frac{3}{2}$.

and Nambu [5] proposed a way out by the introduction of a new quantum number, the color. Now all quarks come in three different colors which are called red, blue and green, accordingly the antiquarks are antired, antiblue or antigreen. All baryons and mesons are colorless combinations of those colored quarks.

In this way the number of fundamental particles got reduced drastically. Only leptons and quarks remain as the universal constituents of matter. They appear in three families and are listed in Table 1.2 together with their basic properties. The definite answer to the question, where the different values of the masses come from, is still not known.

Table 1.2.: Leptons and Quarks.

Leptons	**Charge**	**Mass** [MeV]	**Quarks**	**Charge**	**Mass**[MeV]
ν_e	0	$< 3 \cdot 10^{-6}$	u	$+\frac{2}{3}$	$1 \ldots 5$
e	-1	0.511	d	$-\frac{1}{3}$	$3 \ldots 9$
ν_μ	0	< 0.19	c	$+\frac{2}{3}$	$1115 \ldots 1350$
μ	-1	105	s	$-\frac{1}{3}$	$75 \ldots 170$
ν_τ	0	< 18.2	t	$+\frac{2}{3}$	$169000 \ldots 179000$
τ	-1	1777	b	$-\frac{1}{3}$	$4000 \ldots 4400$

Besides the classification of all fundamental building blocks, a comprehensive description of nature also includes the characterization of the forces that affect those particles. Today all known phenomena can be ascribed to four fundamental forces. They include the familiar gravitation and electromagnetism, which suffice to describe all directly observable effects on earth and in the cosmos, as well as the more unfamiliar forces of the weak and strong interaction.

The gravitational force is the first one, that has been described quantitatively in physics: by NEWTON's theory, published in 1687 in his famous *Philosophiae Naturalis Principia Mathematica*. Special properties of gravitation are its universality and weakness: gravity affects all kinds of matter and energy. At the level of elementary particle physics and at energies that are accessible today it is much too weak, to cause observable effects. Except for the search for an unified theory of all forces it does not play any role.

EINSTEIN's *General Theory of Relativity*, succeeding NEWTON's theory and published in 1916, is a classical gauge theory of gravitation. The gauge freedom of this theory is the possibility of choosing an arbitrary coordinate system, the associated gauge group is the group of diffeomorphisms on the underlying manifold.

Up to now no successful quantum theory of gravitation has been formulated, this remains as one of the most important problems in theoretical physics. During the last years people have been working intensively on theories of supergravity and superstrings, hoping to derive a unified description of gravitation and the three other forces in this way.

For a long time mankind has been acquainted with the electromagnetic force, too. This force differs from gravity, since it is not universal: only charged particles are affected, whereas neutral particles like neutrinos do not feel this force. Between 1855 and 1865 MAXWELL was able to formulate the basic laws of electromagnetism, thereby unifying electric and magnetic interactions. People realised that this theory possesses a residual freedom, the freedom of choosing a definite form of the gauge potential, but did not attach value to this. Nowadays electrodynamics is known to be a $U(1)$ gauge theory, exactly due to this fact.

Since MAXWELL's theory obeys the laws of special relativity from the very beginning, this remained the correct description until 1948. At that time the quantum version of electrodynamics, called quantum electrodynamics (QED), was established independently of each other by TOMONAGA, FEYNMAN [6, 7] and SCHWINGER [8]. In QED the electromagnetic forces between charged particles are mediated by the exchange of virtual photons. In a perturbative approach all processes can be described by FEYNMAN diagrams, like in Figure 1.4, and translated into formulas via the corresponding FEYNMAN rules. After renormalization physical observables can be calculated. This theory turned out to be extraordinary successful. For instance, the calculated magnetic moment of the electron is in agreement with the experimental value with an accuracy of 10^{-10}.

Physicists got the first hint for the existence of the weak force already in 1896, when BECQUEREL discovered radioactivity. The β-decay of the uncharged neutron into a proton and an electron cannot be explained in the context of electromagnetism:

$$n \to p + e^- .$$

The weak interaction turned out to be the reason for this process. The first problem that people encountered, the continuous spectrum of the emitted electrons, was solved by PAULI as stated above by introducing the neutrino, which carries the remaining energy

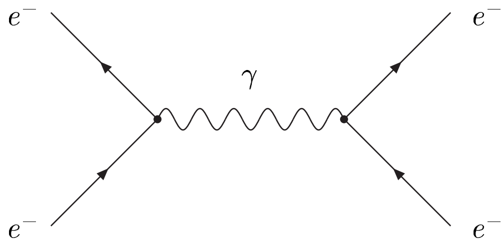

Figure 1.4.: Feynman graph for the collision of two electrons (tree-level).

and the missing angular momentum:

$$n \to p + e^- + \bar{\nu}_e.$$

The weak interaction is very short-ranged, since 1982 we know (from the masses of the Z and $W^\pm$ bosons determined at CERN) that it is mediated only over distances that are smaller than 10^{-18} meters. An effective description of weak interaction processes is given by the FERMI Model.

GLASHOW [9], SALAM [10] and WEINBERG [11] managed to unify electromagnetism and weak interaction within the electroweak model, as a gauge theory with gauge group $SU(2) \times U(1)$.

It was not until 1932 when physicists realised the existence of an additional force: the strong interaction. The discovery of the neutron forced people to introduce this new interaction in oder to explain, how protons and neutrons can form stable nuclei. Obviously, the strong interaction only acts over a very short distance: electromagnetism suffices to explain the observed orbits of the electrons as well as the outcome of RUTHERFORD's scattering experiments. Its range is limited to distances of order of nuclear sizes, typically 10^{-15} meters. Furthermore the strong force does not show universality: particles that interact via this force are called hadrons. In the 1960s the quark model of matter was established. Since that time the strong interaction is understood as the interaction between quarks which binds them to nucleons and other hadrons. The force between nucleons, the nuclear force, is a rudiment of the much stronger force between those quarks. It turned out that the interaction between the quarks is much easier to understand than the complicated force that acts between the nucleons. This theory of quark interaction is quantum chromodynamics (QCD), the gauge theory of strong interactions. The gauge symmetry of QCD is color symmetry, and the corresponding symmetry group is $SU(3)_c$, where c refers to the color (red, green, blue) of the quarks. This symmetry is taken to be an exact symmetry of nature.

The four known interactions between matter particles, gravity, electromagnetism, strong and weak force, reinterpreted as gauge interactions, as well as their basic properties, are listed in Table 1.3.

Table 1.3.: Fundamental Forces.

Force	**Range**	**Affects**	**Gauge Boson**	**Spin**	**Mass**[GeV]
gravitational	infinite	all matter	graviton	2	0
electromagnetic	infinite	electric charges	photon	1	0
weak	10^{-18}	leptons, quarks	$W^{\pm}$, Z	1	80.4 / 91.2
strong	10^{-15}	quarks	8 gluons	1	0

In the course of the 20th century the so-called Standard Model[3] of elementary particle physics has been established. This model turned out to be very powerful in predicting the production of particles in accelerator experiments, cross sections, lifetimes, decay widths and so on.

The Standard Model describes the interaction of the fundamental building blocks of nature, which are quarks and leptons, as quantum gauge interactions with gauge group $SU(3)_c \times SU(2) \times U(1)$.

Recent experiments at LEP indicate that even the ultimate missing particle, the Higgs boson (which is needed to give finite mass to the particles) has been discovered, thereby completing this model. But up to now only four such events have been detected and the results still have to be confirmed.

At the dawning of the 21st century this is the (preliminary) answer of modern physics to the question of the ancient Greeks about the structure of nature.

Now we will focus on the fourth interaction: the gauge theory of strong interactions, quantum chromodynamics. QCD is still under investigation and — since the corresponding field equations are highly nontrivial — many problems are unsolved, in particular the problem of quark confinement: quarks do never occur as single particles, they always form quark-antiquark pairs (mesons) or come as three-quark bound states (baryons). Among the many mechanisms put forward to explain this phenomenon, the most transparent is probably the so-called dual Meissner effect [17, 18], which has recently become popular due to its partial confirmation in lattice experiments and the explicit verification in some supersymmetric models. Furthermore QCD exhibits the spontaneous breakdown of chiral symmetry (χSB): since the masses of the up and down (and to a lesser extent of the strange) quark are very small compared to typical strong interaction energy scales

$$\Lambda_{\mathrm{QCD}} \sim 0.2\mathrm{GeV},$$

[3] For an introduction to the Standard Model and the basic concepts of local gauge theory, see for instance the books by Ne'eman and Kirsh [12], Halzen and Martin [13], Ebert [14] and Geyer [15]. All experimental data are taken from the *2000 Review of Particle Physics* [16].

the theory is approximately invariant under $SU(2)_L \times SU(2)_R$ transformations (or under $SU(3)_L \times SU(3)_R$ transformations, respectively). This is called chiral symmetry. However, we do not see any particle degeneracy patterns ascribable to such symmetries. The resolution to this paradox is that the physical vacuum is not invariant under these symmetries: chiral symmetry is spontaneously broken [19].

In this work we are going to analyse a special class of eigenfunctions of the Dirac operator $\not{D}$, called zero modes. Our zero modes will turn out to be closely related to this spontaneous breakdown: from the experimental data we can determine the value of the quark condensate with the help of QCD sum rules due to SHIFMAN, VAINSHTEIN and ZAKHAROV [20, 21]:

$$< \bar{\psi}\psi >= -(230 MeV)^3.$$

This condensate is related to the spectral density of the Dirac operator $\rho(\lambda)$ near zero eigenvalues by the BANKS-CASHER relation [22]

$$< \bar{\psi}\psi >= -\pi\rho(\lambda = 0).$$

The significance of the quark condensate is the fact that it is an order parameter for the chiral symmetry breaking in the QCD vacuum.

On the other hand, in the ultraviolet limit, the quarks show asymptotic freedom: at high momentum the forces between them vanish and every quark can move almost as a free particle.

In this diploma thesis we study some quantum field theoretical models that might be relevant for realistic quantum field theories. Realistic quantum field theories are difficult to solve because they are governed by nonlinear operator equations. In the usual perturbative treatment, that turned out to be so successful in quantum electrodynamics, we have to start with the solution of the linearized (free) field equations and then to incorporate the effects of interactions as a power series expansion in the coupling constant. For QCD — in which we are mainly interested in — the coupling constant is of order unity and perturbation theory does not work. Furthermore some fundamental properties of quantum field theories cannot be obtained in this approach.

Therefore we proceed in a different way: the operator Euler-Lagrange equations are treated as $\mathbb{C}$-number field equations and are solved by methods of classical mathematical physics. Quantum mechanics is regained either by expanding the quantum theory around the classical solution in a power series of the coupling constant or by quantizing the classical solution in a semiclassical or WKB approximation. In such an approach the nonlinearity of the system is retained at all stages in the calculation. These nonperturbative methods have led to new insights into the properties of quantum field theories.

The classical equations of motion usually yield a certain number of trivial solutions, as well as some nontrivial, solitonic solutions. Often these nontrivial classical solutions suggest a particle interpretation: they have finite energy, are localized in space, are stable and can be boosted to give linearly moving solutions, which carry momentum and display the proper relationship between mass, momentum and energy. These objects are called solitons, even though they are not solitons in the strict sense of soliton theory. Except in the sine-Gordon model, none of the classical solutions encountered in high energy physics really keeps its shape after collision. Nevertheless the notion soliton (instead of the more accurate solitary wave) is used throughout the modern literature. Therefore we will use it in this diploma thesis as well.

In particular we will explore three kinds of nontrivial field configurations: the kink solution in the ϕ^4 theory, the 't Hooft-Polyakov monopole and the instanton solution which occur in Yang-Mills-Higgs and pure Yang-Mills theories, respectively. After an appropriate definition they turn out to carry topological charges.

In a second step we analyse the behaviour of fermions (quarks or leptons, depending on the particular model) in the background of those fields. Of particular interest are so-called zero modes (or Jackiw-Rebbi modes): solutions of the eigenvalue equation

$$H\psi = E\psi,$$

with eigenvalue $E = 0$, where H is the Dirac Hamiltonian and ψ is the fermionic wave function. In the kink and the monopole background we calculate these zero modes explicitly. Furthermore a powerful mathematical theorem, the Callias-Bott-Seeley or CBS index theorem, can be used to calculate the index of appropriate differential operators. This index is equal to the number of left-handed zero modes minus the number of right-handed zero modes. It turns out that in the cases at hand there is always only one species of fermions (either left- or right-handed) present. Therefore the CBS index theorem can be used to determine the absolute number of fermionic zero modes in the given background. The basic statement of this index theorem relates the number of zero modes to the topological charge (thereby disregarding the particular form of the soliton): whenever the background field possesses a topological charge there will be fermionic zero energy modes.

In the instanton case we are dealing with four dimensional Euclidean space. Now we can do the same analysis, but we focus on zero modes of the Dirac Operator $\not{D}$ itself:

$$\not{D}\psi = \gamma_\mu D_\mu \psi = \gamma_\mu(\partial_\mu + A_\mu)\psi = 0.$$

Here the γ_μ matrices form the basis of the standard Clifford algebra and D_μ is the covariant derivative with gauge potential A_μ, which is given by the instanton configuration. In Euclidean space, $\not{D}$ is an elliptic differential operator and the question whether or not $\not{D}$ shows zero modes is a nontrivial one (in Minkowski space $\not{D}$ is hyperbolic and usually there are zero modes). Again those zero modes can be calculated and their explicit shape can be determined. Like kinks and monopoles, instantons possess a topological charge, called Pontryagin index. The celebrated Atiyah-Singer or AS index theorem relates again the number of zero modes (of $\not{D}$) to this topological charge: the higher the charge, the more zero modes are present. Afterwards the zero modes of the Dirac operator can be related to zero modes of the Dirac Hamiltonian (here we use Weyl gauge)

$$H = -i\alpha_i D_i$$

by spectral flow arguments. As will be shown, fermionic zero modes in instanton fields are very important for an understanding of vacuum tuneling processes, the anomaly of the axial current and the change of axial charge.

In this diploma thesis we will use the following recipe again and again:

1. Given a particular field theory, we explore the classical equations of motion that result from the Euler-Lagrange formalism.

2. We analyse in particular the possible vacua and give a description of the entire vacuum structure.

3. Demanding finite energy (or finite Euclidean action) results in a topological classification of all possible static solutions. These solutions are called solitons.

4. Afterwards we use topologically interesting solitons as classical background fields and introduce fermions as quantum objects in these backgrounds.

5. We investigate the existence of zero modes of H (or $\not{D}$ respectively) and determine the number of them as well as their shape by solving the equations of motion explicitly.

6. For all theories at hand there are powerful mathematical theorems, stemming from the theory of differential operators, that predict the existence and number of such zero modes by relating them to topological invariants. We apply those theorems to the particular cases and compare the results.

7. In a final step we analyse the physical consequences that result from the existence of those zero modes.

This work is organized as follows: in Chapter 2 we focus on a toy model, ϕ^4 theory in $1+1$ dimensional Minkowski space. Chapter 3 explains DERRICK's theorem: why gauge fields are necessary in higher dimensions if we want to have nontrivial field configurations. In Chapter 4 we use this insight and investigate $SU(2)$ Yang-Mills-Higgs gauge theory and its solitons, which turn out to be magnetic monopoles. In the simplest case, the 't Hooft-Polyakov monopole of unit charge, we solve the fermionic equations of motion explicitly. In Chapter 5 we examine Euclidean solutions of pure $SU(2)$ gauge theory. The topological solutions are called instantons and are interpreted as tunneling events in Minkowski space. Finally we summarize all calculations and give an outlook, what could or should be done in the future. Appendix A contains some basic definitions and a sketch of the proof of the CBS index theorem. In Appendix B the same is done for the AS index theorem.

2. ϕ^4 Theory

2.1. The Model

Let us consider a scalar field theory in $1+1$ dimensional Minkowski space. Given the potential density $\mathcal{V} = \mathcal{V}(\phi)$, the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \mathcal{V}(\phi) = \frac{1}{2}\dot{\phi}^{\,2} - \frac{1}{2}\phi'^{\,2} - \mathcal{V}(\phi), \tag{2.1}$$

where dot (prime) denotes differentiation with respect to time (space). The total energy and therefore $\mathcal{V}(\phi)$ must be bounded from below, and by adding a suitable constant we can achieve $\mathcal{V}(\phi) \geq 0$, furthermore $\mathcal{V}(\phi)$ should allow for at least two different vacua (absolute minima). The famous ϕ^4- and the Sine-Gordon-Model with potentials

$$\mathcal{V}(\phi) = \frac{1}{4}\lambda\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 \tag{2.2}$$

and

$$\mathcal{V}(\phi) = \frac{m^4}{\lambda}\left(1 - \cos\left(\frac{\sqrt{\lambda}}{m}\phi\right)\right) \tag{2.3}$$

respectively, cf. Figures 2.1 and 2.2, have been studied in detail.

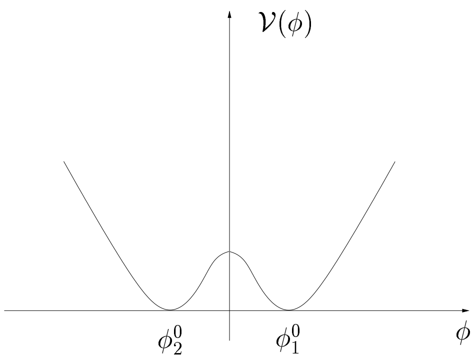

Figure 2.1.: The ϕ^4 potential.

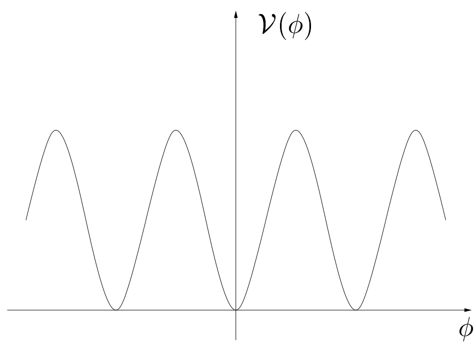

Figure 2.2.: The sine-Gordon potential.

We are looking for static solutions $\phi = \phi(x)$, with energy

$$E[\phi] = \int dx \left(\frac{1}{2}\phi'^{\,2} + \mathcal{V}(\phi(x)) \right) < \infty. \tag{2.4}$$

The Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \tag{2.5}$$

reduce to

$$\phi''(x) = \frac{\partial \mathcal{V}(\phi)}{\partial \phi}(x). \tag{2.6}$$

After multiplying both sides by ϕ' and using the chain-rule we end up with

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2}\phi'^{\,2}\right) &= \frac{d}{dx}\mathcal{V}(\phi(x)) & (2.7) \\ \frac{1}{2}\phi'^{\,2} &= \mathcal{V}(\phi) + c. & (2.8) \end{aligned}$$

In order to have finite energy the solutions must obey

$$\begin{aligned} \lim_{x\to\pm\infty} \mathcal{V}(\phi(x)) &= 0, & (2.9) \\ \lim_{x\to\pm\infty} \phi'(x) &= 0, & (2.10) \end{aligned}$$

implying $c \equiv 0$. There are trivial solutions (often referred to as vacuum solutions and labeled by an index 0): $\phi(x) = \phi_0 = \text{const}$, with $\mathcal{V}(\phi_0) = 0$. But there are non-trivial solutions, too. These solutions are called solitons and are labeled by an index S. Due to this fact different sectors emerge in our theory: a certain number of vacuum sectors as well as soliton sectors. All of them must be treated separately. Accordingly the Hilbert space $\mathcal{H}$ - after quantization - consists of the sum of orthogonal spaces,

$$\mathcal{H} = \mathcal{H}_1^0 \oplus \ldots \oplus \mathcal{H}_n^0 \oplus \mathcal{H}_1^S \oplus \ldots \oplus \mathcal{H}_m^S. \tag{2.11}$$

The equations of motion can be solved by quadrature

$$\begin{aligned} \frac{1}{2}\phi'^{\,2} &= \mathcal{V}(\phi), \\ \frac{d\phi}{dx} &= \pm\sqrt{2\mathcal{V}(\phi)}, \\ x - x_0 &= \pm \int_{\phi(x_0)}^{\phi(x)} \frac{d\tilde{\phi}}{\sqrt{2\mathcal{V}(\tilde{\phi})}}. & (2.12) \end{aligned}$$

The constant x_0 represents the invariance of the Lagrangian $L = \int \mathcal{L} dx$ under translations in x-direction and can be choosen arbitrarily. Sometimes this freedom causes trouble if we try to quantize the theory, the appropriate tool to circumvent these difficulties is the method of collective coordinates [23, 24].

We observe that equation (2.6) is equivalent to the problem of a particle moving along $x = x(t)$ in the potential $-\mathcal{V}(x)$, if we replace $x \to t$ and $\phi \to x$, cf. Figure 2.3. Now the vacuum solutions correspond to particles with zero energy, resting at one of the

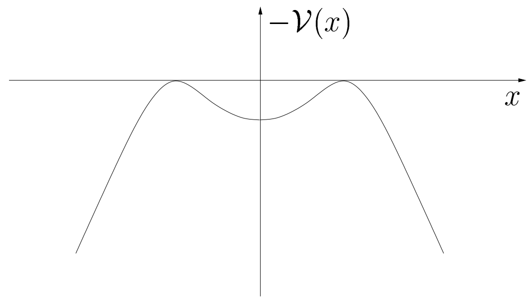

Figure 2.3.: The upside down potential.

maxima of the potential all the time, and the non-trivial solutions can be interpreted as particles being on one of the tops of the potential at the beginning $t \to -\infty$, moving through the valley and ending at the second maximum at late times $t \to +\infty$. By general arguments we can conclude that such non-trivial solutions always appear, if the corresponding potential has at least two minima $\mathcal{V}(\phi) = 0$. These soliton solutions share many properties with usual particles, as we will see soon: they are localized in space, have finite energy (rest mass), under certain conditions they can collide without changing their shape, and by means of a Lorentz-transformation we can give them an arbitrary velocity.

The following explicit calculations are done within the ϕ^4 theory, but the results obtained below can easily be generalized to all theories that have a similar structure of the potential. From the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \mathcal{V}(\phi) = \frac{1}{2}\dot{\phi}^2 - \underbrace{\left(\frac{1}{2}\phi'^{\,2} + \frac{1}{4}\lambda\left(\phi^2 - \frac{m^2}{\lambda}\right)^2\right)}_{\mathcal{U}(\phi)} \tag{2.13}$$

we get

$$L[\phi] = T[\phi] - U[\phi], \tag{2.14}$$

where

$$T[\phi] \equiv \frac{1}{2}\int \mathrm{d}x\, \dot{\phi}^2,\ U[\phi] \equiv \int \mathrm{d}x\, \mathcal{U}(\phi). \tag{2.15}$$

The Euler-Lagrange equation of motion for the static case (2.6) reads

$$-\phi'' - m^2\phi + \lambda\phi^3 = 0, \tag{2.16}$$

and has the solutions

$$\phi^0_{1/2}(x) = \pm\frac{m}{\sqrt{\lambda}} \tag{2.17}$$

and

$$\phi^S_{1/2}(x) = \pm\frac{m}{\sqrt{\lambda}}\tanh\left(\frac{m}{\sqrt{2}}(x-x_0)\right). \tag{2.18}$$

The trivial solutions $\phi^0_{1/2}$ are the two different vacua (zero indicates the vacuum sector), the non-trivial solutions $\phi^S_{1/2}$ are called kink or antikink, respectively (S indicates the soliton sector). They interpolate between the vacuum configurations when x goes from $-\infty$ to $+\infty$ and differ from the trivial solutions only in a small region around x_0, cf. Figures 2.4, 2.5.

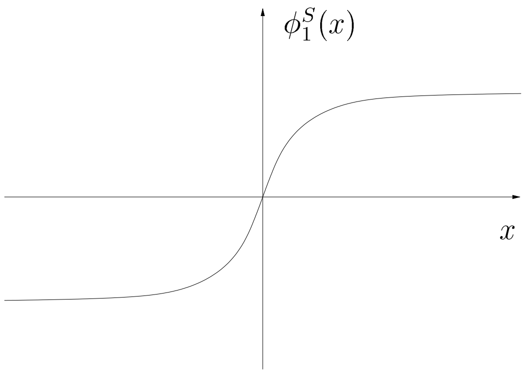

Figure 2.4.: The kink shape ($x_0 = 0$).

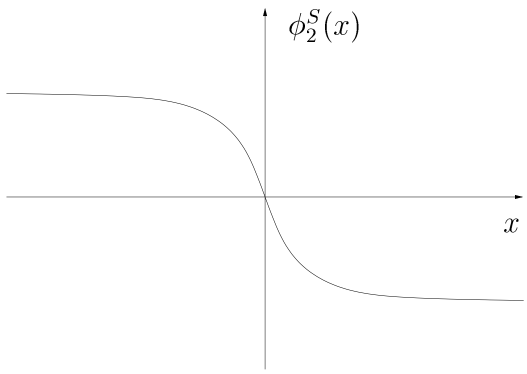

Figure 2.5.: The antikink shape ($x_0 = 0$).

The total energy of the vacuum solutions is zero according to equation (2.4), whereas the solitons carry energy (classical mass)

$$E = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda}. \tag{2.19}$$

The $\frac{1}{\lambda}$ dependence of the energy is characteristic for nonperturbative solutions of the field equations, i.e. these field configurations cannot be found via a power series expansion in λ. Let us reformulate the last statements in a more sophisticated way: The solutions of the field equation that we found are topologically different. We can classify them according to their behaviour at spatial infinity. It is easy to guess, how one has to define an appropriate topological charge Q_{top} in this simple case. Take

$$Q_{\text{top}}(\phi) = \frac{1}{2}\left[\frac{\phi_+}{|\phi_+|} - \frac{\phi_-}{|\phi_-|}\right], \tag{2.20}$$

where

$$\phi_\pm = \lim_{x\to\pm\infty} \phi(x). \tag{2.21}$$

The vacuum solutions are topologically trivial, i.e. $Q_{\text{top}}(\phi^0_{1/2}) = 0$, whereas the kink and antikink carry charge 1 and -1, respectively. The demand for finite energy forces all solutions to take on one of the vacuum values at spatial infinity. Therefore all solutions map the border of space (in this particular case the points $\{-\infty, +\infty\}$) into the set of all possible vacuum values (here this is $\{\frac{m}{\sqrt{\lambda}}, -\frac{m}{\sqrt{\lambda}}\}$). Both manifolds are zero dimensional spheres: S^0_{phys} and S^0_{int}, respectively.

Result: All finite energy solutions of our ϕ^4 theory can be interpreted as mappings $S^0_{\text{phys}} \to S^0_{\text{int}}$ and according to this can be characterized by a number, the topological charge Q_{top}.

2.2. Fermionic Quantization

Now we introduce fermions in all sectors, taking the $\mathbb{C}$-number fields ϕ as (space dependent) masses. This gives rise to a Hamiltonian $H(\phi)$. The Hamiltonian acts on spinors ψ, and in 1+1 dimensions we can realize the Dirac algebra with the help of the Pauli matrices σ^i. We identify

$$\alpha = \sigma^2, \beta = \sigma^1, \gamma^0 = \beta = \sigma^1, \gamma^1 = \beta\alpha = i\sigma^3. \tag{2.22}$$

According to this

$$H(\phi) = \alpha p + g\beta\phi = \sigma^2 p + g\sigma^1\phi, \tag{2.23}$$

with momentum operator $p = \frac{1}{i}\partial_x$ and coupling constant g. Let us rewrite the spinor in components $\psi = \begin{pmatrix} u & v \end{pmatrix}^\top$. Charge conjugation symmetry is mediated by σ^3, since $\{H, \sigma^3\} = 0$. That means σ^3 turns positive (negative) energy solutions in negative (positive) one. We have to solve

$$H(\phi)\psi = E\psi, \begin{pmatrix} 0 & g\phi - \partial_x \\ g\phi + \partial_x & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = E\begin{pmatrix} u \\ v \end{pmatrix}. \tag{2.24}$$

This is equivalent to

$$-Eu + g\phi v - v' = 0, \tag{2.25a}$$
$$g\phi u - Ev + u' = 0. \tag{2.25b}$$

Let us first check whether there are zero modes present in one of these sectors or not, since later on they will turn out to be essential for some unusual and unexpected physical effects. It is easy to see, that there are no normalizable modes in the vacuum sector, if we insist on $E = 0$. Next, look for zero modes in the kink or antikink background. For $E = 0$ the equations decouple, we can integrate both and get

$$u = \exp -g \int^x \mathrm{d}y\ \phi^S_{1/2}(y), \tag{2.26a}$$

$$v = \exp +g \int^x \mathrm{d}y\ \phi^S_{1/2}(y). \tag{2.26b}$$

The trivial solutions are $u = 0$ and $v = 0$. Now we have to use the explicit form of $\phi^S_{1/2}$ and arrive at

$$\begin{aligned} I(x) &= g \int^x \mathrm{d}y\ \phi^S_{1/2}(y) = \pm \frac{mg}{\sqrt{\lambda}} \int^x \mathrm{d}x\ \tanh\left(\frac{m}{\sqrt{2}}x\right) \\ &= \pm g \sqrt{\frac{2}{\lambda}} \log\cosh\left(\frac{m}{\sqrt{2}}x\right) + \text{const.} \end{aligned} \tag{2.27}$$

Therefore

$$u = \exp -I(x) \sim \left(\cosh\left(\frac{m}{\sqrt{2}}x\right)\right)^{\mp\sqrt{\frac{2}{\lambda}}g}, \tag{2.28a}$$

$$v = \exp +I(x) \sim \left(\cosh\left(\frac{m}{\sqrt{2}}x\right)\right)^{\pm\sqrt{\frac{2}{\lambda}}g}. \tag{2.28b}$$

We can combine all these solutions, getting

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v \end{pmatrix}, \begin{pmatrix} u \\ 0 \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}.$$

For the kink only the third combination is both nontrivial and square integrable, i.e. only this is a physical solution, cf. Figure 2.6. We normalize our zero mode

$$\psi_0(x) = \mathcal{N} \begin{pmatrix} u(x) \\ 0 \end{pmatrix}, \tag{2.29}$$

such that

$$1 = \int \mathrm{d}x\ \psi_0^\dagger(x)\psi_0(x). \tag{2.30}$$

Observe that ψ_0 is eigenfunction of σ^3 with $\sigma^3\psi_0 = \psi_0$, i.e. ψ_0 is invariant under charge-conjugation. Zero modes of the form $\begin{pmatrix} u & 0 \end{pmatrix}^\top$ are called left-handed. In the antikink

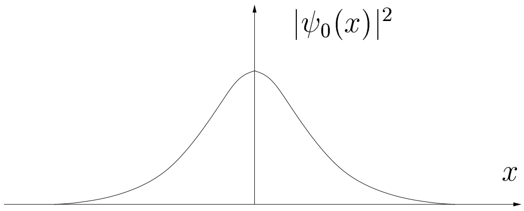

Figure 2.6.: The zero mode shape.

sector we get a similar zero mode with upper and lower components interchanged, i.e. a right-handed one, which is invariant under charge conjugation up to a sign change.

Now let us investigate the remaining spectrum. For $E \neq 0$ we can express v in terms of u (via equation (2.25b)) for both, the vacuum sector $\phi = \phi^0_{1/2}$ and the soliton sector $\phi = \phi^S_{1/2}$:

$$v = \frac{1}{E}(g\phi + \partial_x)u. \tag{2.31}$$

Using this, equation (2.25a) reads

$$\begin{aligned} 0 &= -Eu + \frac{g\phi}{E}(g\phi + \partial_x)u - \frac{1}{E}(g\phi' + \partial_x^2)u - \frac{g\phi}{E}u' \\ &= (-E^2 + g^2\phi^2 - g\phi')u - u''. \end{aligned} \tag{2.32}$$

This is a Schrödinger equation for u with potential $g^2\phi^2 - g\phi'$ and energy E^2:

$$(-\partial_x^2 + g^2\phi^2 - g\phi')u = E^2 u. \tag{2.33}$$

The explicit form of $\phi(x)$ yields

$$g^2\phi^2(x) - g\phi'(x) = \begin{cases} \left(\frac{g^2m^2}{\lambda} \pm \frac{gm^2}{\sqrt{2\lambda}}\right)\tanh^2\left(\frac{m}{\sqrt{2}}x\right) \mp \frac{gm^2}{\sqrt{2\lambda}}, & \phi = \phi^S_{1/2} \\ \frac{g^2m^2}{\lambda}, & \phi = \phi^0_{1/2} \end{cases}.$$

The vacuum sector is trivial: no zero mode, no bound solutions, just plane waves. On the other hand: if the soliton profile is sufficiently weak there are no additional normalizable bound solutions besides the zero mode [25]. This is a restriction on m and λ: if λ is large enough, then there is only one bounded state, the zero mode. In what follows we shall assume that there is exactly one bound state. The generalization to two or more bounded states is straight forward. Furthermore there are scattering states for energies $E^2 \geq \frac{g^2m^2}{\lambda}$. They are given by wave functions u_k and have eigenvalues $E_k^2 = k^2 + \frac{g^2m^2}{\lambda}$. The positive energy solutions of the original Dirac equation (2.24) can be expressed as

$$\psi_E = \begin{pmatrix} \frac{1}{\sqrt{2}}u_k \\ \frac{1}{\sqrt{2}E}(g\phi + \partial_x)u_k \end{pmatrix}, \tag{2.34}$$

the negative energy solutions are

$$\psi_{-E} = \sigma^3 \psi_E = \begin{pmatrix} \frac{1}{\sqrt{2}} u_k \\ -\frac{1}{\sqrt{2}E}(g\phi + \partial_x) u_k \end{pmatrix}. \tag{2.35}$$

Consider $E = -\sqrt{k^2 + \frac{g^2 m^2}{\lambda}} < 0$. The charge density at a given energy E (momentum k) is

$$\begin{aligned}
\rho_k(x) &= \psi_E^\dagger(x)\psi_E(x) = \frac{1}{2}|u_k|^2 + \frac{1}{2E^2}|(\partial_x + g\phi)u_k|^2 \\
&= \frac{1}{2}|u_k|^2 + \frac{1}{2E^2}(|u_k'|^2 + g\phi\partial_x|u_k|^2 + g^2\phi^2|u_k|^2) \\
&= \frac{1}{2}|u_k|^2 + \frac{1}{2E^2}\bigg(\frac{1}{2}\partial_x^2|u_k|^2 - \frac{1}{2}\bar{u}_k'' u_k - \frac{1}{2}\bar{u}_k u_k'' + g\partial_x(\phi|u_k|^2) - \\
&- g|u_k|^2\phi' + g^2\phi^2|u_k|^2\bigg),
\end{aligned} \tag{2.36}$$

we use equation (2.32) in order to express u'' in terms of u:

$$\begin{aligned}
\rho_k(x) &= \frac{1}{2}|u_k|^2 + \frac{1}{2E^2}\bigg(\frac{1}{2}\partial_x^2|u_k|^2 + (E^2 - g^2\phi^2 + g\phi')|u_k|^2 + g\partial_x(\phi|u_k|^2) - \\
&- g|u_k|^2\phi' + g^2\phi^2|u_k|^2\bigg) \\
&= |u_k|^2 + \frac{1}{4E^2}(\partial_x^2|u_k|^2) + \frac{g}{2E^2}\partial_x(\phi|u_k|^2).
\end{aligned} \tag{2.37}$$

However, in the vacuum sector $|u_k|^2$ is a constant, as is ϕ, so that the last two terms in (2.37) vanish. Now we determine the total charge (fermion number) of a given state:

$$Q = \int \mathrm{d}x \int \frac{\mathrm{d}k}{2\pi}\rho_k(x). \tag{2.38}$$

We renormalize this in such a way that the vacuum carries no charge at all. For the empty (i.e. no fermions present) soliton sector we get

$$\begin{aligned}
Q &\equiv \int \mathrm{d}x \int \frac{\mathrm{d}k}{2\pi}(\rho_k^S(x) - \rho_k^0(x)) \\
&= \underbrace{\int \mathrm{d}x \int \frac{\mathrm{d}k}{2\pi}(|u_k^S(x)|^2 - |u_k^0(x)|^2)}_{-1} + \int \frac{\mathrm{d}k}{2\pi}\frac{g}{2E^2}\Big[|u_k^S(x)|^2\phi_k(x)\Big]_{x=-\infty}^{x=+\infty}.
\end{aligned} \tag{2.39}$$

The first integral gives -1, because we integrate over a complete set in the vacuum sector, while in the soliton sector the zero-mode ψ_0 is not included. Now we can evaluate the second term, even without explicit knowledge of the solutions u_k^S. We describe their asymptotical behavior in terms of transmission and reflection coefficients:

$$\begin{array}{rll} u_k^S(x) \to Te^{ikx} & \text{for} & x \to +\infty, \\ u_k^S(x) \to e^{ikx} + Re^{-ikx} & \text{for} & x \to -\infty \end{array} \tag{2.40}$$

with (due to unitarity)

$$|T|^2 + |R|^2 = 1. \tag{2.41}$$

Therefore

$$\begin{aligned} Q &= -1 + \int \frac{\mathrm{d}k}{2\pi} \frac{g}{2E^2} (|T|^2 + (|R|^2 + 1)) \frac{m}{\sqrt{\lambda}} = -1 + \frac{gm}{2\pi\sqrt{\lambda}} \int \mathrm{d}k \frac{1}{E^2} \\ &= -1 + \frac{gm}{2\pi\sqrt{\lambda}} \int \mathrm{d}k \frac{1}{k^2 + \frac{g^2 m^2}{\lambda}} \\ &= -1 + \frac{gm}{2\pi\sqrt{\lambda}} \frac{gm}{\sqrt{\lambda}} \frac{\lambda}{g^2 m^2} \int \mathrm{d}\xi \frac{1}{\xi^2 + 1} \\ &= -1 + \frac{1}{2\pi} \Big[\arctan \xi \Big]_{-\infty}^{+\infty} = -1 + \frac{1}{2} = -\frac{1}{2}. \end{aligned} \tag{2.42}$$

So we encountered a curiosity: the existence of a zero mode within the spectrum of the Hamiltonian causes the fermion number to take on half-integer values - a novel and fascinating quantum mechanical phenomenon, which was previously unsuspected. These results were published for the first time by Jackiw and Rebbi in [26]. The fractional value of Q arises essentially due to the fact that we excluded the zero mode from the definition of Q, corresponding to the rule that in the ground state all negative energy levels are filled, whereas the rest of them remains unoccupied. That this is the correct prescription can be seen from the following: after recognizing the existence of the zero energy mode, we can reformulate the problem of fermion number fractionization in terms of second quantized wave operators. The standard (normal ordered) charge-conjugation-odd fermion charge density is [27]

$$\rho(x) = \frac{1}{2} \Big[\psi^\dagger(x)\psi(x) - \psi(x)\psi^\dagger(x) \Big]. \tag{2.43}$$

Together with the expansion of the wave function

$$\psi(x,t) = \sum_n \Big[a_n f_n(x) e^{-iE_n t} + b_n g_n(x) e^{+iE_n t} \Big] + c\, \psi_0(x), \tag{2.44}$$

this gives the fermion number

$$\begin{aligned} Q &= \int \mathrm{d}x\, \rho(x) \\ &= \frac{1}{2} \sum_n (a_n^\dagger a_n - a_n a_n^\dagger) - \frac{1}{2} \sum_n (b_n^\dagger b_n - b_n b_n^\dagger) + \frac{1}{2}(c^\dagger c - c c^\dagger) \\ &= \sum_n (a_n^\dagger a_n - b_n^\dagger b_n) + c^\dagger c - \frac{1}{2}. \end{aligned} \tag{2.45}$$

It follows that with this choice the zero energy states of the fermion in the solitonic field have a fermion number $+\frac{1}{2}$ or $-\frac{1}{2}$, accordingly as $c^\dagger$ or c annihilates the state. The fermion numbers of all states including the nonzero energy modes are half-integral. For example if there is a second fermion added to the $Q = +\frac{1}{2}$ state, its fermion number would be $Q = +\frac{3}{2}$, while if we took the $Q = -\frac{1}{2}$ state and added a fermion to it, we would get $Q = +\frac{1}{2}$ for that new state. Thus one unique zero energy mode makes all states have a half-integer fermion number and become doubly degenerate.

Remark: in [28] this analysis has been extended to Dirac equations that are not symmetric under charge conjugation, due to the introduction of a symmetry breaking term:

$$H(\phi) = \sigma^2 p + g\sigma^1\phi + \sigma^3\epsilon. \tag{2.46}$$

Now the fermionic charge becomes

$$Q = -\frac{1}{\pi}\arctan\left|\frac{gm}{\epsilon\sqrt{\lambda}}\right|. \tag{2.47}$$

In the conjugation symmetric limit, $\epsilon \to 0$, and the previous result, $Q = -\frac{1}{2}$ is regained.

2.3. The Polyacetylene Story

What does these results imply? Is this just a silly calculation or can one verify its predictions? It turned out, that polymere physics indeed provides the opportunity to do so. To understand this in detail, we have to deal with a very special substance - polyacetylene. Polyacetylene consists of chains of carbon atoms, with electrons moving along the chains. So this is a one dimensional system. There are two kinds of bounds between the carbon atoms: single bounds and double bounds. Let us imagine an infinite long chain.The displacement ϕ of each atom (with respect to the quasi-equilibrium with equal spacing between all atoms) is the so-called phonon field $\phi = \phi(x_i)$.

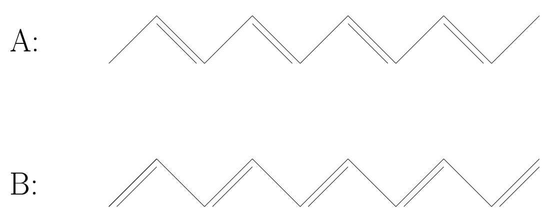

Figure 2.7.: Polyacetylene: A, B.

Detailed dynamical calculations show [29, 30] that the energy density $\mathcal{V}(\phi)$ as a function of constant ϕ has the double-well shape we are familiar with from our ϕ^4-theory. In this case the matrix structure of the Hamiltonian H does not arise from spin. Rather, this structure arises through a linearized approximation and the two-component wave functions that are eigenmodes of H refer to the right-moving and left-moving electrons. The filled negative energy states are the valence electrons, while the conducting electrons populate the positive energy states [31]. Now there are two degenerate vacua called A and B. These correspond to the vacuum solutions $\phi_{1,2}$, cf. Figure 2.7.

Imagine a chain, being in the A(B) vacuum at the very left, $x \to -\infty$, and in the B(A) vacuum for $x \to +\infty$. This is exactly what we called the kink (antikink), cf. Figure 2.8. The circle denotes an unpaired single electron.

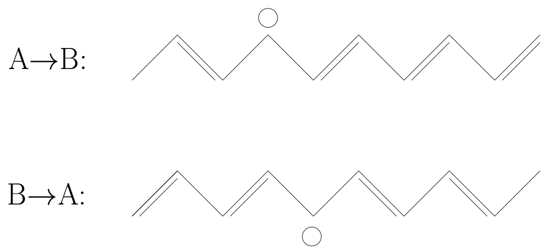

Figure 2.8.: Polyacetylene: kink, antikink.

Finally consider a polyacetylene sample in the B vacuum, but with two solitons along the chain, and compare this with the usual B vacuum by counting the number of links, cf. Figure 2.9.

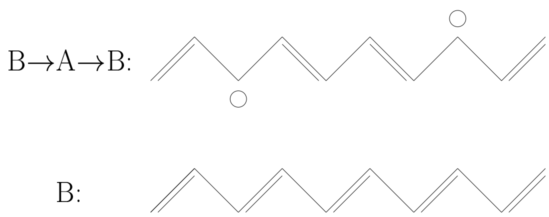

Figure 2.9.: Polyacetylene: BAB vs. B.

Result: the two soliton state exhibits a deficit of one link. If we now imagine separating the two solitons a large distance, so that they are independent of one another, then each soliton carries a deficit of half a link and the quantum numbers are split between the two states. But we must remember that a link corresponds to two states: two electrons with paired spin. Therefore the effect of fractional charge is hidden here by this degeneracy. So in polyacetylene a soliton carries a charge deficit of one unit of electric charge. The soliton state has net charge, but no net spin, since all the electron spins are paired. If an additional electron is inserted into to sample, the charge deficit is extinguished, and one obtains a neutral state, but now there is a net spin. These spin-charge-assignments (charged-without spin, neutral-with spin) have been observed, the same holds for the emergence of a localized electronic mode at mid-gap, i.e. at zero energy [32].

Materials with a slightly different chain structure, with two single bounds and one double bound as fundamental period, have been analyzed in [33]. Now there are three degenerate ground states A, B and C, two types of kinks interpolating between A and B or B and C respectively, as well as the corresponding antikinks. A carbon copy of our analysis now predicts charges $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$. This spectrum (including gap states) is confirmed by numerical calculations and should be obtainable by experiment, too. One candidate possessing an appropriate chain structure is TTF-TCNQ (tetrathiafulvalene-tetracyanoquinodimethane) [33].

2.4. Index Theorem

The occurence of a zero mode in the spectrum of the Dirac Hamiltonian $H(\phi)$ in the kink (or antikink) background of our theory is a consequence of the powerful Callias-Bott-Seeley index theorem [34, 35]. This is a mathematical theorem that counts the number of zero modes of differential operators of a certain class and can be applied to open spacetime manifolds with an odd number of space dimensions. The proof of the theorem is sketched in appendix A[1]. Let us apply this theorem to our model. From the Hamiltonian (2.23) we read off the operator

$$L = -\frac{d}{dx} + g\phi(x), \tag{2.48}$$

where ϕ is either $\phi^0_{1/2}$ or $\phi^S_{1/2}$. The index formula reduces to (A.34)

$$\text{index}\, L = \frac{1}{2}\left[\frac{\phi_+}{|\phi_+|} - \frac{\phi_-}{|\phi_-|}\right] = Q_{\text{top}}. \tag{2.49}$$

In the vacuum sector the index vanishes identically, in accordance with our explicit result that there are no zero modes. In the nontrivial sectors we get

$$\text{index}\, L = \pm 1. \tag{2.50}$$

For the kink background index $L = +1$ means that the number of left-handed zero modes minus the number of right-handed zero modes is equal to one, just as we found it in the explicit calculations: $1 - 0 = 1$. Furthermore in the antikink field the same difference is equal to minus one: $0 - 1 = -1$.

2.5. Results

There are field theories in $1+1$ dimensional Minkowski space, that allow for topologically nontrivial solitonic solutions. For detailed calculations we used the ϕ^4 theory. It contains two vacuum sectors as well as two soliton sectors: the kink and the antikink

[1] For basic definitions, please consult this appendix.

sector. Within the vacuum sectors we can solve the Dirac equation and find that the eigenfunctions of the Hamiltonian are plane waves, starting at energies $E^2 \geq \frac{g^2m^2}{\lambda}$. For smaller values of E we find a gap. In the soliton sectors there are scattering states for sufficiently large energies, again there is a gap around zero energy, but now there is one normalisable eigenstate of H exactly at $E = 0$. The zero energy mode signals quantum mechanical degeneracy, and as a consequence the solitons states are doublets $\left|\phi^S_{1/2}, \pm\right\rangle$. The additional label $\pm$ describes a twofold degeneracy (in addition to the kink/antikink doubling) which is required by the zero energy fermion solution. These explicit results are in agreement with mathematical theorems which state that in nontrivial background fields the Dirac Hamiltonian always exhibits zero energy modes within its spectrum. The effects of fermion fractionization can be observed within the framework of solid state physics.

3. Derrick's Theorem

Now that we have investigated the 1+1 dimensional case in detail, we are ready to generalize our results to higher dimensions. Consider the standard Lagrangian for a set of time independent scalar fields (arranged as a vector) $\phi = \{\phi^a\}$ living in a $D+1$ dimensional Minkowski space

$$L = \int \mathrm{d}^D\mathbf{x}\Big(\frac{1}{2}\partial_i\phi\cdot\partial_i\phi - \mathcal{U}(|\phi|)\Big). \tag{3.1}$$

The potential $\mathcal{U}$ shall be non-negative, and we are looking for static, finite energy solutions. The energy is

$$E[\phi] = \underbrace{\int \mathrm{d}^D\mathbf{x}\Big(\frac{1}{2}\partial_i\phi\cdot\partial_i\phi\Big)}_{U_1[\phi]} + \underbrace{\int \mathrm{d}^D\mathbf{x}\,\mathcal{U}(|\phi|)}_{U_2[\phi]}. \tag{3.2}$$

Both, U_1 and U_2 are non-negative. Now we introduce a one-parameter family of field configurations defined by

$$\phi(\mathbf{x},\lambda) \equiv \phi(\lambda\mathbf{x}). \tag{3.3}$$

For this family, the energy is given by

$$\begin{aligned} E_\lambda[\phi(\mathbf{x},\lambda)] &= U_1[\phi(\mathbf{x},\lambda)] + U_2[\phi(\mathbf{x},\lambda)], \\ &= \lambda^{(2-D)}U_1[\phi] + \lambda^{-D}U_2[\phi]. \end{aligned} \tag{3.4}$$

By Hamilton's principle this must be stationary at $\lambda = 1$. Thus,

$$\begin{aligned} \frac{\partial E_\lambda}{\partial\lambda}\bigg|_{\lambda=1} &= 0, \\ (D-2)U_1[\phi] + DU_2[\phi] &= 0. \end{aligned} \tag{3.5}$$

For $D > 2$ this implies that both U_1 and U_2 must vanish. For $D = 2$ we are left with

$$U_2[\phi] = 0. \tag{3.6}$$

That means, that our field ϕ must be a minimum of the potential everywhere. Therefore the set of minima of the given potential $\mathcal{U}$ must be continuous, otherwise only the trivial solution $\phi = \phi_0 = \text{const}$ is possible.

Result: in three or more dimensions there are no static, finite-energy solutions at all, in two dimensions there are solutions but only under very special circumstances (see for instance the nonlinear σ-model [23]). In order to find solitons also within higher dimensional theories, we have to modify the Lagrangian. This can be achieved by the introduction of higher spin fields: gauge fields. For the time being we will focus on a 3+1 dimensional model. The simplest gauge theory, electrodynamics or $U(1)$ gauge theory, in general does not contain solitonic solutions [23]. Thus we will deal with a generalization of electrodynamics: $SU(2)$ non-Abelian gauge theory, that can be interpreted as a simplified model of quantum chromodynamics. This theory, its solitons and corresponding zero modes will be analysed in the next chapter.

4. The 't Hooft-Polyakov Monopole

4.1. The Model

Consider scalar fields $\phi = \{\phi^a(\mathbf{x},t)\}$ and vector fields $A^a_\mu(\mathbf{x},t)$ with internal space index $a = 1,2,3$, living in 3+1 dimensional Minkowski space. That means for any given a ϕ^a transforms as a scalar and A^a_μ as a vector under Lorentz transformations. From the basic principles of gauge theory we know the Lagrangian [14]

$$\mathcal{L}(\mathbf{x},t) = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \frac{1}{2}(D_\mu\phi)^a(D^\mu\phi)^a - \frac{1}{4}\lambda(\phi^a\phi^a - F^2)^2, \tag{4.1}$$

with field tensor

$$G^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu, \tag{4.2}$$

and covariant derivative

$$(D_\mu\phi)^a \equiv \partial_\mu\phi^a + g\epsilon^{abc}A^b_\mu\phi^c. \tag{4.3}$$

The real constants g, F, λ are parameters of the model. Observe that the potential for the scalar fields ϕ^a is of the ϕ^4 type again. A^a_μ are the $SU(2)$ gauge fields, ϕ^a form the Higgs field. By construction $\mathcal{L}$ is invariant under local $SU(2)$ gauge transformations, which are defined as follows

$$\phi^a(\mathbf{x},t) \quad\rightarrow\quad (U(\mathbf{x},t))^{ab}\phi^b(\mathbf{x},t), \tag{4.4a}$$

$$(A^a_\mu(\mathbf{x},t)L^a)^{bc} \quad\rightarrow\quad (U(\mathbf{x},t))^{bd}(A^a_\mu(\mathbf{x},t)L^a + \frac{i}{g}1\!\!1\partial_\mu)^{de}(U^{-1}(\mathbf{x},t))^{ec}, \tag{4.4b}$$

where

$$(U(\mathbf{x},t))^{bc} \quad\equiv\quad (\exp(-iL^a\theta^a(\mathbf{x},t)))^{bc}, \tag{4.5}$$

$$(L^a)^{bc} \quad=\quad i\epsilon^{abc}. \tag{4.6}$$

L^a are the three generators of $SU(2)$ in 3×3 matrix representation, θ^a are group parameters, varying in group space. To solve the corresponding field equations is a highly nontrivial problem since 15 coupled nonlinear fields are involved. From the Lagrangian we get the equations of motion

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi^a} &= -\lambda(\phi^b\phi^b - F^2)\phi^a + g\epsilon^{acd}A^d_\mu(D^\mu\phi)^c, \\
\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)} &= (D^\mu\phi)^a,
\end{aligned}$$

therefore

$$(D_\mu D^\mu \phi)^a = -\lambda(\phi^b\phi^b - F^2)\phi^a. \tag{4.7a}$$

Furthermore

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A^a_\mu} &= -\frac{1}{2}\epsilon^{dac}A^c_\nu G^{d\mu\nu} - \frac{1}{2}g\epsilon^{eca}A^c_\nu G^{e\nu\mu} + g\epsilon^{fac}\phi^c(D^\mu\phi)^f \\
&= g\epsilon^{bac}\phi^c(D^\mu\phi)^b - g\epsilon^{dca}A^c_\nu G^{d\mu\nu}, \\
\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^a_\mu)} &= G^{a\mu\nu},
\end{aligned}$$

yielding

$$D_\nu G^{a\mu\nu} = g\epsilon^{dac}(D^\mu\phi)^d\phi^c. \tag{4.7b}$$

Due to the possibility of making a gauge transformation via (4.4a) and (4.4b), we can always achieve $A^a_0(\mathbf{x},t) = 0$. This special choice of the fields A^a_μ is called Weyl or temporal gauge. If we restrict ourselves to time-independent, finite energy solutions the equations reduce to

$$(D_i G_{ij})^a = g\epsilon^{abc}(D^j\phi)^b\phi^c, \tag{4.8a}$$

$$(D_i D^i\phi)^a = -\lambda(\phi^b\phi^b)\phi^a + \lambda F^2\phi^a, \tag{4.8b}$$

with $i,j = 1,2,3$. The energy of such a field configuration is

$$E = \int \mathrm{d}^3\mathbf{x}\left(\frac{1}{4}G^a_{ij}G^a_{ij} + \frac{1}{2}(D_i\phi)^a(D_i\phi)^a + \frac{1}{4}\lambda(\phi^a\phi^a - F^2)^2\right). \tag{4.9}$$

It reaches its minimum value $E = 0$ if $A^a_i(\mathbf{x}) = 0$, $\phi^a(\mathbf{x})\phi^a(\mathbf{x}) = F^2$ and $(D_i\phi)^a = 0$, i.e. $\partial_i\phi^a = 0$: the gauge fields vanish and the Higgs field takes on its constant vacuum value. Several other solutions related to $A^a_i = 0$ by gauge transformations, but since (4.9) is gauge invariant, all these solutions have $E = 0$, too. There is a degenerate family of $E = 0$ solutions related by a global $SU(2)$ symmetry, for any solution $\phi = \{\phi^a\}$ must have fixed magnitude $|\phi| = F$ but can point in different ($\mathbf{x}$-dependent) directions in internal space.

Solutions with finite energy must approach vacuum configurations at spatial infinity sufficiently fast:

$$\begin{aligned}
r^{3/2}D_i\phi &\to 0, \\
\phi\cdot\phi &\to F^2,
\end{aligned}$$

but ϕ needs not to go to the same direction in internal space when $r \to \infty$. Why? We require the vanishing of the covariant derivative $D_i\phi$ and not the ordinary derivative $\partial_i\phi$. If we express the covariant derivative in spherical polar coordinates, the θ-component reads

$$(D\phi)^a_\theta = \frac{1}{r}\frac{\partial\phi^a}{\partial\theta} + g\epsilon^{abc}A^b_\theta\phi^c. \tag{4.10}$$

This combination must fall off fast enough, $\frac{\partial\phi^a}{\partial\theta}$ needs not vanish as $r \to \infty$ itself. $A_i^a \sim \frac{1}{r}$ for large r is consistent with $E < \infty$, since

$$E \sim \int \mathrm{d}^3\mathbf{x}\, G_{ij}^a G_{ij}^a \sim \int \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi\, \frac{1}{r^4} r^2 \sin\theta \sim \int \frac{\mathrm{d}r}{r^2} < \infty. \tag{4.11}$$

4.2. Topology

Result: different internal directions are allowed for ϕ at spatial infinity whereas the modulus of ϕ is fixed. We can identify the values of ϕ at spatial infinity with the two dimensional sphere S^2_{int} in internal space, since $\phi \cdot \phi = F^2$. Sometimes this is called vacuum-manifold and we can identify $S^2_{\text{int}} \simeq SU(2)/U(1)$. On the other hand the boundary of the three dimensional physical space is a sphere S^2_{phys} with radius ∞. This is in one-to-one correspondence with the topology of the solutions of the ϕ^4 theory, if we replace S^0 by S^2! As before we can draw the conclusion: the requirement $E < \infty$ permits only those field configurations ϕ that are related to nonsingular mappings $S^2_{\text{phys}} \to S^s_{\text{int}}$.

Again we would like to classify all possible solutions. In order to do so we have to borrow some facts from topology. Let $\pi_n(S^m)$ be the n-th homotopy group associated with mappings $S^n \to S^m$ [23, 36]. Each element of this group corresponds to a whole class of functions $S^n \to S^m$, all functions within this class can be continuously deformed into one another. For small integers n and m the homotopy groups are known and tabulated [37]. It turns out that $\pi_2(S^2)$, the group that is relevant for our considerations, is isomorphic to the group of integers,

$$\pi_2(S^2) \simeq \mathbb{Z}. \tag{4.12}$$

I.e. each finite energy solution belongs to a certain class of functions (referred to as a sector). These classes are numbered serially by integers Q_{top}. As in chapter 2 these integers are called topological charges. Q_{top} counts how often S^2_{int} is covered, when S^2_{phys} is traversed once. According to the famous paper of Arafune, Freund and Goebel [38] we can define a conserved current

$$k_\mu = \frac{1}{8\pi}\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}\partial^\nu\hat{\phi}^a\partial^\rho\hat{\phi}^b\partial^\sigma\hat{\phi}^c, \tag{4.13}$$

where

$$\hat{\phi}^a \equiv \frac{\phi^a}{|\phi|}. \tag{4.14}$$

Because of the antisymmetry of $\epsilon_{\mu\nu\rho\sigma}$ we have $\partial^\mu k_\mu = 0$, this conservation therefore follows by construction, not from the dynamics, k_μ is not a Noether current. Associated with k_μ is a conserved charge

$$Q_{\text{top}} = \int \mathrm{d}^3\mathbf{x}\, k_0 = \frac{1}{8\pi}\int_{S^2_{\text{phys}}} \mathrm{d}^2\sigma_i(\epsilon_{ijk}\epsilon_{abc}\hat{\phi}^a\partial_j\hat{\phi}^b\partial_k\hat{\phi}^a), \tag{4.15}$$

which turns out to be exactly our previously defined topological charge. For a detailed analysis and a proof see RAJARAMAN's book [23].
In the $Q_{\text{top}} = 0$ sector ϕ will tend to the same value as $r \to \infty$ in any direction or to some (θ, φ)-dependent value that can be deformed so as to be (θ, φ)-independent. The trivial vacuum solution $\phi^a = \delta^{3a} F$ belongs to $Q_{\text{top}} = 0$.

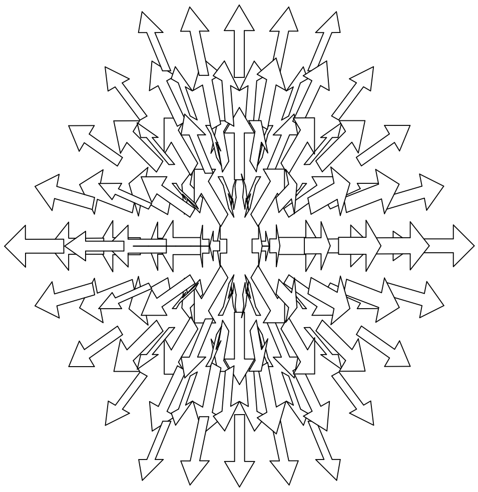

Figure 4.1.: The hedgehog solution.

An example for the $Q_{\text{top}} = 1$ sector is the so called hedgehog solution, cf. Figure 4.1: here ϕ is pointing radially outward, the internal direction of the field is parallel to the coordinate vector.

4.3. Monopoles

Why should we call these solitons magnetic monopoles? To see this, let us first go to electrodynamics. In Maxwell's theory we have the equation of motion

$$\partial_\mu F^{\mu\nu} = 4\pi j^\nu \tag{4.16}$$

and the Bianchi identity

$$\partial^\mu \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} = 0. \tag{4.17}$$

That means, there is an electric current j^ν but no magnetic current j^ν_{mag}. Therefore there are no magnetic monopoles in this theory and the symmetry between electric and magnetic fields somehow is broken. But there is the possibility to introduce magnetic monopoles and a magnetic current by hand into these equations in order to improve the symmetry:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 4\pi j^\nu, \\ \partial_\mu \tilde{F}^{\mu\nu} &= 4\pi j^\nu_{\mathrm{mag}}. \end{aligned} \tag{4.18}$$

The consequences have been studied by DIRAC [39, 40] and SCHWINGER [41]. Quantum theory only permits electric and magnetic charges q and m that fulfill the Dirac quantization condition

$$m \times q = n, \;\; n \in \mathbb{Z}. \tag{4.19}$$

Furthermore a so called Dirac string arises. But these issues will not be discussed here. In non-Abelian $SU(2)$ gauge theory a magnetic current is present without having to alter the Lagrangian or the field equations at all. The Maxwell theory is a theory with a local Abelian $U(1)$ symmetry. This $U(1)$ is a subgroup of our $SU(2)$. Is it possible to imbed an electromagnetic system as part of a richer system? What is the electromagnetic field in this case? Picking A^3_μ as the Maxwell potential is not gauge invariant. 'T HOOFT [42] presented a definition for the electromagnetic field

$$F_{\mu\nu} \equiv \hat{\phi}^a G^a_{\mu\nu} - \frac{1}{g}\epsilon_{abc}\hat{\phi}^a (D_\mu \hat{\phi})^b (D_\nu \hat{\phi})^c, \tag{4.20}$$

which is gauge invariant and in regions where $\hat{\phi}^a = \delta^{a3}$ it reduces to $F_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$. Now we determine the dual of F and its divergence

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\mu\rho\sigma}F^{\rho\sigma}, \tag{4.21}$$

$$\partial^\nu \tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu F^{\rho\sigma} = \frac{1}{2g}\epsilon_{\mu\nu\rho\sigma}\epsilon_{abc}\partial^\nu \hat{\phi}^a \partial^\rho \hat{\phi}^b \partial^\sigma \hat{\phi}^c = \frac{4\pi}{g}k_\mu = 4\pi j^{\mathrm{mag}}_\mu. \tag{4.22}$$

Therefore $\frac{k_\mu}{g}$ is our magnetic current with k_μ being the topological current defined in (4.13). The magnetic field

$$B_i = \frac{1}{2}\epsilon_{ijk}F^{jk} \tag{4.23}$$

has the property

$$\partial_i B_i = \frac{1}{2}\epsilon_{ijk}\partial_i F^{jk} = \frac{4\pi}{g}k_0, \tag{4.24}$$

hence the total magnetic charge is equal to

$$m = \int \mathrm{d}^3\mathbf{x}\, \frac{k_0}{g} = \frac{Q_{\mathrm{top}}}{g}, \tag{4.25}$$

where Q_{top} is the topological charge (4.15).

4.4. The Q_{top}=1 Example of 't Hooft and Polyakov

The previous topological considerations can be done without really solving the equations of motions. This will be the next step. We would like to use symmetry arguments in order to simplify the equations (4.8a) and (4.8b). Our solution shall be invariant under rotations up to gauge transformations, i.e. after a rotation R the fields ϕ and A_i^a are recovered if one makes use of an appropriate global gauge transformation U at the same time. We demand:

$$\begin{aligned} \phi(\mathbf{x}) &= U(R)\phi(R^{-1}\mathbf{x})U^{-1}(R), && (4.26a) \\ \mathbf{A}(\mathbf{x}) &= U(R)R\mathbf{A}(R^{-1}\mathbf{x})U^{-1}(R). && (4.26b) \end{aligned}$$

The most general ansatz obeying this requirement is [43]

$$\begin{aligned} \phi^a(\mathbf{x}) &= \delta_{ia}\frac{x^i}{r}F(r), && (4.27a) \\ A_i^a(\mathbf{x}) &= \epsilon_{aij}\frac{x^j}{r}W(r) + \delta_i^a W_1(r) + x_i x_a W_2(r), && (4.27b) \end{aligned}$$

but in our case this can be reduced to [23]

$$\begin{aligned} \phi^a(\mathbf{x}) &= \delta_{ia}\frac{x^i}{r}F(r), && (4.28a) \\ A_i^a(\mathbf{x}) &= \epsilon_{aij}\frac{x^j}{r}W(r), && (4.28b) \end{aligned}$$

where $F(r)$ and $W(r)$ have to be chosen in such a way, that the field equations are satisfied. With the asymptotics $F(r \to \infty) \to F$ and $W(r \to \infty) \to \frac{1}{gr}$ it matches all earlier requirements including boundary conditions. In the next step we will check that this particular ϕ field belongs to the $Q_{\text{top}} = 1$ sector by calculating the magnetic field at large distances $r \to \infty$. Plugging in our ansatz and the corrensponding asymptotic behaviour of F and W into the equations

$$\begin{aligned} B_i &= \frac{1}{2}\epsilon_{ijk}F^{jk}, && (4.29) \\ F_{ij} &= \hat{\phi}^a\partial_i A_j^a - \hat{\phi}^a\partial_j A_i^a + g\epsilon_{abc}\hat{\phi}^a A_i^b A_j^c - \frac{1}{g}\epsilon_{abc}(D_\mu\hat{\phi})^b(D_\nu\hat{\phi})^c, && (4.30) \end{aligned}$$

yields after a lengthy but straightforward calculation:

$$\begin{aligned} F_{ij} &\to \frac{1}{gr^2}\epsilon_{aji}\hat{x}^a, && (4.31) \\ B_i &\to \frac{1}{2}\epsilon_{ijk}\frac{1}{gr^2}\epsilon_{akj}\hat{x}^a = \frac{1}{gr^2}\hat{x}^i, && (4.32) \end{aligned}$$

in the limit $r \to \infty$. This corresponds to a magnetic monopole of strenght $\frac{1}{g}$ in accordance with $Q_{\text{top}} = 1$. Because $A_0^a = 0$ and all fields are time independent we have $F_{0i} = 0$, therefore no electric field is present. Our solution carries magnetic but no electric charge. We still have to solve the field equations to determine the shape of the functions $F(r)$ and $W(r)$. The particular ansatz reduces them to ordinary differential equations

$$r^2 K''(r) = K(r)(K^2(r)-1) + H^2(r)K(r), \tag{4.33a}$$

$$r^2 H''(r) = 2H(r)K^2(r) + \lambda H(r)(\frac{1}{g^2}H^2(r) - r^2F^2), \tag{4.33b}$$

where

$$K(r) \equiv 1 - grW(r), \tag{4.34a}$$

$$H(r) \equiv grF(r), \tag{4.34b}$$

and prime denotes differentiation with respect to the argument r. This is a set of coupled non-autonomous differential equations. Although much simpler than the parent field equations, these are still not easy to solve. Only in the Bogomolny-Prasad-Sommerfield-limit (BPS-limit) $\lambda \to 0$ the exact solutions are known [44, 45]. In this limit we have

$$K(r) = \frac{rgF}{\sinh(rgF)}, \tag{4.35a}$$

$$H(r) = \frac{rgF}{\tanh(rgF)} - 1. \tag{4.35b}$$

This corresponds to

$$W(r) = \frac{1}{gr} - \frac{F}{\sinh(rgF)}, \tag{4.36a}$$

$$F(r) = \frac{F}{\tanh(rgF)} - \frac{1}{gr}. \tag{4.36b}$$

The shapes of these functions are shown in Figure 4.2, with the special choice of parameters $g = 1$ and $F^{-1} = 1$ unit of length. As it must be, $F(r)$ approaches its vacuum value F for large r, and $W(r)$ goes to zero like $\frac{1}{r}$ in the same limit.
In the BPS-limit, where the potential energy of the Higgs field vanishes with λ, we can deduce a lower bound on the energy [23, 46]. Let us calculate

$$\begin{aligned} E &= \int \mathrm{d}^3\mathbf{x} \left(\frac{1}{4} G^a_{ij} G^a_{ij} + \frac{1}{2} (D_k\phi)^a (D_k\phi)^a \right) \\ &= \frac{1}{4} \int \mathrm{d}^3\mathbf{x} \left(G^a_{ij} - \epsilon_{ijk}(D_k\phi)^a \right)^2 + \underbrace{\frac{1}{2} \int \mathrm{d}^3\mathbf{x}\ \epsilon_{ijk} G^a_{ij} (D_k\phi)^a}_{I} \end{aligned} \tag{4.37}$$

Now we integrate the second integral by parts:

$$\begin{aligned} I &= \frac{1}{2} \int \mathrm{d}^3\mathbf{x}\ \epsilon_{ijk} \partial_k \left(G^a_{ij}\phi^a \right) - \frac{1}{2} \int \mathrm{d}^3\mathbf{x}\ \epsilon_{ijk} \phi^a (D_k G_{ij})^a \\ &= \frac{1}{2} \int \mathrm{d}^3\mathbf{x}\ \epsilon_{ijk} \partial_k \left(G^a_{ij}\phi^a \right) - \int d^3\mathbf{x}\ \phi^a (D_k \tilde{G}_{0k})^a \\ &= \frac{1}{2} \oint_{S^2} \mathrm{d}\sigma_k\ (\epsilon_{kij} G^a_{ij} \phi^a), \end{aligned} \tag{4.38}$$

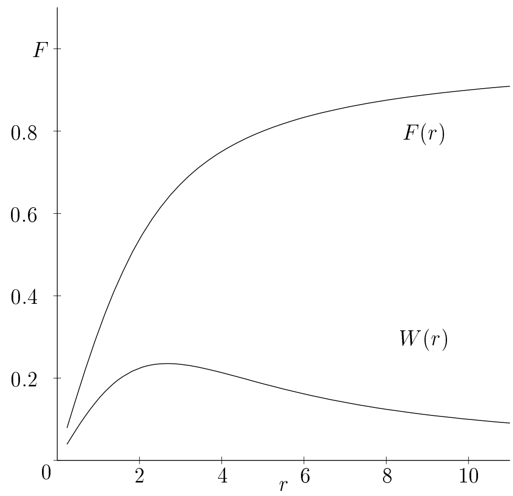

Figure 4.2.: The functions W(r) and F(r).

where we used the Bianchi identity

$$D_\mu \tilde{G}^{\mu\nu} = 0,\ \tilde{G}^{a\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma}. \tag{4.39}$$

The total energy is

$$E = \frac{1}{4}\int \mathrm{d}^3\mathbf{x}\ \left(G^a_{ij} - \epsilon_{ijk} D_k \phi^a\right)^2 + \frac{1}{2}\oint_{S^2} \mathrm{d}\sigma_k\ \left(\epsilon_{kij} G^a_{ij}\phi^a\right), \tag{4.40}$$

and the surface integral can be rewritten again. Consider

$$F_{\mu\nu} = \hat{\phi}^a G^a_{\mu\nu} - \frac{1}{g}\epsilon_{abc}\hat{\phi}^a (D_\mu\hat{\phi})^b (D_\nu\hat{\phi})^c. \tag{4.41}$$

In the limit $r \to \infty$ we have:

$$\begin{aligned} D_\mu \phi^a &\to 0, \\ \hat{\phi}^a &\to \frac{\phi^a}{F}, \\ B_k &= \frac{1}{2}\epsilon_{kij} F_{ij} \to \frac{1}{2F}\epsilon_{kij} G^a_{ij}\phi^a. \end{aligned}$$

Therefore

$$I = F \times \oint_{S^2} \mathrm{d}\sigma_k\ B_k = 4\pi m F = \frac{4\pi Q_{\mathrm{top}} F}{g}, \tag{4.42}$$

thus

$$E = \frac{4\pi Q_{\mathrm{top}} F}{g} + \frac{1}{4}\int \mathrm{d}^3\mathbf{x}\big(G^a_{ij} - \epsilon_{ijk}(D_k\phi)^a\big)^2 \geq \frac{4\pi Q_{\mathrm{top}} F}{g}. \tag{4.43}$$

In any given Q_{top} sector the energy E is minimized if and only if the Bogomolny condition

$$G^a_{ij} = \epsilon_{ijk}(D_k\phi)^a \tag{4.44}$$

is satisfied. If the fields satisfy these equations, then they minimize the static energy in the corresponding Q_{top} sector, therefore they form a classical solution in that sector. We can check that the BPS-solution ((4.36a), (4.36b)) minimizes the energy in the $Q_{\text{top}} = 1$ sector: according to (4.9) the mass of the monopole is

$$M = E = \frac{4\pi F}{g} \times 1 \overset{!}{=} \frac{4\pi F}{g} \times Q_{\text{top}}. \tag{4.45}$$

For $Q_{\text{top}} > 1$ or $\lambda \neq 0$ no explicit solutions are available so far [47]. Numerical work and arguments given by 't Hooft [42] and Polyakov [48] in their original papers indicate that nonsingular, finite energy solutions exist also for $Q_{\text{top}} = 1$ and $\lambda \neq 0$. For explicit calculations we have to restrict ourselves to the monopole field with magnetic charge $Q_{\text{top}} = 1$.

4.5. Fermionic Quantization

Now that we identified the 't Hooft-Polyakov monopoles as particular topologically interesting solutions of the Yang-Mills-Higgs equations of motion ((4.8a), (4.8b)), we will analyse fermions moving in the background of such monopoles, as we did in the ϕ^4 theory. Due to lack of analytical solutions for higher charges Q_{top} we will restrict ourselves to this explicit example, closely following the calculations of Jackiw and Rebbi [26]. Again we use a Yukawa like coupling and can interpret the soliton field as space-dependent mass. We start with the Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{YMH}} + \mathcal{L}_\psi, \tag{4.46}$$

where

$$\begin{aligned} \mathcal{L}_{\text{YMH}} &= -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \frac{1}{2}(D_\mu\phi)^a(D^\mu\phi)^a - \frac{1}{4}\lambda(\phi^a\phi^a - F^2)^2, & (4.47) \\ \mathcal{L}_\psi &= i\bar{\psi}_n\gamma^\mu(D_\mu\psi)_n - gG\bar{\psi}_n T^a_{nm}\psi_m\phi^a. & (4.48) \end{aligned}$$

The covariant derivative acts on spinors ψ as follows:

$$(D_\mu\psi)_n = \partial_\mu\psi_n - igA^a_\mu T^a_{nm}\psi_m. \tag{4.49}$$

Here g is the dimensionless Yang-Mills coupling constant, G characterizes the strength of the Yukawa coupling and is dimensionless, too. F is the vacuum expectation value of the Higgs field $\phi = \{\phi^a\}$ as before. The matrices T^a, $a = 1, 2, 3$, characterize the transformation properties of the fermions with respect to $SU(2)$ isospin rotations. We have

$$\begin{aligned} \delta^a\psi_n &= iT^a_{nm}\psi_m, \\ [T^a, T^b] &= i\epsilon^{abc}T^c. \end{aligned} \tag{4.50}$$

Currently we are interested in the fundamental and adjoint representation, with $T^a_{nm} = \frac{1}{2}\sigma^a_{nm}$ and $T^a_{nm} = i\epsilon_{nam}$ respectively. The Dirac equation in the external potential of the 't Hooft-Polyakov monopole is

$$\left[\boldsymbol{\alpha}\cdot\mathbf{p}\delta_{nm} + gW(r)T^a_{nm}(\boldsymbol{\alpha}\times\hat{\mathbf{r}})^a + gGF(r)T^a_{nm}\hat{r}^a\beta\right]\psi_m = E\psi_n. \tag{4.51}$$

Rewrite the spinor in components

$$\psi_n = \begin{pmatrix} \chi^+_n \\ \chi^-_n \end{pmatrix}, \tag{4.52}$$

then equation (4.51) becomes

$$H_{nm}\chi^\pm_m = \left[\boldsymbol{\sigma}\cdot\mathbf{p}\delta_{nm} + gW(r)T^a_{nm}(\boldsymbol{\sigma}\times\hat{\mathbf{r}})^a \pm igGF(r)T^a_{nm}\hat{r}^a\right]\chi^\pm_m = E\chi^\mp_n, \tag{4.53}$$

since we have chosen the following representation of the Dirac matrices:

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0, \end{bmatrix}, \beta = -i\begin{bmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{bmatrix}. \tag{4.54}$$

This is a quite unusual representation, but suitable for the application of the CALLIAS-BOTT-SEELEY index theorem [34, 35], as we will see afterwards[1].

The operator $\mathbf{J} = \mathbf{j} + \mathbf{I} = \mathbf{l} + \mathbf{s} + \mathbf{I}$, the sum of orbital momentum, spin and isospin commutes with the Hamiltonian H in (4.53). The operators are explicitly given by

$$l_i = \frac{1}{i}\epsilon_{ijk}x_j\partial_k, \tag{4.55a}$$

$$s_i = \frac{1}{2}\sigma_i, \tag{4.55b}$$

$$(I^i)_{nm} = \begin{cases} \frac{1}{2}(\sigma^i)_{nm} & \text{for isospinor fermion fields and} \\ i\epsilon_{imn} & \text{for isovector fermion fields} \end{cases}. \tag{4.55c}$$

The conservation of the total angular momentum follows from the spherical symmetry of the background field, cf. [49] and can be checked by a lengthy calculation.

4.5.1. Isospinor Fermion Fields

With isospinor fermion fields the Dirac equation may be written

$$\begin{aligned} E\chi^\mp_{in} &= (\boldsymbol{\sigma}\cdot\mathbf{p})_{ij}\chi^\pm_{jn} + \frac{1}{2}gW(r)(\boldsymbol{\sigma}\times\hat{\mathbf{r}})^a_{ij}\sigma^a_{nm}\chi^\pm_{jm} \pm \frac{1}{2}igGF(r)\sigma^a_{nm}\hat{r}^a\chi^\pm_{im} \\ &= (\boldsymbol{\sigma}\cdot\mathbf{p})_{ij}\chi^\pm_{jn} + \frac{1}{2}gW(r)(\boldsymbol{\sigma}\times\hat{\mathbf{r}})^a_{ij}\chi^\pm_{jm}((\sigma^a)^{\mathrm{T}})_{mn} \pm \frac{1}{2}igGF(r)\chi^\pm_{im}((\sigma^a)^{\mathrm{T}})_{mn}\hat{r}^a. \end{aligned}$$

Upon defining 2×2 matrices $\mathcal{M}^\pm$ by

$$\chi^\pm_{in} = \mathcal{M}^\pm_{im}\sigma^2_{mn} \tag{4.56}$$

[1] cf. appendix A

and using $\sigma^2\boldsymbol{\sigma}^\top = -\boldsymbol{\sigma}\sigma^2$, one obtains for $\mathcal{M}^\pm$ the matrix equation

$$\boldsymbol{\sigma}\cdot\mathbf{p}\mathcal{M}^\pm - \frac{1}{2}gW(r)(\boldsymbol{\sigma}\times\hat{\mathbf{r}})^a\mathcal{M}^\pm\sigma^a \mp \frac{1}{2}igGF(r)\mathcal{M}^\pm\sigma^a\hat{r}^a = E\mathcal{M}^\mp. \tag{4.57}$$

Now we expand $\mathcal{M}^\pm$ in terms of two scalar and two vector functions (writing them as a sum over the identity and Pauli matrices):

$$\mathcal{M}^\pm_{im}(\mathbf{r}) = g^\pm(\mathbf{r})\delta_{im} + g^\pm_a(\mathbf{r})\sigma^a_{im}. \tag{4.58}$$

The equation (4.57) is then equivalent to the following two equations

$$\begin{aligned}(\partial_a - gW(r)\hat{r}^a \pm \frac{1}{2}gGF(r)\hat{r}^a)g^\pm + i\epsilon_{abc}(\partial_b \mp \frac{1}{2}gGF(r)\hat{r}^b)g^\pm_c &= iEg^\mp_a,\\ (\partial_a + gW(r)\hat{r}^a \pm \frac{1}{2}gGF(r)\hat{r}^a)g^\pm_a &= iEg^\mp. \end{aligned}\tag{4.59}$$

Now we show how the existence of zero-energy solutions can be investigated directly from (4.59). Let us multiply the first equation with $(\partial_a \mp \frac{1}{2}gGF(r)\hat{r}^a)$ and set $E = 0$:

$$(\partial_a \mp \frac{1}{2}gGF(r)\hat{r}^a)(\partial_a - gW(r)\hat{r}^a \pm \frac{1}{2}gGF(r)\hat{r}^a)g^\pm = 0. \tag{4.60}$$

In order to simplify this, define

$$g^\pm(\mathbf{r}) = \exp\left[\frac{1}{2}g\int_0^r \mathrm{d}r'\, W(r')\right]\tilde{g}^\pm(\mathbf{r}), \tag{4.61}$$

then (4.60) takes the form

$$0 = K^{a\dagger}_\pm K^a_\pm \tilde{g}^\pm, \tag{4.62}$$

$$K^a_\pm = p_a + \frac{1}{2}igW(r)\hat{r}^a \mp igGF(r)\hat{r}^a. \tag{4.63}$$

But the operators $K^{a\dagger}_\pm K^a_\pm$ (no sum) are non-negative; it follows that any solution to (4.62) must satisfy

$$K^a_\pm \tilde{g}^\pm = 0, \tag{4.64}$$

which implies

$$\tilde{g}^\pm(\mathbf{r}) = \mathcal{N}^\pm \exp\left[\frac{1}{2}g\int_0^r \mathrm{d}r'\,\big(W(r') \mp GF(r')\big)\right]. \tag{4.65}$$

Since $\tilde{g}^-(\mathbf{r})$ increases exponentially for $r\to\infty$, we must have $\mathcal{N}^- = 0$. Substituting the solution into (4.61) we find

$$g^+(\mathbf{r}) = \mathcal{N}^+ \exp\left[g\int_0^r \mathrm{d}r'\,\big(W(r') - \frac{1}{2}GF(r')\big)\right] \sim Y_{00}. \tag{4.66}$$

In the full partial wave analysis of the problem [26] this corresponds to the $J = 0$ partial wave sector, as it should by symmetry arguments. Therefore our solution to the zero energy equation is

$$\mathcal{M}^+_{im}(\mathbf{r}) \;=\; g^+(\mathbf{r})\delta_{im} = \mathcal{N}^+ \exp\left[g\int_0^r \mathrm{d}r' \, \left(W(r') - \frac{1}{2}GF(r')\right)\right]\delta_{im}. \tag{4.67}$$

Our zero mode wave function is of the form

$$\begin{aligned}\chi^+_{in} &= \mathcal{M}^+_{im}\sigma^2_{mn} = \mathcal{N}^+ \exp\left[g\int_0^r \mathrm{d}r' \, [W(r') - \frac{1}{2}GF(r')]\right]\sigma^2_{in} \\ &= \mathcal{N}^+ \exp\left[-g\int_0^r \mathrm{d}r' \, [\frac{1}{2}GF(r') - W(r')]\right] \times \{s^+_i s^-_n - s^-_i s^+_n\},\end{aligned} \tag{4.68}$$

where

$$s^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, s^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.69}$$

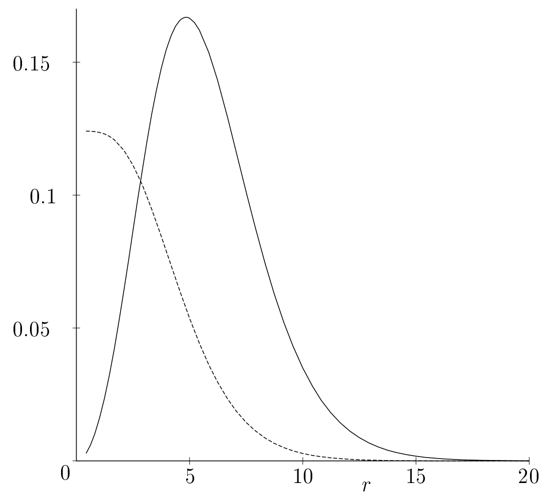

Figure 4.3.: The zero mode profile (dashed line) and its density distribution.

The label i refers to Dirac indices and n to the isospin components. Spin and isospin form an antisymmetric singlet. The degrees of freedom of the spontaneously broken isospin symmetry survive as spin degrees of freedom, and couple to Dirac spin ('spin from isospin', cf. [50]). The radial profile of the zero mode and its density distribution are shown in Figure 4.3, here the specific choice of parameters is $g = G = 1$ and $F^{-1} = 1$ unit of length. Fermion number conjugation is realized by

$$\psi^C_n = \begin{bmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{bmatrix} \sigma^2_{nm}\psi^*_m. \tag{4.70}$$

Our zero mode is fermion number self conjugate, since conjugation simply reverses the sign of the energy in (4.53).

4.5.2. Isovector Fermion Fields

In the isovector example we have $T^a_{nm} = i\epsilon_{nam}$ and n, m take on values $1, 2, 3$. The Dirac equation is of the form

$$\Big((\boldsymbol{\sigma}\cdot\mathbf{p})\delta_{nm} - gW(r)\hat{r}^n\sigma^m + gW(r)\sigma^n\hat{r}^m \mp igGF(r)\epsilon_{nam}\hat{r}^a\Big)\chi^{\pm}_m = iE\chi^{\mp}_n. \quad (4.71)$$

Now we are looking for zero modes $E = 0$ and basically have to repeat the former analysis. Now the equations are more complicated, so the results can not be given in closed analytic form. Jackiw and Rebbi [26] again applied a partial wave decomposition and showed that there are no zero modes for total angular momentum $J > \frac{1}{2}$. However, for $J = \frac{1}{2}$, two linear independent zero modes occur. They have the following form: the lower components vanish as in the isospinor case, and the upper component reads

$$\chi^+_n = \mathcal{N}\Big[f_2(r)\sigma^n + (f_1(r) - f_2(r))\hat{r}^n\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\Big]\chi, \quad (4.72)$$

where either $\chi = s^+$ or $\chi = s^-$, cf. (4.69). $f_1(r)$ and $f_2(r)$ are determined as follows. Let us consider the exponentially decreasing, nonasymptotic part of $W(r)$:

$$\rho(r) \equiv \frac{1}{r} - gW(r), \quad (4.73)$$

and define

$$H(r) = \frac{1}{2}\Big[gGF(r) - \frac{\rho'(r)}{\rho(r)} - \frac{1}{r}\Big]. \quad (4.74)$$

$H(r)$ vanishes at $r = 0$ and tends to a positive constant for large r. Now solve the differential equation

$$-u''(r) + (H^2(r) + H'(r) + 2\rho^2(r))u(r) = 0, \quad (4.75)$$

for $u(r)$ and take the solution that is regular at the origin. The functions $f_1(r)$ and $f_2(r)$ are given in terms of $u(r)$ [26]:

$$f_1(r) = \frac{1}{r^2}u(r)\exp\Big[-\int_0^r \mathrm{d}r'\, H(r')\Big], \quad (4.76)$$

$$f_2(r) = \frac{1}{2r^2\rho(r)}\frac{d}{dr}\big(r^2 f_1(r)\big). \quad (4.77)$$

f_2 increases exponentially, whereas f_1 goes to zero like r^{-2}. Jackiw and Rebbi [26] showed that by construction these spinors are zero energy solutions of the Dirac equation.

The twofold degeneracy of the isovector solution indicates that the solution has spin $\frac{1}{2}$. With our choice of Dirac matrices (4.54), fermion number conjugation is realized by

$$\psi_n^C = \begin{bmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{bmatrix} \psi_n^*. \tag{4.78}$$

As before the only effect of this conjugation, applied to our Dirac equation (4.53), is a change of sign in the energy. Therefore our zero modes are fermion number self-conjugate.

4.6. Index Theorem

We can apply the CALLIAS-BOTT-SEELEY index theorem [34, 35] to our monopole background field, too[2]. For massless fermions, isospinor $T = \frac{1}{2}$ and isovector $T = 1$ case, we get for monopoles within the Q_{top} sector the following results. The index of the operator L, which is constructed out of the Hamiltonian H, the difference in number of left- and right-handed zero modes, is given by (A.50):

$$\text{index } L = \big(T(T+1) - \{m\}(\{m\}+1)\big) Q_{\text{top}}.$$

In the isospinor case

$$T = \frac{1}{2},\ \{m\} = -\frac{1}{2},\ Q_{\text{top}} = 1,\ \text{index } L = \frac{1}{2}\frac{3}{2} + \frac{1}{2}\frac{1}{2} = 1,$$

and indeed we found one left-handed normalizable zero mode (and no right-handed one). In the isovector case

$$T = 1,\ \{m\} = 0,\ Q_{\text{top}} = 1,\ \text{index } L = 1 \cdot 2 - 0 = 2,$$

again in agreement with our explicit results.

Remark: The same calculations can be carried out with mass term $m\bar{\psi}_n\psi_n$ in the Lagrangian. Now the existence of zero modes depends on the relation of the coupling constants. Zero modes are present, if the mass is sufficiently small, $m < gGF$. This can be checked explicitly [51] and on the other hand is contained in the general form of the index theorem [34].

Furthermore the index theorem can be used to determine the number of parameters needed to completely describe a monopole: according to WEINBERG [52] the dimension of the moduli space of a given monopole configuration with charge Q_{top} is equal to twice the number of zero energy modes of fermions in the adjoint representation. Therefore this configuration belongs to a $4Q_{\text{top}} - 1$ parameter family of solutions (after subtraction of an overall charge rotation, which is of no physical significance).

[2] For definitions and detailed calculations see Appendix A.

4.7. Some Remarks on the Julia-Zee Dyon

JULIA and ZEE [53] recognized, that there are also dyons, i.e. electrically and magnetically charged soliton solutions within this model. Instead of $A_0^a = 0$ one takes

$$A_0^a(\mathbf{x}) = \frac{x^a}{gr^2} J(r), \tag{4.79}$$

with $J(r) \to 0$ as $r \to 0$. Now the field equations read

$$\begin{aligned} r^2 K''(r) &= K(r)(K^2(r) - J^2(r) + H^2(r) - 1), && (4.80a)\\ r^2 H''(r) &= 2H(r)K^2(r) + \lambda H(r)(\frac{1}{g^2}H^2(r) - r^2F^2), && (4.80b)\\ r^2 J''(r) &= 2J(r)K^2(r). && (4.80c) \end{aligned}$$

Again these equations can be solved in the BPS-limit $\lambda \to 0$ only. The solutions are [45]

$$\begin{aligned} K(r) &= \frac{rgF}{\sinh(rgF)}, && (4.81a)\\ H(r) &= \cosh\gamma\left(\frac{rgF}{\tanh(rgF)} - 1\right), && (4.81b)\\ J(r) &= \sinh\gamma\left(\frac{rgF}{\tanh(rgF)} - 1\right), && (4.81c) \end{aligned}$$

with an arbitrary real constant γ. The electric charge is

$$q = \int \mathrm{d}^3\mathbf{x}\, \partial_i E^i = -\frac{8\pi}{g}\int_0^\infty \mathrm{d}r\, \frac{J(r)K^2(r)}{r} = \frac{4\pi}{g}\sinh\gamma. \tag{4.82}$$

Nevertheless the asymptotic magnetic field is the same and

$$m = \frac{1}{g}. \tag{4.83}$$

This configuration reduces to the 't Hooft-Polyakov monopole in the limit $\gamma \to 0$. Now we can analyze the properties of fermions within the dyon background as well. This was also done by JACKIW and REBBI [26]. The main results are the following: the Dirac equation (4.53) now acquires on the right-hand side an additional term $T^a_{nm}\hat{r}^a\frac{J(r)}{r}\psi_m$. The complexity of the equations prevents us from solving them explicitly. However, the zero-energy solutions continue to exist, for both isospinor and isovector fermions. The lower components no longer vanish but the upper one keep their shape. Fermion number conjugation remains unaffected by this and the zero energy solutions are self-conjugate. The explicit construction of the zero modes is given in the paper by GONZALEZ-ARROYO and SIMONOV [54].

4.8. Quantum Interpretation

A full quantum field theoretical treatment of the fermion-monopole system is quite difficult [55, 23]. But in analogy to the kink case we can deduce the following properties: the Hilbert space consists of trivial parts, constructed around the vaccum solutions of the Yang-Mills-Higgs equations of motion, and nontrivial parts, constructed around monopoles of charge $Q_{\text{top}} = \pm 1, \pm 2, \ldots$. The 't Hooft-Polyakov monopole of charge $Q_{\text{top}} = 1$ as well as the corresponding dyon change their properties if fermions are present. The monopole becomes a degenerate doublet with fermion number $\pm\frac{1}{2}$. The solitons are spinless, since no spin degree of freedom is found in the classical solution. In the isovector case we find a fourfold degeneracy, because now an additional spin-$\frac{1}{2}$ degree of freedom is present. Therefore we expect to find two operators a_s with $s = \pm\frac{1}{2}$. The basic feature, that the anticommutation relation $\{a_s, a_s^\dagger\} = 1$ for each s requires two states $|\pm\rangle$ carrying fermion number $n = \pm\frac{1}{2}$ remains true also in this case. But since we have now two independent pairs of operators, the soliton states will be product vectors of the form $|i\rangle\,|i'\rangle$ with fermion numbers $+1$ for $|+\rangle\,|+\rangle$, -1 for $|-\rangle\,|-\rangle$ and 0 for $|+\rangle\,|-\rangle$ and $|-\rangle\,|+\rangle$. Thus there are four degenerate soliton states.

4.9. Results

We analysed the $SU(2)$ Yang-Mills-Higgs equations of motion and were able to classify all solutions of the corresponding field equations according to their topological charges Q_{top}. Spherical symmetry allows for an analytical description of the $Q_{\text{top}} = 1$-'t Hooft-Polyakov monopole in the so called BPS-limit. In the background field of such a monopole the Dirac equation for fermions in the fundamental and adjoint representation exhibits one or two zero energy modes, respectively. This leads in close analogy to the ϕ^4 theory to fermion number fractionization, as well as to a degeneracy of the fermion-monopole states. All these explicit results are again in agreement with the Callias-Bott-Seeley index theorem. Furthermore the analysis can be extended to dyons with $Q_{\text{top}} = 1$ and arbitrary electric charge.

5. Instanton Fields

5.1. Euclidean Yang-Mills Theory in $\mathbb{R}^4$

In this chapter we are going to analyse the Euclidean Dirac equation

$$\not{D}\psi = \gamma_\mu(\partial_\mu + A_\mu)\psi = 0, \tag{5.1}$$

in the background of instantons. What are instantons? Instantons are localized finite-action solutions of the classical euclidean field equations of a given theory. In the following sections we will discuss the properties of instantons of pure $SU(2)$ gauge theory in Euclidean four-space. First we are going to describe in detail the model under consideration, then we will classify all possible solutions and finally derive the explicit form of the instanton background fields. Afterwards we analyse how fermions behave in such fields, discuss the zero modes and relate our results again to an important mathematical theorem, the ATIYAH-SINGER index theorem[1].

The Euclidean version of a theory involves replacing the Minkowskian metric $g_{\mu\nu}$ by the Euclidean metric $\delta_{\mu\nu}$. The spacetime vector $(x^\mu)_{\mathrm{Mink}}$ is replaced by $(x_\mu)_{\mathrm{Eucl}}$. Now the theory is left invariant under $O(4)$ rotations rather than Lorentz transformations. Obviously there is no difference between upper and lower components and in what follows we will use only the latter. The requirement of finite energy now is replaced by the demand for finite Euclidean action. Pure $SU(2)$ gauge theory means that - in contrast with the Yang-Mills-Higgs theory - there are no Higgs fields present and the Lagrangian reduces to

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}. \tag{5.2}$$

For the succeeding it is very convenient to choose the gauge field matrices A_μ to be anti-Hermitean and to absorb the coupling constant in the fields. Now g will only appear as a prefactor in the action S. The value of g is unimportant for our classical calculations. We need to take care of it only in the context of quantum theory, where absolute values of S (in units of $\hbar$) play a fundamental role for the calculation, e.g. of transition amplitudes. Let

$$A_\mu = g\frac{\sigma^a}{2i}A^a_\mu, \tag{5.3}$$

[1] cf. Appendix B.

where the generators satisfy

$$\left[\frac{\sigma^a}{2i}, \frac{\sigma^b}{2i}\right] = \epsilon^{abc}\frac{\sigma^c}{2i}. \tag{5.4}$$

The Euclidean action of a given field configuration A_μ is

$$S = \frac{1}{4g^2}\int \mathrm{d}^4x\, G^a_{\mu\nu}G^a_{\mu\nu} = -\frac{1}{2g^2}\int \mathrm{d}^4x\, \mathrm{tr}\,\big(G_{\mu\nu}G_{\mu\nu}\big), \tag{5.5}$$

with this definition S is non-negative. Independent of the choosen gauge (remember the gauge freedom described in chapter 4, the same holds here) we can define the zero action configurations. They are given by

$$G_{\mu\nu}(x) = 0. \tag{5.6}$$

This is realized by

$$A_\mu(x) = 0, \tag{5.7}$$

but while (5.6) is a gauge invariant statement, (5.7) is not. With $A_\mu = 0$ also the gauge transformed field

$$A'_\mu(x) = U(x)(A_\mu(x) + \partial_\mu)U^{-1}(x) = U(x)\partial_\mu U(x)^{-1} \tag{5.8}$$

describes a zero action configuration. Fields of the form (5.8) are called pure gauges. Here $U(x)$ is any element of $SU(2)$ in its 2×2 matrix representation. One can show that $G_{\mu\nu} = 0$ if and only if A_μ is of the form (5.8) [23].

Finite-action solutions must approach such a pure gauge configuration sufficiently fast at spatial infinity. In fact $G_{\mu\nu}$ must fall to zero faster than $\frac{1}{r^2}$, where

$$r^2 = x_\mu x_\mu = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

is the radius in four dimensions. This implies the boundary conditions

$$\lim_{r\to\infty} A_\mu(x) \sim U(x)\partial_\mu U^{-1}(x), \tag{5.9}$$

and we can assign to every finite-action configuration A_μ an $SU(2)$ valued function U at spatial infinity. Spatial infinity corresponds to a three-dimensional sphere with radius $r = \infty$ and is called S^3_{phys}.

Since U depends only on the Euler angles α_1, α_2 and α_3 of S^3_{phys} we cannot define a radial derivative of U, whereas $A_\mu(x)$ may have a nonvanishing radial component at infinity. We can overcome this difficulty by making a gauge transformation, such that the radial component vanishes identically everywhere. Suppose $A_r \neq 0$ and let us construct the gauge function

$$\tilde{U}(x) = \mathcal{P}\bigg(\exp\int_0^r \mathrm{d}r' A_r(x')\bigg), \tag{5.10}$$

where $\mathcal{P}$ denotes path ordering. Now calculate the radial component of the gauge transformed field

$$\begin{aligned} A'_r(x) &= \tilde{U}(x)A_r(x)\tilde{U}^{-1}(x) + \tilde{U}(x)\partial_r\tilde{U}(x)^{-1}, \\ &= \tilde{U}(x)(A_r(x) - A_r(x))\tilde{U}(x)^{-1} = 0. \end{aligned} \tag{5.11}$$

Hence we can rewrite the boundary condition (5.9)

$$A'_\mu(x)\Big|_{S^3_{\mathrm{phys}}} = U(\alpha_1, \alpha_2, \alpha_3)\partial_\mu U^{-1}(\alpha_1, \alpha_2, \alpha_3). \tag{5.12}$$

This enables us to make a homotopy classification[2]. The gauge functions U provide mappings from the boundary of Euclidean four-space S^3_{phys} into the group space of $SU(2)$ which is known to be isomorph to a three dimensional sphere in internal space, since every matrix U in the defining representation of $SU(2)$ can be parametrized by $U = i(a_1\sigma^1 + a_2\sigma^2 + a_3\sigma^3) + a_4\mathbb{1}$, with $\sum_\mu a_\mu a_\mu = 1$. That means

$$U : S^3_{\mathrm{phys}} \to S^3_{\mathrm{int}}, \tag{5.13}$$

and again we refer to topology and borrow the following two facts: first, the third homotopy group of the target sphere S^3 is isomorph to the group of integers $\mathbb{Z}$,

$$\pi_3(S^3) \simeq \mathbb{Z}, \tag{5.14}$$

all functions U can be classified according to their topologial charge, which in this context is called Pontryagin index, $Q_{\mathrm{Pont}} \in \mathbb{Z}$. With this we can also classify all finite-action solutions, according to their behaviour at infinity. Second, this topological charge for a given field configuration A_μ can be calculated via the formula

$$Q_{\mathrm{Pont}} = \int \mathrm{d}^4x\, Q_{\mathrm{Pont}}(x) = -\frac{1}{16\pi^2}\int \mathrm{d}^4x \;\mathrm{tr}\,\big(\tilde{G}_{\mu\nu}G_{\mu\nu}\big), \tag{5.15}$$

where the dual field strength is defined as in chapter 4,

$$\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G_{\rho\sigma}. \tag{5.16}$$

[2]Furthermore the boundary conditions allow for an effective compactification $\mathbb{R}^4 \to S^4$. This will turn out to be important, since the ATIYAH-SINGER index theorem, which we are going to discuss afterwards, is applicable only in the case of compact manifolds.

$Q_{\rm Pont}(x)$ can be rewritten in the following way

$$\begin{aligned}
\text{tr } G_{\mu\nu}\tilde{G}_{\mu\nu} &= \text{tr } \Big((\partial_\mu A_\nu - \partial_\nu A_\mu)\tilde{G}_{\mu\nu} + (A_\mu A_\nu - A_\nu A_\mu)\tilde{G}_{\mu\nu} \Big) \\
&= \text{tr } \Big((\partial_\mu A_\nu - \partial_\nu A_\mu)\tilde{G}_{\mu\nu} + A_\mu [A_\nu, \tilde{G}_{\mu\nu}] \Big) \\
&= \text{tr } \Big((\partial_\mu A_\nu - \partial_\nu A_\mu)\tilde{G}_{\mu\nu} - A_\mu \partial_\nu \tilde{G}_{\mu\nu} \Big) \\
&= \text{tr } \epsilon_{\mu\nu\alpha\beta} \Big((\partial_\mu A_\nu)(\partial_\alpha A_\beta + A_\alpha A_\beta) - \partial_\nu (A_\mu \partial_\alpha A_\beta + A_\mu A_\alpha A_\beta) \Big) \\
&= \text{tr } \epsilon_{\mu\nu\alpha\beta} 2\partial_\mu \Big(A_\nu \partial_\alpha A_\beta + \frac{2}{3} A_\nu A_\alpha A_\beta \Big), \qquad (5.17)
\end{aligned}$$

where we used $D_\mu \tilde{G}_{\mu\nu} = 0$ and

$$\text{tr } \epsilon_{\mu\nu\alpha\beta}(\partial_\mu A_\nu) A_\alpha A_\beta = \frac{1}{3} \text{tr } \epsilon_{\mu\nu\alpha\beta} \partial_\mu (A_\nu A_\alpha A_\beta).$$

Finally

$$\begin{aligned}
Q_{\rm Pont}(x) &= \partial_\mu k_\mu, \qquad (5.18) \\
k_\mu &= -\frac{1}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr } A_\nu \Big(\partial_\rho A_\sigma + \frac{2}{3} A_\rho A_\sigma \Big). \qquad (5.19)
\end{aligned}$$

In regular gauge (i.e. no singularities in the interior) we can use Stokes theorem to get

$$Q_{\rm Pont} = \int \mathrm{d}^4 x \; Q_{\rm Pont}(x) = \oint_{S^3_{phys}} \mathrm{d}\sigma_\mu k_\mu. \qquad (5.20)$$

On the surface at infinity we have $G_{\mu\nu} = 0$, therefore

$$0 = \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma} = 2\epsilon_{\mu\nu\rho\sigma}(\partial_\rho A_\sigma + A_\rho A_\sigma), \qquad (5.21)$$

leading to

$$\begin{aligned}
Q_{\rm Pont} &= \frac{1}{24\pi^2} \oint_{S^3_{phys}} \mathrm{d}\sigma_\mu \epsilon_{\mu\nu\rho\sigma} \text{tr } (A_\nu A_\rho A_\sigma) \\
&= \frac{1}{24\pi^2} \oint \mathrm{d}\sigma_\mu \epsilon_{\mu\nu\rho\sigma} \text{tr } \Big(U(\partial_\nu U^{-1}) U(\partial_\rho U^{-1}) U(\partial_\sigma U^{-1}) \Big). \qquad (5.22)
\end{aligned}$$

A conceptional proof that $Q_{\rm Pont}$ really counts, how often the target space $S^3_{\rm int}$ is covered when the basis space, i.e. the boundary $S^3_{\rm phys}$, is traversed once, is given in [23].

Remark: we can distort the boundary S^3 into a large cylinder with spacelike hypersurfaces $\mathbb{R}^3$ corresponding to the coordinates x_i, $i = 1, 2, 3$, cf. Figure 5.1. In Weyl gauge, $A_4 = 0$, there are only contributions from the abutting faces and the topological

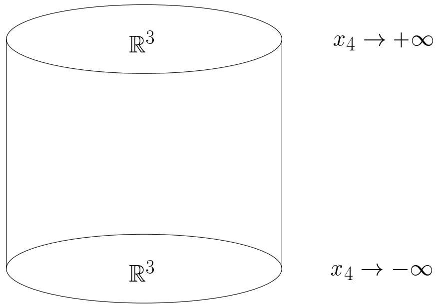

Figure 5.1.: Boundary of space.

charge can be calculated as the difference of two winding numbers (Chern-Simons numbers) of the gauge field configuration at $x_4 = -\infty$ and $x_4 = +\infty$, respectively.

$$Q_{\text{Pont}} = Q_{\text{CS}}(+\infty) - Q_{\text{CS}}(-\infty), \tag{5.23}$$

where

$$Q_{\text{CS}} = \frac{1}{16\pi^2} \int \mathrm{d}^3x \epsilon_{ijk} \mathrm{tr} \left(A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k \right). \tag{5.24}$$

In this language a more suggestive interpretation is possible: field configurations with Pontryagin index Q_{Pont} start at a certain field configuration $A_i(-\infty)$ with Chern-Simons number $Q_{\text{CS}}(-\infty)$. As the Euclidean *time* x_4 goes by, the gauge fields evolve and end up at a different configuration $A_i(+\infty)$, now with Chern-Simons number $Q_{\text{CS}}(+\infty) = Q_{\text{CS}}(-\infty) + Q_{\text{Pont}}$. In order for the whole gauge field to have finite action, both configurations $A_i(\pm\infty)$ have to be pure gauges. This field configurations, reinterpreted as tunneling events in Minkowski space, are called instantons. Now we are going to derive the explicit form of these instantons. Like in the case of the 't Hooft-Polyakov monopole we use a trick to solve the highly nontrivial equations of motion.

5.2. Instanton Configurations

Consider the inequality

$$-\int \mathrm{d}^4x \, \mathrm{tr} \left((G_{\mu\nu} \pm \tilde{G}_{\mu\nu})^2 \right) \geq 0. \tag{5.25}$$

With

$$\mathrm{tr} \, (G_{\mu\nu} G_{\mu\nu}) = \mathrm{tr} \, (\tilde{G}_{\mu\nu} \tilde{G}_{\mu\nu}) \tag{5.26}$$

this is equivalent to

$$-\int \mathrm{d}^4x \,\mathrm{tr}\,(G_{\mu\nu}G_{\mu\nu}) \geq \pm \int \mathrm{d}^4x \,\mathrm{tr}\,(G_{\mu\nu}\tilde{G}_{\mu\nu}), \tag{5.27}$$

that is

$$S \geq \frac{8\pi^2}{g^2}|Q_{\mathrm{Pont}}|. \tag{5.28}$$

The field equations are derived from the action principle $\delta S[A_\mu] = 0$. This variation can be done separately in every homotopy sector. S reaches its absolute minimum

$$S = \frac{8\pi^2}{g^2}|Q_{\mathrm{Pont}}| \tag{5.29}$$

if and only if

$$\tilde{G}_{\mu\nu} = \pm G_{\mu\nu}. \tag{5.30}$$

Field configurations, that satisfy (5.30) are called selfdual fields or anti-selfdual fields, respectively. Every (anti-)selfdual field configuration is a solution of the equations of motion, since they minimize the action S. It is much easier to find a solution of the duality equations than to solve the equations of motion.

Remark: observe that in the (anti-)selfdual case action S and topological charge Q_{Pont} coincide up to a factor. According to the review article by Schäfer and Shuryak [56] all solutions of the equations of motion that are neither selfdual nor anti-selfdual are just saddle-points and not extrema of the action.

Following the book by Rajaraman [23], we make the ansatz

$$A_\mu(x) = i\bar{\Sigma}_{\mu\nu}\partial_\nu \log\rho(x) \tag{5.31}$$

where

$$\bar{\Sigma}_{\mu\nu} \equiv \frac{1}{2}\bar{\eta}_{a\mu\nu}\sigma^a,$$

and

$$\bar{\eta}_{a\mu\nu} \equiv \epsilon_{a\mu\nu} - \delta_{a\mu}\delta_{\nu 4} + \delta_{a\nu}\delta_{\mu 4}$$

are the so-called 't Hooft symbols. Let us calculate the field strength and its dual

$$\begin{aligned} G_{\mu\nu} &= i\bar{\Sigma}_{\nu\sigma}(\partial_\mu\partial_\sigma \log\rho - (\partial_\mu \log\rho)(\partial_\sigma \log\rho)) \\ &- i\bar{\Sigma}_{\mu\sigma}(\partial_\nu\partial_\sigma \log\rho - (\partial_\nu \log\rho)(\partial_\sigma \log\rho)) - i\bar{\Sigma}_{\mu\nu}(\partial_\sigma \log\rho)^2, \qquad (5.32\mathrm{a}) \\ \tilde{G}_{\mu\nu} &= i\bar{\Sigma}_{\nu\sigma}(\partial_\sigma\partial_\mu \log\rho - (\partial_\sigma \log\rho)(\partial_\mu \log\rho)) \\ &- i\bar{\Sigma}_{\mu\sigma}(\partial_\sigma\partial_\nu \log\rho - (\partial_\sigma \log\rho)(\partial_\nu \log\rho)) + i\bar{\Sigma}_{\mu\nu}\partial_\sigma\partial_\sigma \log\rho, \qquad (5.32\mathrm{b}) \end{aligned}$$

where we used

$$\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{\Sigma}_{\alpha\beta} = -\bar{\Sigma}_{\mu\nu}. \tag{5.33}$$

Requiring $\tilde{G}_{\mu\nu} = G_{\mu\nu}$ is equivalent to two equations, the first one gives an identity, the second one reads

$$\partial_\sigma\partial_\sigma \log\rho + (\partial_\sigma \log\rho)^2 = 0. \tag{5.34a}$$

This can be written as

$$\frac{\Box\rho}{\rho} = 0. \tag{5.35}$$

The only nonsingular solution for ρ is $\rho = \text{const}$ and therefore $A_\mu = 0$. But singular ρ will yield in addition nontrivial, nonsingular gauge fields A_μ.
Example: for

$$\rho = \frac{1}{|x|^2} \tag{5.36}$$

we calculate

$$\begin{aligned} \Box\rho &= -4\pi^2\delta^4(x), \\ \frac{\Box\rho}{\rho} &= 0. \end{aligned} \tag{5.37}$$

The same result holds for the more general form

$$\rho(x) = 1 + \sum_{i=1}^{Q_{\mathrm{Pont}}} \frac{\lambda_i^2}{(x-a_i)^2}, \tag{5.38}$$

with real constants $a_{i\mu}$ and λ_i. After a gauge transformation this will yield the Q_{Pont}-instanton solution. In the simplest nontrivial case we get the one-instanton solution, $Q_{\mathrm{top}} = 1$. Using

$$y_\mu \quad = \quad x_\mu - a_\mu, \tag{5.39}$$

we have

$$\rho(x) \quad = \quad 1 + \frac{\lambda^2}{y^2}. \tag{5.40}$$

The gauge field reads

$$A_\mu(x) = -2i\lambda^2\bar{\Sigma}_{\mu\nu}\frac{y_\nu}{y^2(y^2+\lambda^2)} = -i\bar{\eta}_{a\mu\nu}\sigma^a\frac{\lambda^2}{y^2(y^2+\lambda^2)}, \tag{5.41}$$

and is singular at $y = 0$. The singularity can be removed by a gauge transformation mediated by

$$U_1(y) = \frac{1}{|y|}(y_4 1\!\!1 + i y_j \sigma_j). \tag{5.42}$$

We calculate

$$U_1^{-1}\partial_\mu U_1 = -2i\bar{\Sigma}_{\mu\nu}\frac{y_\nu}{y^2}, \tag{5.43}$$

therefore A_μ can be written

$$A_\mu = \frac{\lambda^2}{y^2+\lambda^2}U_1^{-1}\partial_\mu U_1, \tag{5.44}$$

and after a gauge transformation we get

$$\begin{aligned} A'_\mu &= U_1(A_\mu + \partial_\mu)U_1^{-1} = \left(\frac{\lambda^2}{y^2+\lambda^2} - 1\right)(\partial_\mu U_1)U_1^{-1} \\ &= -\frac{y^2}{y^2+\lambda^2}(\partial_\mu U_1)U_1^{-1} = \frac{y^2}{y^2+\lambda^2}U_1\partial_\mu U_1^{-1}. \end{aligned} \tag{5.45}$$

With the abbreviations

$$\begin{aligned} \Sigma_{\mu\nu} &\equiv \frac{1}{2}\eta_{a\mu\nu}\sigma^a, \\ \eta_{a\mu\nu} &\equiv \epsilon_{a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{\mu 4}, \end{aligned}$$

we can express

$$U_1\partial_\mu U_1^{-1} = -2i\Sigma_{\mu\nu}\frac{y_\nu}{y^2}, \tag{5.46}$$

and finally have

$$A'_\mu(x) = -2i\Sigma_{\mu\nu}\frac{y_\nu}{y^2+\lambda^2} = -2i\Sigma_{\mu\nu}\frac{(x-a)_\nu}{(x-a)^2+\lambda^2} = -i\eta_{a\mu\nu}\sigma^a\frac{(x-a)_\nu}{(x-a)^2+\lambda^2}. \tag{5.47}$$

This is the gauge transformed instanton solution which is non-singular everywhere, provided that $\lambda \neq 0$. It has the following properties: the selfdual field strength is

$$G'_{\mu\nu} = 2i\eta_{a\mu\nu}\sigma^a\frac{\lambda^2}{((x-a)^2+\lambda^2)^2}, \tag{5.48}$$

for $x \to \infty$ the field reduces to a pure gauge $A'_\mu \to U_1(x)\partial_\mu U_1^{-1}$ and the action is

$$S = -\frac{1}{2g^2}\int \mathrm{d}^4x\, \mathrm{tr}\left(G'_{\mu\nu}G'_{\mu\nu}\right) = \frac{48\lambda^4}{g^2}\int \mathrm{d}^4x\frac{1}{(y^2+\lambda^2)^4} = \frac{8\pi^2}{g^2}, \tag{5.49}$$

therefore $Q_{\mathrm{Pont}} = 1$.

The same analysis can be done for the anti-instanton, the anti-selfdual solution of the Yang-Mills equations of motion with $Q_{\text{Pont}} = -1$. In this case the gauge fields read

$$A'_\mu(x) = -2i\bar{\Sigma}_{\mu\nu}\frac{(x-a)_\nu}{(x-a)^2+\lambda^2} = -i\bar{\eta}_{a\mu\nu}\sigma^a\frac{(x-a)_\nu}{(x-a)^2+\lambda^2}. \tag{5.50}$$

Remark: the identification instanton and anti-instanton, as well as $Q_{\text{Pont}} = \pm 1$ is merely a matter of definition.

The field equations are invariant under translations, this is reflected by the four free parameters a_μ, scale invariance leads to the emergence of one parameter λ, global gauge rotations correspond to three free parameters. In total there are eight free parameters. Brown, Carlitz and Lee [57] proved, that a solution in the Q_{Pont} sector has exactly $8Q_{\text{Pont}}$ degrees of freedom[3]. Usually the overall gauge orientation is fixed, so effectively the Q_{Pont} instanton solution exhibits $8Q_{\text{Pont}} - 3$ degrees of freedom. For the Q_{Pont}-instanton solution

$$A_\mu(x) = i\bar{\Sigma}_{\mu\nu}\partial_\nu\left[\log\left(1+\sum_{i=1}^{Q_{\text{Pont}}}\frac{\lambda_i^2}{y_i^2}\right)\right], \tag{5.51}$$

we find

$$S = \frac{8\pi^2}{g^2}\times Q_{\text{Pont}}. \tag{5.52}$$

The action of an Q_{Pont}-instanton solution is equal to Q_{Pont} times the action of the single instanton solution. This is a remarkable property for solutions of non-linear field equations.

5.3. Fermions in Instanton Fields

Now we are ready to study the behaviour of fermions within the background of such instanton configurations. In particular we are interested in zero modes of the Euclidean Dirac operator. We use the chiral representation for the γ matrices. In Euclidean space we can choose all of them to be anti-Hermitean

$$\gamma_i \equiv \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},\ \gamma_4 \equiv i\begin{pmatrix} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix},\ \gamma_5 \equiv -\gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} -1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix}. \tag{5.53}$$

They obey the following relations

$$\gamma_\mu^\dagger = -\gamma_\mu,\ \{\gamma_\mu,\gamma_\nu\} = -2\delta_{\mu\nu}. \tag{5.54}$$

[3] They point out a remarkable connection between the dimension of the moduli space of an instanton configuration, i.e. the number of free parameters, and the number of zero modes of fermions in the adjoint representation of $SU(2)$: the number of free parameters is exactly twice the number of those zero modes. In Appendix B we show, how to count the number of zero modes and we find that there are $\frac{2}{3}\cdot 1\cdot(1+1)\cdot(2+1)\cdot Q_{\text{Pont}} = 4Q_{\text{Pont}}$ zero modes, therefore the dimension of the moduli space is $8Q_{\text{Pont}}$.

The Hermitean Dirac operator is

$$\not{D} = \gamma_\mu D_\mu = \gamma_\mu(\partial_\mu + A_\mu). \tag{5.55}$$

Let

$$\sigma_\mu \equiv (\sigma^i, i\mathbb{1}),\ \bar{\sigma}_\mu \equiv (\sigma^i, -i\mathbb{1}),$$

with the properties

$$\begin{aligned} 2\delta_{\mu\nu} &= \sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu, & (5.56a)\\ 2i\bar{\Sigma}_{\mu\nu} &= i\bar{\eta}_{a\mu\nu}\sigma^a = \frac{1}{2}(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu), & (5.56b)\\ 2i\Sigma_{\mu\nu} &= i\eta_{a\mu\nu}\sigma^a = \frac{1}{2}(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu). & (5.56c) \end{aligned}$$

The Dirac operator can be written as

$$\not{D} = \begin{pmatrix} 0 & iD_4 + \sigma_i D_i \\ iD_4 - \sigma_i D_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_\nu D_\nu \\ -\bar{\sigma}_\mu D_\mu & 0 \end{pmatrix}. \tag{5.57}$$

With the projectors

$$\mathbb{P}_\pm \equiv \frac{1}{2}(\mathbb{1} \pm \gamma_5),\ \mathbb{P}_+ = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix},\ \mathbb{P}_- = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.58}$$

the corresponding Weyl operators read

$$\begin{aligned} D_+ &= \not{D}\mathbb{P}_+ = \begin{pmatrix} 0 & \sigma_\nu D_\nu \\ 0 & 0 \end{pmatrix}, & (5.59a)\\ D_- &= \not{D}\mathbb{P}_- = D_+^\dagger = \begin{pmatrix} 0 & 0 \\ -\bar{\sigma}_\mu D_\mu & 0 \end{pmatrix}, & (5.59b) \end{aligned}$$

and according to this we define the Laplacians

$$\begin{aligned} \triangle_+ &= D_- D_+ = \begin{pmatrix} 0 & 0 \\ 0 & -\bar{\sigma}_\mu\sigma_\nu D_\mu D_\nu \end{pmatrix}, & (5.60a)\\ \triangle_- &= D_+ D_- = \begin{pmatrix} -\sigma_\nu\bar{\sigma}_\mu D_\nu D_\mu & 0 \\ 0 & 0 \end{pmatrix}. & (5.60b) \end{aligned}$$

Since $[\not{D}, \gamma_5] = 0$ on the space S of all zero modes, we can choose all of them to be eigenfunctions of γ_5 as well. Let

$$S_\pm = \{\psi :\ \not{D}\psi = 0,\ \gamma_5\psi = \pm\psi\} \tag{5.61}$$

be the set of all zero modes with positive or negative chirality, respectively. In our representation zero modes of positive chirality have components $\begin{pmatrix} 0 & \chi \end{pmatrix}^\top \in S_+$ and are called right-handed, whereas $\begin{pmatrix} \chi & 0 \end{pmatrix}^\top \in S_-$ have negative chirality and are called left-handed.

In order to analyse $\triangle_-$, we calculate

$$\begin{aligned} -\sigma_\nu\bar\sigma_\mu D_\nu D_\mu &= \frac{1}{2}(-\sigma_\nu\bar\sigma_\mu - \sigma_\nu\bar\sigma_\mu - \sigma_\mu\bar\sigma_\nu + \sigma_\mu\bar\sigma_\nu)D_\nu D_\mu \\ &= \frac{1}{2}(-2\delta_{\mu\nu} + 4i\bar\Sigma_{\mu\nu})D_\nu D_\mu \\ &= -D_\mu D_\mu + 2i\bar\Sigma_{\mu\nu}[D_\nu, D_\mu] \\ &= D^2 - 2i\bar\Sigma_{\mu\nu}G_{\mu\nu} \end{aligned} \tag{5.62}$$

For selfdual fields $\bar\Sigma_{\mu\nu}G_{\mu\nu}$ vanishes since $\bar\Sigma_{\mu\nu}$ is anti-selfdual. $D^2 = -(iD)^2 < 0$ is a negative operator[4], as a consequence $\triangle_-$ and D_- do not have any zero modes. The index of D_+ equals the total number of zero modes in an instanton field (the index of D_- equals the number of zero modes in an anti-instanton field). In an instanton field all zero modes have positive chirality and are right-handed, in an anti-instanton field they have negative chirality and are left-handed. Now we want to derive the explicit form of those zero modes.

5.4. Explicit Form of Zero Modes

To find the zero modes in an instanton field we still have to solve the equation

$$\sigma_\mu D_\mu \chi = 0. \tag{5.63}$$

χ is a 2×2 matrix because it carries spin and isospin indices. In this derivation we follow closely the work of Grossman [61]. Using the fact that $\sigma_\mu^\top = -\sigma^2\bar\sigma_\mu\sigma^2$ we can bring the σ-matrices to the right an get

$$(\partial_\mu + A_\mu)\varphi\bar\sigma_\mu = 0, \tag{5.64}$$

where

$$\varphi = \chi\sigma^2. \tag{5.65}$$

Now we take the Q_{Pont}-instanton solution of the form

$$A_\mu = i\bar\Sigma_{\mu\nu}b_\nu,\ b_\nu = \partial_\nu \log\rho,\ \frac{1}{\rho}\Box\rho = 0, \tag{5.66}$$

and expand φ in terms of the σ^μ (because the σ^μ form a basis of all 2×2-matrices)

$$\varphi \equiv -iM_\mu\sigma_\mu. \tag{5.67}$$

[4] First of all D^2 is a non-positive operator, $D^2 \le 0$. But by the very definition there are no zero modes of D_μ in the $Q_{\mathrm{Pont}} \neq 0$ sector, since the instanton number of any reducible connection vanishes [58, 59, 60], therefore $D^2 < 0$.

Then equation (5.63) reads

$$2i\bar{\Sigma}_{\mu\nu}(\partial_\nu M_\mu - \frac{1}{2}b_\nu M_\mu) + (\partial^\mu M_\mu + \frac{3}{2}b^\mu M_\mu) = 0. \tag{5.68}$$

Since $\bar{\Sigma}_{\mu\nu}$ is a traceless tensor, we can take the trace of (5.68) and get the following:

$$\begin{aligned} \bar{\Sigma}_{\mu\nu}(2\partial_\nu M_\mu - b_\nu M_\mu) &= 0, & (5.69a) \\ \partial^\mu M_\mu + \frac{3}{2}b_\mu M_\mu &= 0. & (5.69b) \end{aligned}$$

Define

$$N_\mu \equiv \rho^{-1/2} M_\mu, \tag{5.70}$$

so the corresponding equations are

$$\begin{aligned} \partial_\mu N_\nu - \partial_\nu N_\mu - \epsilon_{\mu\nu\rho\sigma}\partial_\rho N_\sigma &= 0, & (5.71a) \\ \partial_\mu(\rho^2 N_\mu) &= 0. & (5.71b) \end{aligned}$$

If we make the ansatz

$$N_\mu \equiv \partial_\mu h + g_\mu, \tag{5.72}$$

with $\partial_\mu g_\mu = 0$, we can derive g_μ from an antisymmetric (and because of the additional three free parameters also anti-selfdual) tensor $g_\mu = \partial_\nu X_{\nu\mu}$. From (5.71a) it follows

$$\Box\, X_{\mu\nu} = 0, \tag{5.73}$$

admitting only singular contributions or contributions that are non-vanishing at infinity. Therefore we have $g_\mu = 0$. If we furthermore set

$$h \equiv \frac{\omega}{\rho}, \tag{5.74}$$

then equation (5.71b) implies $\Box\ \omega = 0$. Now we specify our ρ and list the possible harmonic solutions ω. Taking the form that exhibits $5Q_{\text{Pont}} + 4$ degrees of freedom for the Q_{Pont} instanton configuration

$$\rho = \sum_{i=1}^{Q_{\text{Pont}}+1} \frac{\lambda_i^2}{(x-a_i)^2}, \tag{5.75}$$

one obtains $Q_{\text{Pont}} + 1$ solutions (yielding non-singular wave-functions) of the form

$$\omega^{(k)} = \frac{\lambda_k^2}{(x-a_k)^2}, \quad k = 1, 2, \ldots, Q_{\text{Pont}} + 1. \tag{5.76}$$

Although each $\omega^{(k)}$ as well as ρ are singular, the singularities match, so that the resulting

$$M_\mu^{(k)} = \rho^{1/2}\partial_\mu\left(\frac{\omega^{(k)}}{\rho}\right) \tag{5.77}$$

are non-singular and normalizable. Finally

$$\chi^{(k)} = -iM_\mu^{(k)}\sigma^\mu\sigma^2, \quad k = 1, 2, \ldots Q_{\text{Pont}}. \tag{5.78}$$

Remark: these are only Q_{Pont} independent solutions, since $\sum_{k=1}^{Q_{\text{Pont}}+1} M_\mu^{(k)} = 0$.

Now let us concentrate on the $Q_{\text{Pont}} = 1$ instanton sector again. With $\lambda_2 \to \infty$, $|a_2| \to \infty$ such that $\frac{\lambda_2}{|a_2|} = 1$ and $a_1 \equiv a$, $\lambda_1 \equiv \lambda$ and $y \equiv x - a$, we have

$$A_\mu = -i\bar\eta_{a\mu\nu}\sigma^a \frac{\lambda^2 y_\nu}{y^2(y^2+\lambda^2)}, \tag{5.79}$$

$$\chi = -2U_1^{-1}(y)\frac{\lambda^2}{(y^2+\lambda^2)^{3/2}}\sigma^2. \tag{5.80}$$

This is the gauge field and the corresponding zero mode in the singular gauge, cf. (5.41). After a gauge transformation back to regular gauge via U_1 we end up with the regular form

$$A_\mu = -i\eta_{a\mu\nu}\sigma^a \frac{y_\nu}{y^2+\lambda^2}, \tag{5.81}$$

$$\chi = -2\frac{\lambda^2}{(y^2+\lambda^2)^{3/2}}\sigma^2. \tag{5.82}$$

This agrees with the result by 'T HOOFT [62]. The full right-handed spinor $\psi(x)$ with its four Dirac and two isospin components is given by

$$\psi(x) \sim \frac{\lambda^2}{((x-a)^2+\lambda^2)^{3/2}} \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}. \tag{5.83}$$

5.5. Index Theorem

The number of fermionic zero modes in the background of a given Euclidean Yang-Mills field configuration with topological charge Q_{Pont} can be determined with the help of the ATIYAH-SINGER index theorem [63, 64]. The number of zero modes with positive chirality (n_+) minus the number of zero modes with negative chirality (n_-), i.e. the index of the Weyl operator D_+, can be expressed as[5]:

$$\text{index } D_+ = \int_{M^D} \text{ch}(G) = \frac{1}{(D/2)!}\left(\frac{i}{2\pi}\right)^{D/2}\int_{M^D} \text{tr } G^{D/2}, \tag{5.84}$$

[5] For definitions and a proof see appendix B.

which in our case ($D = 4$) reduces to

$$\begin{aligned}
\text{index } D_+ &= \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \int_{S^4} \text{tr } G^2 = -\frac{1}{8\pi^2}\int_{S^4} \text{tr } G^2 \\
&= -\frac{1}{16\pi^2}\int \mathrm{d}^4x \text{tr } G_{\mu\nu}\tilde{G}_{\mu\nu} \\
&= Q_{\text{Pont}} = n_+ - n_-. \qquad (5.85)
\end{aligned}$$

With the help of our vanishing theorem (see section 5.4) we can conclude, that in an Q_{Pont} instanton field there are exactly Q_{Pont} zero modes and all of them are of positive chirality, whereas in a $|Q_{\text{Pont}}|$ anti-instanton field there are $|Q_{\text{Pont}}|$ zero modes but all of them are of negative chirality. The calculations à la GROSSMAN are in agreement with the predictions of the ATIYAH-SINGER index theorem.

5.6. Quantum Interpretation

The existence of zero modes of the Dirac operator in the instanton fields implies some astonishing physical effects. Massless fermions will lead to a suppression of the tunneling amplitude between gauge field configurations with different Chern-Simons numbers. Furthermore zero modes give rise to the so-called level crossing, the eigenvalues of the Dirac Hamiltonian vary with time, some of them cross zero and change their sign. These effects will be discussed in the next subsections.

5.6.1. Suppresion of Tunneling

Interpret the instantons in $\mathbb{R}^4$ as tunneling events in $3+1$ dimensional Minkowski space and let us consider the transition from a gauge field configuration with Chern-Simons number $Q_{\text{CS}}(-\infty)$ at $t = -\infty$ to $Q_{\text{CS}}(+\infty)$ at $t = +\infty$ in the presence of massless fermions. Quantization via the path integral formalism [65, 66, 67] results in the transition amplitude

$$\langle Q_{\text{CS}}(-\infty)|Q_{\text{CS}}(+\infty)\rangle = \int \mathcal{D}A'_\mu \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp S. \qquad (5.86)$$

Gauge fixing terms are understood to be included. The prime denotes integration only over fields A_μ with appropriate topological charge $Q_{\text{top}} = Q_{\text{CS}}(+\infty) - Q_{\text{CS}}(-\infty)$. The combined action of Yang-Mills and Fermi fields is given by

$$\begin{aligned}
S &= S_A + S_\psi, \qquad (5.87) \\
S_A &= -\frac{1}{2g^2}\int \mathrm{d}^4x \text{ tr } G_{\mu\nu}G_{\mu\nu}, \\
S_\psi &= \int \mathrm{d}^4x\ \bar{\psi}\not{D}\psi = \int \mathrm{d}^4x\ \psi^\dagger\gamma_\mu(\partial_\mu + A_\mu)\psi.
\end{aligned}$$

The fermions can be integrated out exactly, since the action depends on those fields in a bilinear fashion. So we get

$$\langle Q_{\mathrm{CS}}(-\infty)|Q_{\mathrm{CS}}(+\infty)\rangle = \mathcal{N}\mathrm{det}\not{D} \int \mathcal{D}A'_\mu \exp S_A. \tag{5.88}$$

This transition amplitude vanishes identically since the Dirac operator $\not{D}$ has zero modes. Massless fermions suppress the tunneling between topologically distict vacua of the Yang-Mills fields. If the fermions carry mass m, all eigenvalues are shift, the determinant no longer vanishes and tunneling is possible again.

5.6.2. The Spectral Flow

The equations of motion for a massless Dirac field formally conserve the axial-vector current $j^5_\mu(x) = \bar{\psi}(x)\gamma_\mu\gamma_5\psi(x)$ as well as the vector current $j_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$. This would imply chiral $U(1) \otimes U(1)$ symmetry. But the bilinear product $\bar{\psi}(x)\psi(y)$ diverges in quantum field theory as x approaches y. Therefore one has to define these currents more carefully, and in doing that we choose a gauge invariant regularization. According to Schwinger [68] this can be done by separating the two points slightly

$$j^5_\mu = \bar{\psi}\left(x + \frac{1}{2}\epsilon\right)\gamma_\mu\gamma_5\mathcal{P}\left[\exp\left(-\int_{x-\frac{1}{2}\epsilon}^{x+\frac{1}{2}\epsilon} A_\mu \mathrm{d}x'_\mu\right)\right]\psi\left(x - \frac{1}{2}\epsilon\right), \tag{5.89}$$

similarly for j_μ. If one calculates the divergence of both redefined currents one gets [23]

$$\begin{aligned} \partial_\mu j_\mu(x) &= 0, & (5.90)\\ \partial_\mu j^5_\mu(x) &= \frac{i}{8\pi^2}\mathrm{tr}\ G_{\mu\nu}\tilde{G}_{\mu\nu} = -2i\partial_\mu k_\mu = -2iQ_{\mathrm{Pont}}(x), & (5.91) \end{aligned}$$

where $Q_{\mathrm{Pont}}(x)$ is the Pontryagin density. The divergence of the axial-vector current no longer vanishes at the quantum level, the classical symmetry is violated and we entcountered what is called an anomaly. The appropriate Noether charge, which is not conserved anymore, is the so-called axial charge Q^5, which is equal to the number of particles with positive chirality minus the number of particles with negative chirality:

$$Q^5 = \int \mathrm{d}^3x j^5_0. \tag{5.92}$$

For the change in Q^5 we get from (5.91)

$$\triangle Q^5 \equiv Q^5(t = +\infty) - Q^5(t = -\infty) = 2Q_{\mathrm{Pont}}. \tag{5.93}$$

This change is equal to two times the Pontryagin index of the background field: instantons cause the axial charge to change. How can one understand this?

Consider the Dirac Hamiltonian H in Weyl gauge, which is given by

$$H\psi = -\partial_4\psi = -\gamma_4\gamma_i(\partial_i + A_i)\psi = -i\alpha_i D_i\psi, \tag{5.94}$$

and depends on x_4 via the gauge fields. For each fixed value of x_4 we can solve the eigenvalue equation

$$H(x_4)\psi_{x_4}(\mathbf{x}) = \lambda(x_4)\psi_{x_4}(\mathbf{x}). \tag{5.95}$$

We know that the Hamiltonians at $x_4 = -\infty$ and $x_4 = +\infty$ in the instanton background differ only by a unitary gauge transformation. Therefore they have the same spectrum. But as the 'time' x_4 goes by these eigenvalues are subject to change and a particular eigenmode needs not to come back to its starting value but may be shifted upwards or downwards.

The spectral flow of H is defined as the number of modes changing their negative energy eigenvalues to positive ones minus the number of modes changing their eigenvalues the other way round. A generalization of the ATIYAH-SINGER index theorem by ATIYAH, PATODI and SINGER immediately leads to the following

Theorem: *Spectral Flow*
The number of zero modes of the Dirac operator is equal to the spectral flow of the Dirac Hamiltonian.

A rigorous mathematical proof can be found in the literature [69]. Here we are going to use some physical arguments in order to substantiate this theorem. Let us assume that the background fields are slowly-varying and allow for an adiabatic approximation[6]. We rewrite the wave function, by separating the x_4 coordinate, as the product of a function F which depends solely on x_4 and a function ψ_{x_4} which depends on the spatial coordinates $\mathbf{x} = \{x_1, x_2, x_3\}$ and parametrically on x_4:

$$\psi(\mathbf{x}, x_4) = F(x_4)\psi_{x_4}(\mathbf{x}), \tag{5.96}$$

we have

$$\begin{aligned} -\partial_4 F(x_4)\psi_{x_4}(\mathbf{x}) &= HF(x_4)\psi_{x_4}(\mathbf{x}) = \lambda(x_4)F(x_4)\psi_{x_4}(\mathbf{x}), \\ -\frac{dF}{dx_4} &= \lambda(x_4)F(x_4), \end{aligned} \tag{5.97}$$

and the solution is

$$F(x_4) = F(0) \times \exp\left(-\int_0^{x_4} \mathrm{d}\tau\lambda(\tau)\right). \tag{5.98}$$

Obvioulsy $\psi(\mathbf{x}, x_4)$ is normalizable if and only if λ is positive for $x_4 \to +\infty$ and negative for $x \to -\infty$.

The existence of Q_{Pont} zero modes of the Dirac operator (with positive chirality) in an Q_{Pont}-instanton field necessarily implies that Q_{Pont} fermionic levels flow from negative to positive values. We have the spectrum indicated in Figure 5.2. Since the spectrum of

[6]For a more general proof, which does not require the fields to change adiabatically, see the paper by CHRIST [70].

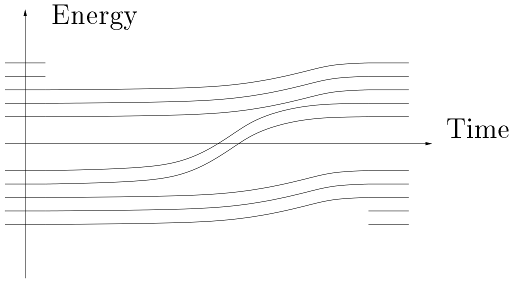

Figure 5.2.: Level crossing in the sector with positive chirality, $Q_{\text{Pont}} = 2$.

H is symmetric [71], there are also Q_{Pont} fermionic modes with negative chirality, that interpolate between positive eigenvalues at $x_4 = -\infty$ and negative values at $x_4 = +\infty$, cf. Figure 5.3.

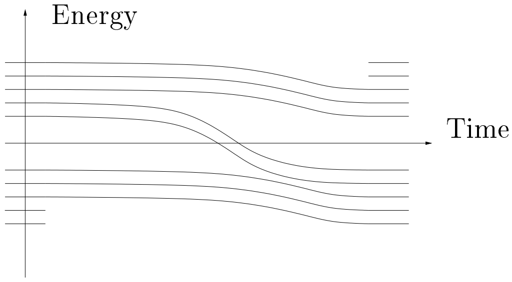

Figure 5.3.: Level crossing in the sector with negative chirality, $Q_{\text{Pont}} = 2$.

For a $|Q_{\text{Pont}}|$ anti-instanton field it is the other way round. The explicit form of the wave function $\psi_{x_4}(\mathbf{x})$ at the cross-over point in the one instanton field, where $H\psi = E\psi = 0$, has been calculated by Kiskis [72].

Interpretation: Processes that change the winding number are accompanied by the absorbtion and emission of fermions, depending on their chirality. In terms of the second quantization the one particle state corresponds to a situation where all negative energy states and the lowest positive energy state are filled and all other positive energy states are empty. Now in the presence of an instanton one of the negative energy states is shifted to positive values, one particle with positive chirality emerges. At the same time one of the positive energy states with negative chirality turns into a negative energy state. The particle vanishes in the Dirac Sea. As an illustrative example we could imagine the spectrum which is indicated in Figure 5.4.

In total the instanton field can turn a negative chirality particle into a positive chirality

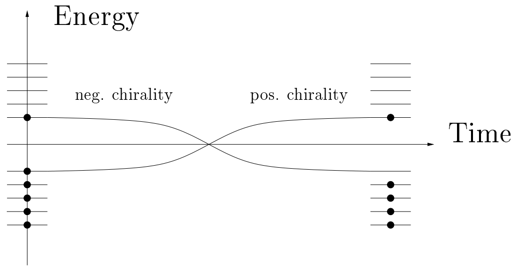

Figure 5.4.: Level crossing and change of axial charge, $Q_{\mathrm{Pont}} = 1$.

one (by winding from one vacuum configuration to another one). We find that the total charge is conserved, since $\triangle Q = 1 - 1 = 0$, but the axial charge changes $\triangle Q^5 = 1 - (-1) = 2$, in accordance with the anomaly equation (5.93). In fact no fermion ever changes its chirality, all of them just move one level up or down. The axial charge is said to come *from the bottom of the Dirac Sea* [56]. In this way one can understand how instantons can change the total axial charge of the system.

Remark: these effects can explicitly be shown using a toy-model (the Schwinger model on a circle), see the book by Bertlmann [73] pp. 227-233. 't Hooft [74] gives the following pictorial description of how an instanton affects fermions:

An instanton is like a little door, that suddenly appears, opens to let one or several particles through, to or from this infinite reservoir called the 'Dirac sea', and then closes and disappears.[...] What an instanton does in quantum chromodynamics is the following: it turns the energy of one right helicity particle state from positive to negative and does the opposite to one left helicity state. So one right helicity particle will seem to disappear and a left helicity particle pops up. It is as if a right helicity particle transmuted into a left helicity one! This is why helicity is no longer conserved, and consequently the algebra that was associated with it breaks down.

6. Summary and Outlook

In this diploma thesis we analysed a variety of field theoretical models: the ϕ^4 model in $1+1$ dimensions, the $SU(2)$ Yang-Mills-Higgs theory in $3+1$ dimensions and pure $SU(2)$ Yang-Mills theory in Euclidean four-space.

All models exhibit nontrivial classical solutions, which are called kink, 't Hooft-Polyakov monopole and instanton, respectively. The kink and the monopole are solitons, i.e. solutions of the classical field equations with particle-like properties: they have finite energy, are localized in space, can be boosted and display the correct relationship between energy, momentum and mass. Furthermore they cannot be found in a perturbative expansion since they depend in an nonanalytical fashion on the coupling constant. The instanton turned out to be a tunneling event in Minkowski space and a *pseudo-particle* in Euclidean space.

All those field configurations are stable for topological reason, they can be classified according to the homotopy groups $\pi_0(S^0)$, $\pi_2(S^2)$ and $\pi_3(S^3)$ respectively, and therefore carry topological charge. In these configurations, the fields approach different degenerate vacua as one approaches spatial infinity in different directions. The vacua are chosen in such a fashion that they cannot be continuosly deformed to a single vacuum. This guarantees the stability of the soliton, and gives rise to a new type of quantum number: the topological charge. Due to conservation of this charge these objects are stable.

In a second step we used the soliton and instanton configurations as background fields and analysed the behaviour of fermions in these fields. Since the corresponding equations of motion are quite simple (in the case of the kink solution) or can be reduced drastically (like in the case of the 't Hooft-Polyakov monopole and the instanton) due to symmetry arguments, an explicit solution is possible.

In the models with an odd number of space dimensions we investigated the Dirac Hamiltonian and its zero modes

$$H\psi = 0.$$

Now the total number of zero modes can simply be counted. On the other hand we used the CALLIAS-BOTT-SEELEY index theorem to determine their number and afterwards compared both results.

In the kink and the monopole case the existence of fermionic zero energy modes leads to some important physical effects. First of all the soliton states become multiply degenerate: soliton plus empty fermionic zero mode and soliton plus filled zero mode carry the same energy. As a second effect the fermion number no longer takes on only integer values but becomes fractional.

The predictions from the kink model can be tested in solid state physics: the phonon field of polyacetylen exhibits a ϕ^4 potential, the electrons show a zero energy mode (at mid-gap) and the fractionization is reflected in a *wrong* spin-charge assignment: neutral chains of polyacetylen carry spin, whereas charged chains are spinless. Both effects have been observed experimentally.

Since the relevant properties are the same, the 't Hooft-Polyakov monopole is assumed to show these effects, too.

In the instanton field we did a similar analysis, but now we started with the Dirac operator $\not{D}$ and examined its zero modes

$$\not{D}\psi = 0.$$

Zero modes of the corresponding Dirac Hamiltonian can be related to the zero modes of the Dirac operator by spectral flow arguments. We have shown, how the Atiyah-Singer index theorem can be used to count their number: the number of zero modes is proportional to the topological charge of the background field.

Also in the instanton case those zero modes have important physical consequences: massless fermions suppress the tunneling between topologically distinct vacua, in the massive case, the process in Minkowski space that corresponds to the instanton field, is accompagnied by a change of axial charge $\triangle Q^5$. Therefore the $U(1)$ axial symmetry of the theory breaks down, and this solves the famous $U(1)$ problem.

Perhaps the procedures, theorems and results that have been given in this diplom thesis can be aplied to some — up to now — unresolved problems.

Studying chiral symmetry breaking requires an understanding of quasi-zero modes, the spectrum of the Dirac operator near the $\lambda = 0$ eigenvalue, since the order parameter for this phase transition, the quark condensate $< \bar{\psi}\psi >$, is related to the spectral density $\rho(\lambda)$ by the Banks-Casher relation [22]

$$< \bar{\psi}\psi >= -\pi\rho(\lambda = 0).$$

If there is only one instanton the spectrum consists of a single zero mode, plus a continuous spectrum of non-zero modes. But if there is a finite density of instantons, the spectrum is complicated, even if the ensemble is very dilute. The zero modes are expected to mix, so that the eigenvalues spread over some range $\triangle\lambda$. A precise description of the faith of zero modes within such an ensemble would contribute to a better understanding of the ground state of QCD as well as of the chiral symmetry breaking. The fermions could be simulated on a lattice and furthermore one could try to calculate the exact eigenmodes of $\not{D}$ in an instanton-antiinstanon background.

The question, whether or not the two main effects of low temperature QCD, chiral symmetry breaking and confinement are related, is not answered yet. One of the physical scenarios of color confinement is based on the idea of monopole-antimonopole pair condensation in the vacuum state of quantum Yang-Mills theory. The chiral symmetry breaking is supposed to happen due to the influence of instanton configurations.

Recently a new decomposition of the Yang-Mills connection A_μ has been proposed by Cho [75, 76, 77], Fadeev and Niemi [78, 79, 80] and Shabanov [81, 82]. This is a generalization of the Abelian projections introduced by 'T HOOFT and is supposed to give a new effective description of the low energy phase of QCD. There we rewrite

$$A_\mu^a = \alpha_\mu^a + C_\mu n^a + W_\mu^a,$$

with coupling constant g and

$$\begin{aligned} \alpha_\mu^a &= g^{-1}\epsilon^{abc}(\partial_\mu n^b)n^c, \\ W_\mu^a n^a &= 0. \end{aligned}$$

This gives an effective theory for the unit vector field n^a. The projected gauge field α_μ^a depends on A_μ^a and n^a.

Fermionic zero modes of this connection are still under investigation. Naively the Pontryagin index of the α-field vanishes because this is an reducible connection, but since the Higgs field $\mathbf{n}$ turns out to be singular, a careful analysis is needed.

For the time beeing we have to analyse the one instanton configuration. The unit vector in 3-direction, $n^a = \delta^{a3}$, is related to the standard Hopf map

$$n_H = \begin{pmatrix} 2x_1x_2 + 2x_3x_4 \\ 2x_1x_4 - 2x_2x_3 \\ x_1^2 + x_3^2 - x_2^2 - x_4^2 \end{pmatrix}$$

by the same gauge transformation U_1 (cf. chapter 5) that takes the singular form of the instanton field to the regular form and back [60]. Hopf maps are maps

$$S^3 \to S^2, \tag{6.1}$$

and can be characterized by a topological invariant, the Hopf index.

The connection between this Hopf index, the topological charge of the instanton configuration and the magnetic charge of monopoles that arise after projection, as well as the connection between confinement and chiral symmetry breaking in this FADDEEV-NIEMI decomposition are subject of current research [83, 84, 85, 86, 87, 88].

A. Callias-Bott-Seeley Index Theorem

A.1. Introduction - The Problem

In this appendix we give some basic ideas, how to derive index theorems for Dirac operators on open spaces of odd dimension, closely following the work of CALLIAS [34]. The derivation is not straight forward but consists of many Lemmata and Propositions that are needed in order to substantiate the main theorems. Some of those intermediate steps are sketched, for the remaining details see [34].

We are interested in Dirac equations in Minkowski space with non-degenerate static (time-independent) modes. Such a Dirac equation can be written in the form

$$H\psi = \begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix} \psi = i\partial_t \psi, \tag{A.1}$$

where L is an elliptic operator on odd-dimensional Euclidean space. We will see that L has a nonvanishing index. The general idea is to use traces of the type

$$\mathrm{Tr}\ (e^{-tL^\dagger L} - e^{-tLL^\dagger}) \tag{A.2a}$$

or

$$\mathrm{Tr}\ \Big[\Big(\frac{z}{L^\dagger L + z}\Big)^s - \Big(\frac{z}{LL^\dagger + z}\Big)^s\Big], \tag{A.2b}$$

where the trace is taken in the Hilbert space as well as over Dirac and internal indices. On a compact manifold either of these traces gives the index for any value of t or z, because all eigenvalues are discrete and the spectrum of $LL^\dagger$ and $L^\dagger L$ is the same up to a different number of zero modes.

Proof: let ψ_λ be an eigenfunction of $L^\dagger L$:

$$L^\dagger L \psi_\lambda = \lambda \psi_\lambda,$$

then we find a corresponding eigenfunction of $LL^\dagger$ with the same eigenvalue

$$LL^\dagger(L\psi_\lambda) = L(L^\dagger L)\psi_\lambda = \lambda(L\psi_\lambda).$$

On an open space we get the index by taking the limit $t \to \infty$ for (A.2a) or $z \to 0$ for (A.2b), cf. [89]. We will use the second one with $s = 1$.

We study Dirac operators that arise in Yang-Mills theories with both gauge and Higgs fields. The most general Dirac equation in $D+1$ dimensional Minkowski space is

$$\big(i\alpha^i\partial_i \otimes 1\!\!1_m + \alpha^i \otimes A_i(x) - \beta \otimes \Phi(x)\big)\psi(x,t) = -i\partial_t\psi(x,t). \tag{A.3}$$

Here $\psi(x,t)$ is a $2pm$-component spinor. The $2p \times 2p$ Dirac matrices are given by

$$\alpha^i = \begin{pmatrix} 0 & \delta^i \\ \delta^i & 0 \end{pmatrix},\ \beta = i\begin{pmatrix} 0 & -1\!\!1_p \\ 1\!\!1_p & 0 \end{pmatrix}, \tag{A.4}$$

where the D $p \times p$ matrices δ^i satisfy an Euclidean Dirac algebra

$$\delta^i\delta^j + \delta^j\delta^i = 2\delta^{ij}\,1\!\!1_p. \tag{A.5}$$

The coefficients are given by Hermitean $m \times m$ matrices $A_i(x)$ and $\Phi(x)$. They are assumed differentiable and bounded in x and

$$\lim_{|x|\to\infty} A_i(x) = 0, \tag{A.6}$$

and $\Phi(x)$ approaches a constant as $|x| \to \infty$. Now separate the time variable

$$\psi(x,t) = \psi(x)e^{iEt}, \tag{A.7}$$

and express (A.3) as an eigenvalue problem

$$H\psi = \begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix}\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = E\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{A.8}$$

where L is a first order $pm \times pm$ matrix differential operator on $\mathbb{R}^n$:

$$L = i\delta^i\partial_i \otimes 1\!\!1_m + \delta^i \otimes A_i(x) + i\,1\!\!1_p \otimes \Phi(x), \tag{A.9}$$

$L^\dagger$ is the Hilbert space adjoint of L. We are interested in static (t-independent) solutions, i.e. the $E=0$ eigenspace. For these solutions we have

$$L\,\psi_- = 0, \tag{A.10a}$$
$$L^\dagger\,\psi_+ = 0. \tag{A.10b}$$

The dimension of the $E=0$ space is given by

$$k \equiv k_+ + k_-, \tag{A.11}$$

where

$$k_+ \equiv \text{dim ker } L^\dagger,\ k_- \equiv \text{dim ker } L. \tag{A.12}$$

We cannot in general determine k, but we can find a formula for

$$\text{index } L \equiv k_- - k_+ = \text{dim ker } L - \text{dim ker } L^\dagger, \tag{A.13}$$

in terms of the behaviour of the operator L at infinity. Sometimes one can find either k_+ or k_- a priori, see [89] and chapter 5. Then our formula will determine k.

Definition : L is Fredholm if both k_- and k_+ are finite and L is closed [90, 91].

If L is Fredholm, so are $L^\dagger, LL^\dagger$ and $L^\dagger L$. In what follows we will restrict ourselves to this special class of operators.

A property of the index which will turn out to be extremly useful in the derivation of the index formula is its homotopy invariance: If $t \to L(t)$ is a norm continuous map of the interval [0,1] into the space of Fredholm operators then index $L(0) =$ index $L(1)$. It is also invariant under perturbations that are compact relative to the original operator, cf. [91], p. 445.

Definiton: If $\mathcal{H}$ is a Hilbert space, $B : \mathcal{H} \to \mathcal{H}$ is compact relative to $L : D(L) \to \mathcal{H}, D(L) \subset \mathcal{H}$, if B is compact as an operator $D(L) \to \mathcal{H}$, where D(L) is equipped with the norm $\|.\| + \|L.\|$.

We need precise conditions that tell us, when an operator of the form (A.9) is Fredholm. For more general cases one can use the

Theorem 1: (Seeley) *Let $A = \sum_{|\alpha|\leq m} a_\alpha(x)\left(i\frac{\partial}{\partial x}\right)^\alpha$ be a differential operator, where the $a_\alpha(x)$ are bounded and their derivatives are continuous and vanish at ∞. Then A is Fredholm if there are constants c and C such that*

$$|\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha| \geq c|\xi|^m \qquad \forall x \in \mathbb{R}^n,$$

(i.e. A is uniformly elliptic) and

$$|\sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha|$$

is bounded away from 0 for $|x| \geq C$. Conversely, if A is Fredholm then there exist such constants c and C.

Proof: see [92].

Restricted to the form (A.9) of L, we get the

Corollary: *The operator L defined by (A.9) is Fredholm if $|\Phi(x)| \geq B$ for $|x| \geq C$ where B and C are positive constants. The index of L is equal to the index of $\tilde{L}$ if $\tilde{L}$ is an elliptic operator such that*

$$\tilde{L} = i\delta^i\partial_i \otimes 1\!\!1_m + i1\!\!1_p \otimes U(x) \tag{A.14}$$

for $|x| > C$, where $U(x)$ is the Hermitean unitary matrix

$$\begin{aligned} U(x) &\equiv |\Phi(x)|^{-1}\Phi(x), \\ |\Phi(x)| &= (\Phi^\dagger(x)\Phi(x))^{1/2}. \end{aligned} \tag{A.15}$$

Remark: That means that the index of L can be expressed solely in terms of the Higgs field Φ. The particular properties of the gauge fields, determined by the equations of motion, are nonrelevant in this regard.

Proof: Let

$$L_1 = i\delta^i\partial_i \otimes 1\!\!1_m + i1\!\!1_p \otimes \Phi(x). \tag{A.16}$$

L is Fredholm if and only if L_1 is Fredholm since the fact that the $A_i(x)$ are bounded and vanish as $|x| \to \infty$ implies that the term $\delta^i \otimes A_i(x)$ is L_1-compact, by the RELLICH Lemma. Notice that

$$L_1^\dagger L_1 = -\partial^2 1\!\!1 - \delta^i \otimes \Phi_{,i}(x) + 1\!\!1_p \otimes |\Phi(x)|^2. \tag{A.17}$$

Since $\Phi(x)$ is C^∞ and asymptotically homogeneous we have

$$L_1^\dagger L_1 \geq -\partial^2 1\!\!1 - \frac{\kappa}{|\kappa|} 1\!\!1 + B 1\!\!1, \tag{A.18}$$

where B is such that $|\Phi(x)|^2 > B$. The operator on the righthand side has a discrete spectrum of eigenvalues contained in $(-\infty, B)$, so that if $B > 0$ it is Fredholm. Then so is $L_1^\dagger L_1$ and therefore L_1 as well. The one parameter family of operators

$$L(t) = tL_1(t) + (1-t)\tilde{L} = i\delta^i \partial_i \otimes 1\!\!1_m + i 1\!\!1_p \otimes [t\Phi(x) + (1-t)U(x)], \tag{A.19}$$

for $0 \leq t \leq 1$, is a homotopy of L_1 to $\tilde{L}$ within the class of Fredholm operators. Thus $\tilde{L}$ has the same index as L_1 and L.

A.2. General First Order Operators

We now derive some general formulas for first order elliptic operators with arbitrary coefficients. These formulas readily yield the index theorem. Consider an arbitrary operator L, which is assumed to be closed on a dense domain $D(L)$ in a Hilbert space $\mathcal{K}$, which is the direct sum of M copies of another Hilbert space $\mathcal{H}$, $\mathcal{K} = \bigoplus_{i=1}^M \mathcal{H}$. L is a matrix of operators on $\mathcal{H}$, $L = [L_{ij}]$, $i, j = 1, 2 \ldots M$.

Definition: Given an operator $A = [A_{ij}]$ on $\mathcal{K}$ we define the **internal trace** of A, tr A to be the following operator on $\mathcal{H}$

$$\text{tr } A = \sum_i A_{ii}, \tag{A.20}$$

with domain $\bigcap_{i=1}^M D(A_{ii})$.

For L as in (A.9) the operators $LL^\dagger$ and $L^\dagger L$ are selfadjoint and positive. If z is a non-negative real number, $(LL^\dagger + z)^{-1}$ and $(L^\dagger L + z)^{-1}$ are bounded operators on $\mathcal{K}$ and

$$B_z \equiv z \text{ tr } \left((L^\dagger L + z)^{-1} - (LL^\dagger + z)^{-1} \right) \tag{A.21}$$

is a bounded operator on $\mathcal{H}$. Let $f(z) = \text{Tr } B_z$, where Tr denotes the trace in the Hilbert space $\mathcal{H}$: if $\{\phi_k\}_{k=1}^\infty$ is an orthonormal basis for $\mathcal{H}$, then Tr $B = \sum_{k=1}^\infty (\phi_k, B\phi_k)$. Now, under certain assumptions, the index of L can be expressed in terms of the trace of B_z on $\mathcal{H}$.

Lemma 1: *Suppose $\mathcal{K}$, $\mathcal{H}$, L, B_z are as above and furthermore $L : D(L) \to \mathcal{K}$ is*

Fredholm and B_z is trace-class on $\mathcal{H}$, and Tr $|B_z|$ is bounded for z in a domain C in the complex plane having $z = 0$ as a limit point. Then

$$\textit{index}\ L = \lim_{z\to 0} f(z). \tag{A.22}$$

Proof: Since L is Fredholm, so are $L^\dagger L$ and $LL^\dagger$ and the zero eigenvalues of $L^\dagger L$ and $LL^\dagger$ are isolated. Obviously ker $L^\dagger L$ = ker L and ker $LL^\dagger$ = ker $L^\dagger$. Let $\mathbb{P}_+$ be the projection on ker $L^\dagger L$ and $\mathbb{P}_-$ the projection on ker $LL^\dagger$. Then the operator

$$\tilde{B}_z = \mathrm{tr}\ \frac{z}{L^\dagger L + z} - \mathrm{tr}\ \mathbb{P}_+ - \mathrm{tr}\ \frac{z}{LL^\dagger + z} + \mathrm{tr}\ \mathbb{P}_- = B_z - \mathrm{tr}\ \mathbb{P}_+ + \mathrm{tr}\ \mathbb{P}_- \tag{A.23}$$

is trace-class since B_z is and $P_\pm$ are finite dimensional projections. Further $\lim_{z\to 0} \tilde{B}_z = 0$ strongly. Let $\{\phi_k\}_{k=1}^\infty$ be an orthonormal basis. Then the series

$$\mathrm{Tr}\ \tilde{B}_z = \sum_{k=1}^{\infty} (\phi_k, \tilde{B}_z \phi_k) \tag{A.24}$$

converges absolutely and uniformly for $z \in C$ and the limit of each term as $z \to 0$ is 0. Thus

$$\begin{aligned} \lim_{z\to 0} \mathrm{Tr}\ \tilde{B}_z &= 0, \\ \lim_{z\to 0} f(z) &= \lim_{z\to 0} \mathrm{Tr}\ \tilde{B}_z + \mathrm{Tr}\ P_+ - \mathrm{Tr}\ P_- = \mathrm{index}\ L. \end{aligned} \tag{A.25}$$

Now we find the bridge between the region $z \to 0$ where the index is computed (according to Lemma 1), and $z \to \infty$ where Tr B_z is computed explicitly.

Lemma 2: *With all the assumptions and definitions preceding Lemma 1, suppose B_z is trace-class for z in a domain C. Then $f(z) = Tr\, B_z$ is analytic for $z \in C$.*

Proof: Let $f_N(z) \equiv \sum_{k=1}^N (\phi_k, B_z \phi_k)$. Then each $f_N(z)$ is analytic and $f_N(z)$ is bounded for all N and all z in a compact subset of C. Thus $f(z) = \lim_{N\to\infty} f_N(z)$ is analytic in C.

Now the analytic function $f(z)$ can be expressed as

$$2f(z) = \lim_{R\to\infty} \int_{S_R^{D-1}} \mathrm{d}S_i J_z^i(x,x) + \int \mathrm{d}^D x A_z(x,x), \tag{A.26}$$

with suitable defined functions J_z and A_z, see [34], Proposition 1. In the special case of (A.9) that we are interested in, the bulk contribution of A_z vanishes and we are left with an integral over the boundary S_∞^{D-1}:

$$f(z) = \frac{1}{(1+z)^{D/2}} Q[U], \tag{A.27}$$

$$Q[U] = \frac{1}{2(\frac{D-1}{2})!} \left(\frac{i}{8\pi}\right)^{\frac{D-1}{2}} \lim_{R\to\infty} \int_{S_R^{D-1}} \mathrm{tr}\ \Big(U(x) \big(\mathrm{d}U(x)\big)^{D-1} \Big). \tag{A.28}$$

For a proof see [34].

A.3. An Index Formula for Dirac Operators

Together with Lemma 1 this gives the final result:
Theorem 2: *Let L be a first order differential operator on $\mathbb{R}^D$, D odd, which up to C^∞ zero order terms vanishing at infinity is of the form*

$$L = i\delta^i \partial_i \otimes 1\!\!1_m + i\,1\!\!1_p \otimes \Phi(x), \tag{A.29}$$

where the δ^i are constant $p \times p$ matrices, $p = 2^{(D-1)/2}$, satisfying the algebra

$$\delta^i \delta^j + \delta^j \delta^i = 2\delta^{ij}\,1\!\!1_p. \tag{A.30}$$

$\Phi(x)$ is a $m \times m$ Hermitean matrix of C^∞ functions such that $|\Phi(x)| \geq B > 0$ for $|x| \geq C$, where B and C are constants, further $\Phi(x)$ homogeneous of order 0 as $x \to \infty$. Let $U(x) \equiv |\Phi(x)|^{-1}\Phi(x)$. Then the index of L is given by

$$\text{index}\, L = \frac{1}{2(\frac{D-1}{2})!}\left(\frac{i}{8\pi}\right)^{\frac{D-1}{2}} \lim_{R\to\infty} \int_{S_R^{D-1}} tr\left(U(x)(dU(x))^{D-1}\right), \tag{A.31}$$

where $(dU)^{D-1}$ is the $(D-1)$st power of the matrix dU with the differentials being multiplied by exterior multiplication.
Remark: The formula (A.31) remains essentially the same if D is even, and it gives trivially that index $L = 0$ in that case, for any L of the form (A.9).

A.4. Example: the Kink

Consider $D = 1$ space dimension, only one internal degree of freedom and L of the form

$$L = -\frac{d}{dx} + \phi(x), \tag{A.32}$$

where $\phi(x)$ is a real valued function on $\mathbb{R}$. Observe that this exactly coincides with the upper right part of the Hamiltonian of our ϕ^4 theory in chapter 2. Let

$$\lim_{x\to\pm\infty} \phi(x) = \phi_\pm < \infty. \tag{A.33}$$

Then we can apply (A.31) and get

$$\text{index}\, L = \frac{1}{2}\left[\frac{\phi_+}{|\phi_+|} - \frac{\phi_-}{|\phi_-|}\right]. \tag{A.34}$$

The vaccum sectors have vanishing index, whereas kink and antikink carry index $+1$ and -1, respectively.

A.5. Example: $SU(2)$ Monopole

We study the Dirac equation for an isospin T particle in the field of a static system of $SU(2)$ magnetic monopoles in $3+1$ dimensional Minkowski space. We have the gauge potentials and Higgs field

$$\begin{aligned} A_i(\mathbf{x}) &= A_i^a(\mathbf{x})T^a, \\ \Phi(\mathbf{x}) &= \phi^a(\mathbf{x})T^a + m, \end{aligned} \tag{A.35}$$

with a running from 1 to 3. ϕ^a is a vector in internal space and takes on its fixed vaccum expectation value $\phi^a\phi^a = F^2$ as $|\mathbf{x}|$ goes to infinity. m is the mass of the fermions. The generators of isospin rotations are

$$[T^a, T^b] = i\epsilon^{abc}T^c, \; T^aT^a = T(T+1)\mathbb{1}. \tag{A.36}$$

The configuration

$$A_0^a = 0, \; A_i^a = A_i^a(\mathbf{x}), \; \phi^a = \phi^a(\mathbf{x}), \tag{A.37}$$

could arise as a static finite energy solution of the coupled Yang-Mills-Higgs equations in the absence of fermions. If this is the case, A and $\phi = \{\phi^a\}$ meet the earlier requirements[1]. This configuration represents a system of total magnetic charge (Kronecker index, Brouwer degree, Poincare-Hopf index, homotopy number) [38]

$$Q_{\text{top}} = -\frac{1}{8\pi}\int_{S^2_\infty} \epsilon^{abc}\phi^a \mathrm{d}\phi^b \mathrm{d}\phi^c. \tag{A.38}$$

Q_{top} is essentially the degree of the mapping $\phi : S^2_{\text{phys}} \to S^2_{\text{int}}$, where S^2_{phys} corresponds to the boundary of the physical space and S^2_{int} to the possible values of the field ϕ with fixed length. The index formula reduces to ($D=3$)

$$\text{index } L = \frac{i}{16\pi}\int_{S^2_\infty} \text{tr } U \mathrm{d}U \mathrm{d}U, \tag{A.39}$$

where $U = |\Phi|^{-1}\Phi = (\Phi^\dagger\Phi)^{-1/2}\Phi$. What is left is to evaluate this (A-independent) integral. Let therefore $\lambda_\alpha(\mathbf{x})$, $\psi_\alpha(\mathbf{x})$ be the eigenvalues and eigenvectors of $\Phi(\mathbf{x})$:

$$\Phi(\mathbf{x})\psi_\alpha(\mathbf{x}) = (\lambda_\alpha(\mathbf{x}) + m)\psi_\alpha(\mathbf{x}). \tag{A.40}$$

At $|\mathbf{x}| \to \infty$ those λ_α are just $-T, -T+1, \ldots, T-1, T$ (for the moment we can take $F=1$, since the index depends only on the ratio $\frac{m}{F}$, finally we can go back to arbitrary F). Now we have to verify [93] the formula

$$\partial_j U = \sum \left[\frac{2}{\lambda_\beta(\mathbf{x}) - \lambda_\alpha(\mathbf{x})} \text{sign}\lambda_\beta(\mathbf{x})(\psi_\alpha(\mathbf{x}), \partial_j\Phi(\mathbf{x})\psi_\beta(\mathbf{x}))\right]\psi_\alpha(\mathbf{x})(\psi_\beta(\mathbf{x}))^\dagger, \tag{A.41}$$

[1] For a discussion of the asymptotic behaviour cf. chapter 4.

where $(.\,,.)$ denotes the inner product in the finite-dimensional space of the matrix $\Phi(\mathbf{x})$ and the sum is over all α and β where the product $(\lambda_\alpha(\mathbf{x})+m)(\lambda_\beta(\mathbf{x})+m)$ is negative.

In the next step we perform matrix multiplication and trace operation in order to calculate the index via (A.39). This can be done as follows: at each point $\mathbf{x}$ let $\phi^a(\mathbf{x})$, $\phi_1^a(\mathbf{x})$, $\phi_2^a(\mathbf{x})$ be an orthonormal set of three-vectors. The following calculations are performed at spatial infinity. There ϕ^a is a vector with fixed length and we have

$$\begin{aligned}\partial_j\phi^a &= c_{1j}\phi_1^a + c_{2j}\phi_2^a\\ \partial_j\Phi(x) &= \partial_j(\phi^a T^a + m) = c_{1j}T_1 + c_{2j}T_2,\end{aligned} \tag{A.42}$$

with

$$T_0 \equiv T^a\phi^a = \Phi - m, \tag{A.43a}$$

$$T_i \equiv T^a\phi_i^a,\ i=1,2. \tag{A.43b}$$

These T_i can be arranged to form raising and lowering operators

$$T_\pm \equiv T_1 \pm iT_2, \tag{A.44}$$

with

$$T_\pm\psi_\alpha = \sqrt{T(T+1)-\lambda_\alpha(\lambda_\alpha\pm 1)}\,\psi_{\alpha\pm1}. \tag{A.45}$$

Conversely

$$T_1 = \frac{1}{2}(T_+ + T_-),\ T_2 = \frac{1}{2i}(T_+ - T_-). \tag{A.46}$$

Then it is easy to calculate the following matrix elements

$$\begin{aligned}(\psi_\alpha, T_1\psi_\beta) &= \frac{1}{2}\delta_{\lambda_\alpha,\lambda_\beta-1}\sqrt{T(T+1)-\lambda_\alpha(\lambda_\alpha+1)}\\ &+ \frac{1}{2}\delta_{\lambda_\alpha,\lambda_\beta+1}\sqrt{T(T+1)-\lambda_\alpha(\lambda_\alpha-1)},\end{aligned} \tag{A.47a}$$

$$\begin{aligned}(\psi_\alpha, T_2\psi_\beta) &= \frac{1}{2i}\delta_{\lambda_\alpha,\lambda_\beta+1}\sqrt{T(T+1)-\lambda_\alpha(\lambda_\alpha-1)}\\ &- \frac{1}{2i}\delta_{\lambda_\alpha,\lambda_\beta-1}\sqrt{T(T+1)-\lambda_\alpha(\lambda_\alpha+1)}.\end{aligned} \tag{A.47b}$$

Let $\{m\}$ be the largest eigenvalue of $\phi^a T^a$ smaller than m, or, if there is no such eigenvalue, the smallest eigenvalue of $\phi^a T^a$ minus one. Then only $(\lambda_\alpha,\lambda_\beta) = (\{m\},\{m\}+1)$ and $(\lambda_\alpha,\lambda_\beta) = (\{m\}+1,\{m\})$ contribute, since λ_α and λ_β have to differ exactly by ±1 (otherwise all matrix elements vanish due to the Kronecker δ in (A.47a) and (A.47b)) and the values $\lambda_\alpha + m$ and $\lambda_\beta + m$ have different sign, therefore $(\lambda_\alpha+m)(\lambda_\beta+m) < 0$, so they appear within the sum (A.41). With this information, a short calculation gives

$$\mathrm{tr}\; U\mathrm{d}U\mathrm{d}U = 2i\big(T(T+1) - \{m\}(\{m\}+1)\big)c_{1i}c_{2j}\,\mathrm{d}x^i\mathrm{d}x^j. \tag{A.48}$$

Using

$$\epsilon^{abc}\phi^a \mathrm{d}\phi^b \mathrm{d}\phi^c = c_{1i}c_{2j}\mathrm{d}x^i \mathrm{d}x^j \tag{A.49}$$

and formula (A.38) for Q_{top} we get

$$\begin{aligned}
\operatorname{index} L &= \frac{i}{16\pi}\int_{S^2_\infty} \operatorname{tr} U\mathrm{d}U\mathrm{d}U \\
&= -\frac{1}{8\pi}\big(T(T+1) - \{m\}(\{m\}+1)\big)\int_{S^2_\infty}\epsilon^{abc}\phi^a \mathrm{d}\phi^b \mathrm{d}\phi^c \\
&= \big(T(T+1) - \{m\}(\{m\}+1)\big)Q_{\mathrm{top}}.
\end{aligned} \tag{A.50}$$

For arbitrary F replace m by $\frac{m}{F}$ in this formula. For $Q_{\mathrm{top}} = 1$ and $m = 0$ two cases have been studied by Jackiw and Rebbi [26]:

- Isospinor case

  $$T = \frac{1}{2},\ \left\{\frac{m}{F}\right\} = -\frac{1}{2},$$

  with index $\frac{1}{2}\cdot\frac{3}{2} + \frac{1}{2}\cdot\frac{1}{2} = 1$.

- Isovector case

  $$T = 1,\ \left\{\frac{m}{F}\right\} = -1,$$

  with index $1\cdot 2 - 0 = 2$.
  Remark: Here $\frac{m}{F}$ and an eigenvalue of Φ coincide, and therefore the continuum spectrum extends down to zero. So zero is no longer isolated, L not Fredholm. But Weinberg [94] argues, that there are no contributions from the continuum to the index and (A.31) is still applicable.

The general formula (A.31) is in agreement with the explicit calculations.

B. The Atiyah-Singer Index Theorem

B.1. Basic Definitions

Consider the eigenvalue equation of the Euclidean self-adjoint Dirac operator

$$\not{D}\varphi_n(x) = \gamma_\mu(\partial_\mu + A_\mu)\varphi_n(x) = \lambda_n\varphi_n(x). \tag{B.1}$$

The φ_n form an orthonormal basis. Since $\{\gamma_\mu, \gamma_5\} = 0$ we have

$$\not{D}\gamma_5\varphi_n(x) = -\lambda_n\gamma_5\varphi_n(x), \tag{B.2}$$

so γ_5 takes eigenfunctions with positive eigenvalues into eigenfunctions with negative eigenvalues and vice versa. In the subspace S of zero modes

$$S \equiv \{\varphi_n^0 : \not{D}\varphi_n^0 = 0\}, \tag{B.3}$$

we have

$$[\not{D}, \gamma_5]\varphi_n^0 = (\not{D}\gamma_5 - \gamma_5\not{D})\varphi_n^0 = 0, \tag{B.4}$$

and therefore can choose the zero modes to be eigenfunctions of γ_5 with positive or negative chirality. Let

$$\mathbb{P}_\pm \equiv \frac{1}{2}(\mathbb{1} \pm \gamma_5), \tag{B.5}$$

and construct

$$\varphi_{n\pm}^0(x) \equiv \mathbb{P}_\pm\varphi_n^0(x), \tag{B.6}$$

with

$$\gamma_5\varphi_{n\pm}^0(x) = \pm\varphi_{n\pm}^0(x),\ \not{D}\varphi_{n\pm}^0(x) = 0. \tag{B.7}$$

The index of each self-adjoint operator vanishes by definition,

$$\text{index } \not{D} = \text{dim ker } \not{D}^\dagger - \text{dim ker } \not{D} = 0, \tag{B.8}$$

and nothing can be said about the number of zero modes of $\not{D}$. Instead of $\not{D}$ we analyse the Weyl operators

$$D_\pm \equiv \not{D}\mathbb{P}_\pm. \tag{B.9}$$

We have

$$D_\pm^\dagger = D_\mp, \tag{B.10}$$

furthermore define Laplace operators according to

$$\triangle_+ \equiv D_+^\dagger D_+ = D_- D_+, \tag{B.11a}$$
$$\triangle_- \equiv D_-^\dagger D_- = D_+ D_-. \tag{B.11b}$$

On the spaces of zero modes $S_\pm$ of positive and negative chirality,

$$S_\pm = \{\varphi^0_{n\pm} : \gamma_5 \varphi^0_{n\pm} = \pm\varphi^0_{n\pm}\}, \tag{B.12}$$

they act as shown in Figure B.1, for instance

$$\begin{aligned} D_+\varphi_+ &= \not{D}\mathbb{P}_+\mathbb{P}_+\varphi = \not{D}\mathbb{P}_+\varphi \\ &= \frac{1}{2}\not{D}(\mathbb{1}+\gamma_5)\varphi = \frac{1}{2}(\mathbb{1}-\gamma_5)\not{D}\varphi = \mathbb{P}_-\not{D}\varphi \\ &= \lambda\mathbb{P}_-\varphi = \lambda\varphi_-. \end{aligned} \tag{B.13}$$

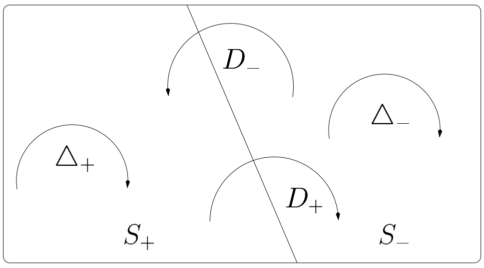

Figure B.1.: The zero mode space.

The index of the Weyl operator is given by

$$\begin{aligned} \text{index}\, D_+ &= \dim\ker D_+ - \dim\ker D_+^\dagger \\ &= \dim\ker D_+ - \dim\ker D_- \\ &= n_+ - n_-, \end{aligned} \tag{B.14}$$

the index of D_+ is the number of zero modes with positive chirality (n_+) minus the number of zero modes with negative chirality (n_-). Furthermore

$$\text{index}\, D_- = -\text{index}\, D_+. \tag{B.15}$$

Now we want to find a connection between the index of a differential operator D (later $D = D_+$) and the heat kernel of the corresponding Laplace operators $\triangle_+ = D^\dagger D$, $\triangle_- =$

$DD^\dagger$. In the case of a compact manifold $\mathcal{M}$, the spectrum of $\triangle$ is discrete and each eigenvalue has finite degeneracy.
Lemma 1: *The spectrum of nonzero eigenvalues of $\triangle_+$ and $\triangle_-$ is the same.*
Proof: suppose the eigenvalue equation

$$\triangle_+\phi_\lambda = D^\dagger D\phi_\lambda = \lambda\phi_\lambda. \tag{B.16}$$

Then $D\phi_\lambda = \psi_\lambda$ is an eigenfunction of $\triangle_-$ with the same eigenvalue

$$\triangle_-\psi_\lambda = DD^\dagger D\phi_\lambda = \lambda D\phi_\lambda = \lambda\psi_\lambda. \tag{B.17}$$

Remark: this argumentation does not hold for zero modes, the number of zero modes may be different for both operators.
Lemma 2: *$ker\,\triangle_+ = ker\,D, ker\,\triangle_- = ker\,D^\dagger$.*
Proof: We have ker $\triangle_+ = \{f : \triangle_+ f = 0\}$ and ker $D = \{f : Df = 0\}$. If $Df = 0$ then automatically $\triangle_+ f = D^\dagger Df = 0$ and if $f \in$ ker $\triangle_+$, so we have $0 = (D^\dagger Df, f) = (Df, Df)$ and consequently $Df = 0$, i.e. $f \in$ ker D.

B.2. An Index Formula for Euclidean Dirac Operators

With the definitions $E_+ = \{\phi_\lambda\}$ (eigenfunctions of $\triangle_+$) and $E_- = \{\psi_\lambda\}$ (eigenfunctions of $\triangle_-$) we have the following
Theorem 1: *index and heat kernel*

$$index\,D = tr\,_{E_+}e^{-t\triangle_+} - tr\,_{E_-}e^{-t\triangle_-}, \quad \forall t > 0. \tag{B.18}$$

Proof:

$$\begin{aligned}
\mathrm{tr}\,_{E_+}e^{-t\triangle_+} - \mathrm{tr}\,_{E_-}e^{-t\triangle_-} &= \sum_{\lambda,\phi_\lambda}\left\langle\phi_\lambda\middle|e^{-t\triangle_+}\middle|\phi_\lambda\right\rangle - \sum_{\lambda,\psi_\lambda}\left\langle\psi_\lambda\middle|e^{-t\triangle_-}\middle|\psi_\lambda\right\rangle \\
&= \sum_\lambda e^{-t\lambda}\left[\sum_{\phi_\lambda}\langle\phi_\lambda|\phi_\lambda\rangle - \sum_{\psi_\lambda}\langle\psi_\lambda|\psi_\lambda\rangle\right] \\
&= \sum_\lambda e^{-t\lambda}\big[\dim E_+(\lambda) - \dim E_-(\lambda)\big]
\end{aligned}$$

and, since for $\lambda \neq 0$ the dimensions of $E_+(\lambda)$ and $E_-(\lambda)$ are equal,

$$\begin{aligned}
\mathrm{tr}\,_{E_+}e^{-t\triangle_+} - \mathrm{tr}\,_{E_-}e^{-t\triangle_-} &= e^{-t\cdot 0}\big[\dim E_+(0) - \dim E_-(0)\big] \\
&= \dim \ker \triangle_+ - \dim \ker \triangle_- \\
&= \dim \ker D - \dim \ker D^\dagger \\
&= \mathrm{index}\ D.
\end{aligned} \tag{B.19}$$

In the case of our Weyl operator we have the
Theorem 2:

$$index\,D_+ = tr\,_S\gamma_5 e^{-t\not{D}^2}, \quad \forall t > 0. \tag{B.20}$$

Proof:

$$\begin{aligned}
\text{index } D_+ &= \text{tr }_{S_+} e^{-tD_- D_+} - \text{tr }_{S_-} e^{-tD_+ D_-} \\
&= \text{tr }_{S_+} e^{-t\not{D}^2}\mathbb{P}_+ - \text{tr }_{S_-} e^{-t\not{D}^2}\mathbb{P}_- \\
&= \text{tr }_{S=S_+\oplus S_-} e^{-t\not{D}^2}(\mathbb{P}_+ - \mathbb{P}_-) \\
&= \text{tr }_S \gamma_5 e^{-t\not{D}^2} \equiv \text{tr } \gamma_5 e^{-t\triangle}. \qquad \text{(B.21)}
\end{aligned}$$

Here we used a power series expansion of the exponential function as well as the cyclic property of the trace operation.

What is left is to evaluated the right-hand side of equation (B.21). In order to do this, we expand the function $e^{-t\triangle}$ into eigenfunctions $\chi_n(x)$ of $\triangle$ [73]. Applied to a square integrable test function φ we get

$$\begin{aligned}
e^{-t\triangle}\varphi(x) &= \int \mathrm{d}y\; e^{-t\triangle} \sum_n \chi_n(x)\chi_n^*(y)\varphi(y) \\
&= \int \mathrm{d}y\; \sum_n e^{-\lambda_n t}\chi_n(x)\chi_n^*(y)\varphi(y) \\
&\equiv \int \mathrm{d}y\; G_\triangle(x,y,t)\varphi(y). \qquad \text{(B.22)}
\end{aligned}$$

The operator $e^{-t\triangle}$ has a kernel function, the heat kernel

$$G_\triangle(x,y,t) = \sum_n e^{-\lambda_n t}\phi_n(x)\phi_n^*(y) = \left\langle x \middle| e^{-t\triangle} \middle| y \right\rangle, \qquad \text{(B.23)}$$

which satifies the so-called heat equation

$$\triangle G_\triangle(x,y,t) = -\frac{\partial}{\partial t} G_\triangle(x,y,t). \qquad \text{(B.24)}$$

This allows for the calculation of the index via the Fujikawa procedure [95, 73]: expand the heat kernel into Seeley coefficients a_n,

$$G_\triangle(x,y,t) = \frac{1}{(4\pi t)^{D/2}} \exp\left[-\frac{(x-y)^2}{4t}\right] \sum_n a_n(x,y)t^n, \qquad \text{(B.25)}$$

pick up the t-independent part and perform $t \to 0$. In D dimensions only the coefficient $a_{D/2}$ contributes. For the Dirac operator we find $a_0 \sim 1\!\!1$, $a_1 \sim G$, $a_2 \sim G^2, \ldots$.

For general even dimensional compact manifolds $\mathcal{M}^D$ follows the Atiyah-Singer index theorem. Atiyah and Singer have shown [63, 64], that the analytic index defined in (B.14) equals another index which is fully determined by topology and therefore called topological index. This is a topological invariant. Moreover, it can be expressed as an integral over certain characteristic classes, which can be found explicitly for a given differential operator.

In the case of the Dirac Operator $\not{D}$ containing the Yang-Mills gauge potential $A = A_\mu^a dx^\mu T^a$ the characteristic classes are determined by the Chern character.
Definition: the Chern character $\mathrm{ch}(G)$ is given by

$$\mathrm{ch}(G) = \mathrm{tr}\ \exp\left[\frac{i}{2\pi}G\right] = r + \frac{i}{2\pi}\mathrm{tr}\ G + \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \mathrm{tr}\ G^2 + \dots, \tag{B.26}$$

where r is the dimension of the group, and G is the curvature two-form

$$G = dA + A^2. \tag{B.27}$$

Theorem: ATIYAH-SINGER index theorem

$$\mathrm{index}\ D_+ = \int_{\mathcal{M}^D} \mathrm{ch}(G). \tag{B.28}$$

The integral is taken over the compact manifold $\mathcal{M}^D$ with dimension D, so the $D/2$-th term is picked up

$$\mathrm{index}\ D_+ = \frac{1}{(D/2)!}\left(\frac{i}{2\pi}\right)^{D/2} \int_{\mathcal{M}^D} \mathrm{tr}\ G^{D/2}. \tag{B.29}$$

B.3. Examples: 2 and 4 Dimensions

Example: in $D = 2$ dimensions we get

$$\mathrm{index}\ D_+ = -\frac{1}{4\pi}\int \mathrm{d}x\ \epsilon_{\mu\nu}F_{\mu\nu} = -\frac{1}{2\pi}\int_{S^2} F. \tag{B.30}$$

This equation can be used to determine the number of zero modes localized near vortex-like configurations in planar electrodynamics [96], but will not be discussed here.
Example: in $D = 4$ dimensions we get in the same way

$$\mathrm{index}\ D_+ = \frac{1}{2!}\left(\frac{i}{2\pi}\right)^2 \int_{S^4} \mathrm{tr}\ G^2 = -\frac{1}{8\pi^2}\int_{S^4} \mathrm{tr}\ G^2, \tag{B.31}$$

observe that the index is equal to the topological charge of the Yang-Mills background field defined in chapter 5:

$$\mathrm{index}\ D_+ = Q_{\mathrm{Pont}}. \tag{B.32}$$

Interpretation: in a background field with Pontryagin index Q_{Pont} the number of zero modes with positive chirality minus the number of zero modes with negative chirality is equal to Q_{Pont}.

Due to the trace operation the index depends on the representation of the gauge group. In the fundamental representation of $SU(2)$, where $\mathrm{tr}\ T^aT^b = -\frac{1}{2}\delta^{ab}$ the index of D_+ is equal to Q_{Pont}, in the adjoint representation we have $\mathrm{tr}\ T^aT^b = -2\delta^{ab}$ and the index of D_+ is equal to $4Q_{\mathrm{Pont}}$. In general, for fermions in the representation with isospin T, we have [97, 98]

$$\mathrm{index}\ D_+ = n_+ - n_- = \frac{2}{3}T(T+1)(2T+1)Q_{\mathrm{Pont}}. \tag{B.33}$$

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Erklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Andreas Kirchberg   Jena, 28. September 2000