Functional Renormalization Group Approach to the 3-Dimensional $\mathcal{N} = 2$ Wess-Zumino Model

seit 1558

Master thesis submitted in partial fulfillment
of the requirements for the degree

Master of Science

in Physics at the
Institute for Theoretical Physics,
Faculty of Physics and Astronomy,
Friedrich Schiller University Jena

Polina Feldmann
Jena, November 1, 2016

Supervisors:
Prof. Dr. Andreas Wipf, Dr. Luca Zambelli

For my grandfather.

Wess-Zumino models constitute a class of particularly simple supersymmetric quantum field theories. This makes them well suitable for studying the basic implications of supersymmetry. Thus, they pave the way for more sophisticated theories which are relevant e. g. with regard to particle physics beyond the standard model. Furthermore, Wess-Zumino models arise in the context of emergent supersymmetry. Recently, a conformal bootstrap study has provided results supposed to characterize the three-dimensional $\mathcal{N} = 2$ Wess-Zumino model. The authors could find but little to compare their findings with. We analyze this model, now using the functional renormalization group approach. Particularly, we provide approximate values of the critical exponents at the non-trivial fixed point. Our work follows prior successful applications of the functional renormalization group to Wess-Zumino models in two and three dimensions.

Contents

1. Introduction

Starting from the early 1970s, supersymmetry has received a great deal of attention. This is due to several appealing properties. The Coleman-Mandula theorem proves that internal symmetries cannot be included non-trivially into the algebra of spacetime transformations assumed in quantum field theory. However, it a priori excludes symmetry groups with a graduated algebra. Dropping this restriction, the Haag–Lopuszanski–Sohnius theorem shows that supersymmetry provides the only non-trivial extension of the spacetime algebra. Thus, supersymmetry is fundamentally outstanding. In addition to this, it is beloved for its mathematical beauty. This, in turn, entails numerous applications.

Supersymmetry is a pivotal ingredient to string theories. Furthermore, it is extensively employed in theories going beyond the standard model. It provides solutions to problems such as the hierarchy of energy scales, the smallness of the cosmological constant, and the renormalizability of supergravity. Moreover, it has been successfully applied to disordered condensed matter systems. Finally, during the last decade another topic — emergent supersymmetry — has come up. Introductions to supersymmetry can be found e. g. in [1–4]. Emergent supersymmetry is addressed in [5–8] and others.

Wess-Zumino models are particularly simple quantum field theories exhibiting supersymmetry. They trace back to the first explicit formulation of a supersymmetric Lagrangian density in [9]. The literature contains numerous lattice studies of Wess-Zumino models; see [10] for a review of different approaches to supersymmetry on the lattice. The non-trivial fixed point of the three-dimensional $\mathcal{N} = 2$ Wess-Zumino model considered in the present thesis constitutes the simplest non-trivial superconformal theory. Furthermore, it explicitly occurs in the context of emergent supersymmetry [7]. This fixed point model has been studied by conformal bootstrapping in [11] and [12]. The authors feel the need of further comparative results. The present thesis addresses this gap of knowledge. We analyze the concerned fixed point of the three-dimensional $\mathcal{N} = 2$ Wess-Zumino model using the functional renormalization group (FRG) approach.

The renormalization group (RG) made its first appearance in the 1950s within the scope of quantum field theories. During its further development Kadanoff proposed his “blocking” procedure for the treatment of critical phenomena in thermodynamics. Wilson demonstrated in his famous paper [13] from 1971 that the Kadanoff picture and the quantum field theoretical renormalization group approach are two sides of the same coin. This and subsequent papers thoroughly deepened the understanding of the renormalization group.

The underlying idea is to describe the very same system at different length or momentum (energy) scales. The renormalization group approach has proven itself to be a very flexible, versatile tool. It is applied to topics as diverse as quantum phase transitions, gauge theories, quantum gravity, equilibrium critical phenomena, dynamical criticality, disordered models, and others. Particularly, the functional renormalization group has already been used for the study of supersymmetric models. Significant contributions go back to Rosten, see e. g. [14], and Wipf et al., e. g. [15–17]. Introductions to the (functional) renormalization group are provided e. g. by [18–21].

In [22–25], different Wess-Zumino models are analyzed by means of a particular formulation of the FRG — the Wetterich equation. The present thesis closely follows these works. Our computations provide approximate values of the critical exponents at the fixed point of interest. Particularly, we come fairly close to the critical exponent conjectured in [11]. Presumably, the agreement could be enhanced by further exploitation of the FRG approach.

This thesis is organized as follows: In chapter 2, some basics of quantum field theory and supersymmetry are recapitulated. We introduce the four-dimensional $\mathcal{N} = 1$ Wess-Zumino model and derive the three-dimensional $\mathcal{N} = 2$ Wess-Zumino model from it. Chapter 3 gives an idea of critical phenomena and the (functional) renormalization group approach. Furthermore, the Wetterich equation is discussed. In chapter 4, we present our computation of critical exponents at the non-trivial fixed point of the three-dimensional Wess-Zumino model. Each section is dedicated to a different approximation. The results are compared to [11]. Chapter 5 summarizes our findings and gives an outlook on pending work.

2. The Supersymmetric Model

2.1. Quantum Fields and Spin

The fundamental objects of a quantum field theory in d spatial and t temporal dimensions are fields $F\colon \mathbb{R}^{d+t} \to \mathbb{H}^n(\mathbb{C}^n, \mathbb{R}^n)$ which are defined to exhibit a specific behavior under coordinate transformations. Let Λ be an arbitrary element of the connected Lie group $SO^{\uparrow}(d,t)$. Then, if $D := d + t > 2$, $\Lambda[F](x) \equiv F'(x) = r(\Lambda)F(\Lambda^{-1}x)$ for some linear representation r of the universal covering group of $SO^{\uparrow}(d,t)$, which is a double cover isomorphic to $Spin(d,t)$. For smaller dimensions, $D \leq 2$, the universal covering groups do not exhibit a covering index of two. The index is preferred over universality. With respect to $SO^{\uparrow}(d,t)$ the set of all possible r becomes the set of all actual and projective representations. Concerning translations $\tau_y\colon x \mapsto x + y$ all fields are set to stay invariant — $\tau_y[F](x) \equiv F'(x) = F(x - y)$.

If $SO(2)$ is a subgroup of $SO^{\uparrow}(d,t)$, it is possible to assign a spin to the fields. Any complex representation of $SO(2) \cong U(1)$ is equivalent to a direct sum of its irreducible, one dimensional ones, which are canonically labeled by integer numbers. Restricting the potentially projective r to $SO(2)$ defines some set of half integers. What is called spin is their greatest absolute value [26]. While true representations describe bosons with integer spin, the projective ones correspond to a half integer spin and thus fermions. In this thesis, we will consider only spin-0 scalar fields and spin-1/2 Dirac and Majorana spinors.

All elements $\Lambda \in SO^{\uparrow}(d,t)$ can be parametrized by a generic antisymmetric matrix $\omega_{\mu\nu}$, assigning [1]

$$\Lambda(\omega) := \mathrm{e}^{(\omega^{\rho}{}_{\sigma})}\,. \tag{2.1}$$

This is equivalent to writing

$$\Lambda(\omega) = \mathrm{e}^{\frac{i}{2}(\omega_{\mu\nu}(M^{\mu\nu})^{\rho}{}_{\sigma})} \quad \text{with} \quad (M^{\mu\nu})^{\rho}{}_{\sigma} = -i(\eta^{\mu\rho}\eta^{\nu}{}_{\sigma} - \eta^{\nu\rho}\eta^{\mu}{}_{\sigma}), \tag{2.2}$$

where $\eta^{\mu\nu}$ is the metric of the considered spacetime. These $M^{\mu\nu}$ define the Lie algebra $\mathfrak{so}(d,t)$:

$$i[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\sigma\mu}M^{\rho\nu} + \eta^{\sigma\nu}M^{\rho\mu} \quad \text{with} \quad M^{\mu\nu} = -M^{\nu\mu} \tag{2.3}$$

The (projective) representations of $SO^{\uparrow}(d,t)$ can be obtained using the one-to-one correspondence

$$r(\Lambda) = \mathrm{e}^{\frac{i}{2}\omega_{\mu\nu}r(M^{\mu\nu})} \tag{2.4}$$

between (projective) representations of a connected Lie group and true ones of its algebra [27].

The trivial representation defines a scalar field. It maps all elements of the algebra to zero and thus any Λ to one. Consequently, the corresponding field has only one component and fulfills $\Lambda[\phi](x) = \phi(\Lambda^{-1}x)$ or, equivalently, $\phi'(x') = \phi(x)$. According to the above definition its spin is 0. The transformation prescription can be reformulated by means of the orbital angular momentum

$$L^{\mu\nu} = -i(x^\mu \partial^\nu - x^\nu \partial^\mu) \tag{2.5}$$

constituting a faithful representation of $\mathfrak{so}(d, t)$ to get

$$\phi'(x) = \mathrm{e}^{\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}}\phi(x). \tag{2.6}$$

Let us now introduce the Clifford algebra whose generators, the Dirac or Gamma matrices Γ^μ, satisfy

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}. \tag{2.7}$$

Setting

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu] \tag{2.8}$$

we find that also the $\Sigma^{\mu\nu}$ behave as generators of $\mathfrak{so}(d, t)$. The size of the Gamma matrices in $D = d + t$ dimensions is bound from below by $2^{\lfloor \frac{D}{2} \rfloor}$ [27]. Commonly, only such minimal size representations of the Clifford algebra, which are necessarily irreducible, are considered. Additionally, one can choose a realization of the Gamma matrices which respects $\Gamma^{j\dagger} = -\Gamma^j$ for all spatial and $\Gamma^{s\dagger} = \Gamma^s$ for all temporal directions [1]. Throughout the succeeding computations, this hermiticity property will be assumed. A field transforming via

$$\psi'(x') = \exp^{\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}\psi(x) =: S(\omega)\psi(x) \tag{2.9}$$

is called a Dirac spinor. Its spin equals 1/2. With the above prescription, one can easily verify this result for a particular spacetime, e. g. $(3, 1)$, and Clifford algebra representation, e. g. the common Weyl representation. As for the scalar field, we can rewrite the transformation rule, employing the orbital angular momentum:

$$\psi'(x) = \exp^{\frac{i}{2}\omega_{\mu\nu}(L^{\mu\nu}+\Sigma^{\mu\nu})}\psi(x). \tag{2.10}$$

All $[L^{\mu\nu}, \Sigma^{\rho\sigma}] = 0$ with the result that also the sum $L^{\mu\nu} + \Sigma^{\mu\nu}$ represents the algebra $\mathfrak{so}(d, t)$. Due to the spin-statistics theorem, in contrast to bosons fermionic fields or, to be more precise, their components anticommute.

Let the Dirac conjugate of a spinor ψ be defined as

$$\bar{\psi} = \psi^\dagger A \quad \text{with} \quad A = \mathbb{1}\Gamma^0 \cdots \Gamma^{t-1}. \tag{2.11}$$

This ensures that when ψ transforms via $\psi' = S\psi$, $\bar{\psi}$ goes to $\bar{\psi}S^{-1}$.

We call every matrix C fulfilling

$$\Gamma^{\mu T} = -\eta C^{-1} \Gamma^\mu C \quad \text{and} \quad C^T = -\epsilon C \tag{2.12}$$

for some $\eta, \epsilon \in \{\pm 1\}$ a charge conjugation matrix and

$$\psi_C = C \bar{\psi}^T, \tag{2.13}$$

which transforms the same way as ψ, the charge conjugated spinor. Depending on $D \bmod 8$ one or two particular combinations of η and ϵ are viable [1]. For some sets of η, ϵ, and t, one finds $(CA^T)(CA^T)^* = \mathbb{1}$. This allows for a consistent restriction of Dirac spinors to $\psi_C = \psi$, thus defining Majorana spinors. Simultaneously, a representation of the Gamma matrices admitting $C = (A^T)^{-1}$ called Majorana representation can be chosen. Then $\psi_C = \psi^*$ and the Majorana condition simply constrains Dirac spinors to $\mathbb{R}^n$. For example, a representation constituted of a purely imaginary or purely real set of matrices and still fulfilling the above hermiticity property will always be Majorana.

Finally, let us have a look at a useful form of the Fierz identity. The matrices

$$\Gamma_{\mu_1, \dots, \mu_p} := \Gamma_{[\mu_1} \cdots \Gamma_{\mu_p]} \tag{2.14}$$

with $1 \leq p \leq D$ for D even and $1 \leq p \leq (D-1)/2$ for D odd constitute an orthogonal basis of the complex $2^{\lfloor \frac{D}{2} \rfloor}$-dimensional matrix space. Therefore, for two D-dimensional spinors ψ, χ one can derive [1]

$$2^{\lfloor \frac{D}{2} \rfloor} \psi \bar{\chi} = -\sum_p \frac{1}{p!} (-1)^{p(p-1)/2} \Gamma_{\mu_1, \dots, \mu_p} (\bar{\chi} \Gamma^{\mu_1, \dots, \mu_p} \psi). \tag{2.15}$$

The sum is supposed to include all elements of the particular basis.

2.2. The Algebra of Supersymmetry

Let us consider an arbitrary local, classical field theory involving some set of field components F_i. The corresponding Lagrangian density $\mathcal{L}$ will depend on $x \in \mathbb{R}^{d+t}$, on $F_i(x)$, and on derivatives of F_i at x, defining the action

$$S = \int \mathrm{d}^D x \; \mathcal{L}(x, F_i(x), \partial_\mu F_i(x), \partial_\rho \partial_\sigma F_i(x), \dots). \tag{2.16}$$

As customary, we will assume that when $|x^\mu| \to \infty$, the $|F_i(x)|$ descend to zero fast enough to make all occurring surface integrals over the boundary of the spacetime vanish.

The dynamics of the field theory are determined by the Euler-Lagrange equations

$$0 = \frac{\delta S}{\delta F_i(y)} = \frac{\delta \mathcal{L}}{\delta F_i(y)} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu F_i(y)} + \partial_\rho \partial_\sigma \frac{\delta \mathcal{L}}{\delta \partial_\rho \partial_\sigma F_i(y)} + \dots \tag{2.17}$$

Now, let us recall the Noether theorem for classical fields. Consider the infinitesimal form

$$F_i(x) \to F_i(x) + \alpha_m \delta^m[F]_i(x) \tag{2.18}$$

of continuous transformations $F_i(x) \to F_i'(x) = T_\alpha[F]_i(x)$ parametrized by an n-dimensional α. The action is invariant under the infinitesimal transformation if and only if $\delta^m[F]_i$ changes the Lagrangian density at most by adding a total divergence $\partial_\mu (V^m)^\mu$. If S is invariant, we call T a symmetry of the Lagrangian, action, and considered theory. Employing the Euler-Lagrange equations, one can show that every n-dimensional symmetry corresponds to n conserved currents

$$\begin{aligned} &(j^m)^\mu = \sum_i \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu F_i} \delta^m[F]_i - \partial_\nu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \partial_\nu F_i} \right) \delta^m[F]_i + \frac{\delta \mathcal{L}}{\delta \partial_\mu \partial_\nu F_i} \partial_\nu \delta^m[F]_i + \dots - (V^m)^\mu \right), \\ &\partial_\mu (j^m)^\mu = 0. \end{aligned} \tag{2.19}$$

Solving $\partial_\mu (j^m)^\mu = 0$ for $\partial_0 (j^m)^0$ and integrating over all x^μ except for x^0 defines a set of conserved, linearly independent charges

$$c^m = \int \mathrm{d}x^1 \cdots \mathrm{d}x^D \, (j^m)^0, \;\; \partial_0 c^m = 0. \tag{2.20}$$

If, as most commonly, a spacetime with one temporal direction is considered, x^0 is assumed to be the time coordinate. After introducing the conjugate momentum

$$\pi_i = \frac{\delta \mathcal{L}}{\delta \partial_0 F_i} \tag{2.21}$$

for each field component, we can use the common definition of Poisson brackets to restore the infinitesimal field transformation from the corresponding charge:

$$\delta^m F(x) = \{F(x), c^m\}_p. \tag{2.22}$$

As long as the symmetries of an action constitute a connected Lie group, transformations of the form

$$T_\alpha[F](x) = r(T_\alpha) F(T_\alpha^{-1} x) \tag{2.23}$$

can always be rewritten as

$$T_\alpha[F](x) = \mathrm{e}^{i \alpha_m r(\mathcal{T}^m)} \, F(x), \tag{2.24}$$

where $r(\mathcal{T}^m)$ is a linear representation of the particular Lie algebra. For an example, see the discussion of $SO^{\uparrow}(d,t)$ in the previous section. Consequently, $\delta^m = ir(\mathcal{T}^m)$. To trace the recurrence of the symmetry's algebra, we introduce the following

Notation: When writing $\delta^m\delta^n F$ we assume that δ^m acts directly on the field content of $\delta^n F$. This particularly implies $\delta^m\delta^n F = -r(\mathcal{T}^n)r(\mathcal{T}^m)F$.

Thus, the $-ic^m$ obey, in combination with Poisson brackets, the same algebraic relations as the $i\delta^m$ and $r(\mathcal{T}^m)$ along with the usual commutator. To see this, one can consider

$$\begin{aligned}\{\{F,c^m\}_p,c^l\}_p - \{\{F,c^l\}_p,c^m\}_p &= ir(\mathcal{T}^m)\{F,c^l\}_p - ir(\mathcal{T}^l)\{F,c^m\}_p \\ &= [ir(\mathcal{T}^m), ir(\mathcal{T}^l)]F \\ &= [\delta^l,\delta^m]F\end{aligned} \tag{2.25}$$

and rewrite the left hand side by means of the Jacobi identity. Of course, the algebras generated by the δ^m and charges c^m are isomorphic to the one of the symmetry group.

We postulate that a physical model comes with an action invariant under coordinate transformations. In the previous section we have specified which kind of transformations we assume to connect inertial reference frames. The (d,t)-spacetime group of translations and $SO^{\uparrow}(d,t)$ are joined by a semidirect product $\mathbb{R}^{d+t} \rtimes SO^{\uparrow}(d,t)$. The algebra of this Lie group is an extension of $\mathfrak{so}(d,t)$ including the generator of translations P^μ. Its defining commutation relations read [27]:

$$\begin{aligned} i[J^{\mu\nu},J^{\rho\sigma}] &= \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu} \quad \text{with} \quad J^{\mu\nu} = -J^{\nu\mu}, \\ i[P^\mu,J^{\rho\sigma}] &= \eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho, \\ [P^\mu,P^\nu] &= 0. \end{aligned} \tag{2.26}$$

Adding $r(P^\mu) = i\partial^\mu$ to the representations of $J^{\rho\sigma}$ — $r(J^{\rho\sigma}) = L^{\rho\sigma}$ or $r(J^{\rho\sigma}) = L^{\rho\sigma} + \Sigma^{\rho\sigma}$ — discussed above, we can restate the transformation under translations as

$$\tau_y[F](x) = F(x-y) = \mathrm{e}^{-y_\mu\partial^\mu}F(x) = \mathrm{e}^{iy_\mu r(P^\mu)}F(x). \tag{2.27}$$

The Noether theorem provides us with the angular momentum $c_J^{\rho\sigma}$ — the charge generating $SO^{\uparrow}(d,t)$ transformations,

$$F \to F + \delta_\omega F, \;\; \delta_\omega F = \frac{1}{2}\omega_{\rho\sigma}\{F,c_J^{\rho\sigma}\}_p = \frac{i}{2}\omega_{\rho\sigma}(L^{\rho\sigma}+\Sigma^{\rho\sigma})F = \frac{i}{2}\omega_{\rho\sigma}r(J^{\rho\sigma})F, \tag{2.28}$$

where $\Sigma^{\rho\sigma}$ applies only if F is a spinor, and the momentum charge c_P^μ,

$$F \to F + \delta_y F, \;\; \delta_y F = y_\mu\{F,c_P^\mu\}_p = -y_\mu\partial^\mu F = iy_\mu r(P^\mu)F. \tag{2.29}$$

Remark: The factor 1/2 in Eq. (2.28) compensates our transition from the $D(D-1)/2$ independent elements of both ω and J to antisymmetric $\omega_{\mu\nu}$, $J^{\mu\nu}$.

A supersymmetric theory always contains both bosonic and fermionic fields ϕ_i and ψ_j. Its action is, in addition to the $\mathbb{R}^{d+t} \rtimes SO^{\uparrow}(d,t)$ symmetry, invariant under infinitesimal transformations of the form

$$\begin{aligned} &F(x) \to F(x) + \delta_\alpha F(x), \\ &\delta_\alpha \phi_i = \bar{\alpha}_{1\mu}\, f^{\mu}_{1i}(\psi_j) + \cdots + \bar{\alpha}_{N\mu}\, f^{\mu}_{Ni}(\psi_j), \\ &\delta_\alpha \psi_j = g^{\mu}_{1j}(\phi_i)\, \alpha_{1\mu} + \cdots + g^{\mu}_{Nj}(\phi_i)\, \alpha_{N\mu}. \end{aligned} \tag{2.30}$$

Here, f and g are some functions, and α_i are “anticommuting parameters” which equal spinors aside from the fact that they are no fields and thus are not affected by any transformations. Each α_i carries $\lfloor \frac{D}{2} \rfloor$ real degrees of freedom. It is said that supersymmetry transforms bosons into fermions and vice versa.

Extending the algebra of the P^μ and $J^{\rho\sigma}$ to include the generators Q^m of supersymmetry transformations we arrive at a $\mathbb{Z}_2$ graded Lie algebra, whose elements can be classified as even e (bosonic) or odd o (fermionic) [28]. The generalized Lie bracket $\langle \cdot, \cdot \rangle$ respects the grading:

$$\langle e, e\rangle = [e, e] = e, \;\; \langle e, o\rangle = [e, o] = o, \;\; \langle o, o\rangle = \{o, o\} = e. \tag{2.31}$$

Here, $\{\cdot, \cdot\}$ defines a bilinear, symmetric operation. We will always imply the usual anticommutator. Assigning a grade $\deg(e) = 0$ and $\deg(o) = 1$ to all elements g of the algebra, we can state the axiomatic generalized Jacobi identity as [1]

$$\begin{aligned} (-1)^{\deg(g_1)\deg(g_3)} \langle g_1, \langle g_2, g_3\rangle\rangle + (-1)^{\deg(g_2)\deg(g_1)} \langle g_2, \langle g_3, g_1\rangle\rangle \\ + (-1)^{\deg(g_3)\deg(g_2)} \langle g_3, \langle g_1, g_2\rangle\rangle = 0. \end{aligned} \tag{2.32}$$

As can be guessed from Eq. (2.31), the P^μ and $J^{\rho\sigma}$ are even and the Q^m odd. The latter can always be chosen to constitute $\mathcal{N}$ Q^μ_i which are called spinors because of their transformation behavior under the adjoint representation of $SO^{\uparrow}(d,t)$:

$$[J^{\mu\nu}, Q_i] = -\Sigma^{\mu\nu} Q_i \tag{2.33}$$

For the remaining algebraic relations defining the algebra of supersymmetry, it can be proven that [28]

$$[P^\mu, Q^\nu_i] = 0, \;\; \text{and} \;\; \{Q^\mu_i, Q^\nu_j\}, \; \{Q^\mu_i, \bar{Q}^\nu_j\} \propto P^\rho. \tag{2.34}$$

The particular results of the anticommutators depend on the considered theory.

Remark: The $\bar{Q}_i$ do not add anything new to the algebra of P^μ, $J^{\rho\sigma}$ and Q^ν_i because they are not independent of the Q_i.

If the number $\mathcal{N}$ of Q_i is greater than 1, the supersymmetry is usually said to be extended. For the sake of completeness, let us mention that a generic supersymmetric theory may include additional even generators. These are called central charges, because they commute with the entire algebra of supersymmetry.

Which impact does the quantization of a field theory have on these results [1]? The fields F become operators $\hat{F}$ acting on a Hilbert space of states. Furthermore, symmetry transformations continuously connected to the identity can be represented, according to Wigner's theorem, by linear unitarian operators $\hat{U}$ transforming the fields via

$$\hat{F}' = \hat{U}\hat{F}\hat{U}^{-1}. \tag{2.35}$$

Using the exponential map, the unitarian operators can be expressed in terms of Hermitian ones. For the transformations discussed in this section we write

$$\hat{U}(\omega) = \mathrm{e}^{\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}}, \quad \hat{U}(y) = \mathrm{e}^{iy_\mu \hat{P}^\mu}, \quad \text{and} \quad \hat{U}(\alpha) = \mathrm{e}^{i\sum_j (\bar{\alpha}_j)_\mu (\hat{Q}_j)^\mu}. \tag{2.36}$$

Thus, the Hermitian operators generate the same algebra as their namesakes introduced before. The infinitesimal formulation of Eq. (2.35) reads

$$\begin{aligned} &\hat{F} \to \hat{F} + \delta\hat{F}, \\ &\delta_\omega \hat{F} = \frac{i}{2}\omega_{\mu\nu}[\hat{J}^{\mu\nu}, \hat{F}], \quad \delta_y \hat{F} = iy_\mu[\hat{P}^\mu, \hat{F}], \quad \text{and} \quad \delta_\alpha \hat{F} = i\sum_j [(\bar{\alpha}_j)_\mu (\hat{Q}_j)^\mu, \hat{F}]. \end{aligned} \tag{2.37}$$

Remark: Note that the anticommuting parameters $\bar{\alpha}_j$ are inside the commutator.

If we identify the classical charges with the Hermitian operators, this becomes the Dirac equivalent[1] of Eq. (2.22), which derives infinitesimal symmetry transformations from charges. From here on, we will omit the hats over operators and call P^μ, $J^{\rho\sigma}$, and Q_i^ν both generators and charges of the corresponding symmetry transformations. Particularly, the Q_i will be usually denoted as supercharges.

2.3. The Standard Wess-Zumino Model: $\mathrm{D} = 4$ and $\mathcal{N} = 1$

Assume a $(3,1)$-spacetime with metric $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Then $O(d,t)$ becomes the Lorentz and $\mathbb{R}^{d+t} \rtimes O(d,t)$ the Poincaré group. Referring to Lorentz or Poincaré transformations we, however, will consider only elements of $SO^\uparrow(d,t) \subset O(d,t)$ and $\mathbb{R}^{d+t} \rtimes SO^\uparrow(d,t) \subset \mathbb{R}^{d+t} \rtimes O(d,t)$, respectively.

A Wess-Zumino model is a supersymmetric quantum field theory built from the minimal number of field degrees of freedom admitting supersymmetry in $(3,1)$ dimensions. If we insist on a

[1]We assume the observation that commutators on operators of a quantized theory, $[\hat{X}, \hat{Y}]$, correspond to i times the Poisson bracket of the respective classical objects, $i\{X, Y\}_p$.

standard kinetic term, an analytic formulation which apart from this restriction is generic [14] reads

$$\mathcal{L} = \partial_m \phi \partial^m \phi^\dagger + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi + ff^\dagger + \left\{ \frac{\partial W(\phi)}{\partial \phi} f - \frac{1}{4}\bar{\psi}(1-\Gamma_5)\frac{\partial^2 W(\phi)}{\partial \phi^2}\psi + \text{h. c.} \right\}, \tag{2.38}$$

where ϕ and f are complex scalar fields and ψ is a Majorana spinor. $W(\phi)$ is a holomorphic function called the superpotential. We assume that its couplings c_n from

$$W(\phi) = \sum_{n=1}^{\infty} \frac{c_n}{n}\phi^n \tag{2.39}$$

are real. As $\mathcal{L}$ depends only on derivatives of W, c_0 is arbitrary and can be set to zero. The Latin indexes run from zero to three, while

$$\partial\!\!\!/ = \Gamma^\mu \partial_\mu \quad \text{and} \quad \Gamma_5 = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3. \tag{2.40}$$

Eq. (2.38) gives an off-shell formulation of the Lagrangian density: The Euler-Lagrange equations for f and $f^\dagger$ are algebraic. Thus, f can be eliminated from $\mathcal{L}$, which is why f is called an auxiliary field. It has several computational advantages to rely on this off-shell formulation. Particularly, the infinitesimal supersymmetry transformations become linear. We consider

$$\begin{aligned}
\delta_\alpha \phi &= \frac{1}{\sqrt{2}}\bar{\alpha}(1-\Gamma_5)\psi, \\
\delta_\alpha F &= -\frac{i}{\sqrt{2}}\bar{\alpha}(1+\Gamma_5)\partial\!\!\!/\psi, \\
\delta_\alpha \psi &= \frac{1}{\sqrt{2}}(F - i\partial\!\!\!/\phi^\dagger)(1-\Gamma_5)\alpha + \frac{1}{\sqrt{2}}(F^\dagger - i\partial\!\!\!/\phi)(1+\Gamma_5)\alpha,
\end{aligned} \tag{2.41}$$

where α is Majorana.

Because $\mathcal{L}$ transforms as a scalar field, integration by substitution immediately shows that the corresponding action S is invariant under Poincaré transformations. As for the supersymmetry, one can explicitly compute

$$\delta_\alpha L = \bar{\alpha}\partial_\mu (V_1^\mu + V_2^\mu + V_2^{\dagger\mu}) \tag{2.42}$$

with

$$\begin{aligned}
V_1^\mu &= -\frac{i}{2\sqrt{2}}\Gamma^\mu \Big((F + i\partial\!\!\!/\phi)(1+\Gamma_5) + (F^\dagger + i\partial\!\!\!/\phi^\dagger)(1-\Gamma_5) \Big)\psi, \\
V_2^\mu &= -\frac{i}{\sqrt{2}}\Gamma^\mu \frac{\partial W}{\partial \phi}(1-\Gamma_5)\psi.
\end{aligned} \tag{2.43}$$

Before deriving the algebraic relations concerning the supercharge Q, we redefine $r(Q)$, setting

$$\delta_\alpha = \bar{\alpha} r(Q). \tag{2.44}$$

A look at the generic properties of an algebra of supersymmetry, see section 2.2, reveals that this solely flips the sign of $\{r(Q_i^\mu), r(Q_j^\nu)\}$. In return, apart from its operator role, $r(Q)$ becomes a Majorana anticommuting parameter, just as α.

All (anti)commutators involving $r(Q)$ can be computed, applying

$$\begin{aligned}
&[\bar{\beta} r(Q), r(\bar{Q})\alpha] \\
&= [\bar{\beta} r(Q), \bar{\alpha} r(Q)] = [\delta_\alpha, \delta_\beta], \\
&[-iyr(P), \bar{\alpha} r(Q)] = [\delta_\alpha, \delta_y], \\
&[\frac{i}{2}\omega r(J), \bar{\alpha} r(Q)] = [\delta_\alpha, \delta_\omega]
\end{aligned} \tag{2.45}$$

to each field. As an example, let us evaluate the less trivial

$$\begin{aligned}
[\delta_\alpha, \delta_\omega]\phi &= \delta_\alpha \frac{i}{2}\omega L\phi - \delta_\omega \frac{1}{\sqrt{2}}\bar{\alpha}(1-\Gamma_5)\psi \\
&= \left(\frac{i}{2}\omega L\right)\left(\frac{1}{\sqrt{2}}\bar{\alpha}(1-\Gamma_5)\right)\psi - \left(\frac{1}{\sqrt{2}}\bar{\alpha}(1-\Gamma_5)\right)\left(\frac{i}{2}\omega(L+\Sigma)\right)\psi \\
&= -\frac{i}{2\sqrt{2}}\bar{\alpha}(1-\Gamma_5)\omega\Sigma\psi \\
&= -\frac{i}{2\sqrt{2}}\omega_{\mu\nu}\bar{\alpha}\Sigma^{\mu\nu}(1-\Gamma_5)\psi \\
&= -\frac{i}{2}\omega_{\mu\nu}\bar{\alpha}\Sigma^{\mu\nu} r(Q)\phi.
\end{aligned} \tag{2.46}$$

Remark: For the computation of $[\delta_\alpha, \delta_\omega]$ on ψ, it is advantageous to use $\delta_\alpha = r(\bar{Q})\alpha$. Then one can easily derive $[J^{\mu\nu}, \bar{Q}]$. The sought $[J^{\mu\nu}, Q]$ is gained by Hermitian conjugation.

Because the algebra of a full-dimensional representation must be identical to the original and the exceptional $\{r(Q)^\mu, r(Q)^\nu\}$ turns out to vanish, we can omit r, getting

$$\begin{aligned}
\{Q^\mu, \bar{Q}^\nu\} &= -2(\Gamma_\rho)^{\mu\nu} P^\rho, \\
\{Q^\mu, Q^\nu\} &= 0, \\
[P^\mu, Q^\nu] &= 0, \\
[J^{\mu\nu}, Q] &= -\Sigma^{\mu\nu} Q.
\end{aligned} \tag{2.47}$$

This result perfectly agrees with our generic statements from section 2.2. Deviations from [1], are due to different conventions.

2.4. Dimensional Reduction

Dimensional reduction establishes an identification of field theories defined in spacetimes of different dimensions. Imagine that we have a D-dimensional supersymmetric model $\mathcal{L}$. We can modify it by compactifying one or more coordinate directions, e. g.

$$\mathbb{R}^D \to S^1_R \times \mathbb{R}^{D-1}. \tag{2.48}$$

S_R denotes a circle with radius R. This neither destroys the supersymmetry of $\mathcal{L}$, nor does it interfere with the invariance under coordinate transformations within the $\mathbb{R}^{D-1}$ subspace. Now, any field can be expanded in modes with a $2\pi R/n$-periodic or absent dependence on the compactified coordinate y. For all n-modes the y-derivative goes as $1/R$. Thus, sending R to zero causes the kinetic term of a y-dependent field to diverge.

All quantum field theoretical observables can be computed by means of path integrals, which sum over all fields, weighting them with $\exp(iS)$ where S is the action. In case of a y-dependence, the phase S diverges along with the kinetic term. Thus, the corresponding contributions to the integral are averaged out. Another way to see that they are eliminated arises from applying a Wick rotation to the path integral. Then the exponential weight becomes real — a divergent kinetic term damps the integrand away. Omitting y-dependent fields we can immediately evaluate the y-integration in S. Thus, we are left with a $(D-1)$-dimensional supersymmetric Lagrangian density and action.

2.5. Key Model: $\mathrm{D} = 3$ and $\mathcal{N} = 2$

The present thesis focuses on the Euclidean three-dimensional $\mathcal{N} = 2$ Wess-Zumino model. We use dimensional reduction, to deduce it from the four-dimensional standard one introduced in section 2.3. As we are interested in a Euclidean formulation, we compactify the time coordinate.

Accordingly, only fields independent of x^0 are considered. This means that ∂_0 can be omitted. The time integral is shifted from the action to the Lagrangian density. We regard it as the $T \to \infty$ limit of an integration over a finite interval of length T. Before taking the limit, we absorb T by the fields and coupling constants:

$$\phi \to \frac{\phi}{\sqrt{T}}, \quad \psi \to \frac{\psi}{\sqrt{T}}, \quad f \to \frac{f}{\sqrt{T}}, \quad \text{and} \quad W \to \frac{W}{T}. \tag{2.49}$$

To turn the resulting Lagrangian into a familiar form, we relate the four-dimensional Gamma matrices to two-dimensional ones. For the sake of simplicity, we decide on a particular representation, straightaway. In four dimensions we choose

$$\Gamma^0 = \sigma^2 \otimes \mathbb{1}, \quad \Gamma^j = i\sigma^3 \otimes \sigma^j, \tag{2.50}$$

and in three dimensions we set

$$\gamma^j = i\sigma^j. \tag{2.51}$$

Here, σ^j denotes the Pauli matrices. Hence,

$$\Gamma^j = \sigma^3 \otimes \gamma^j \quad \text{and} \quad [\Gamma^j, \Gamma^k] = \mathbb{1} \otimes [\gamma^j, \gamma^k]. \tag{2.52}$$

The last equality ensures that $\psi_{1/2}$ defined by

$$\psi = e_1 \otimes \psi_1 + e_2 \otimes \psi_2 \tag{2.53}$$

where ψ is an arbitrary 4D spinor and $e_{1/2}$ are the canonical two-dimensional basis vectors are spinors themselves.

The four-dimensional Majorana condition

$$\psi_C = C\bar{\psi}^T = C\Gamma^{0T}\psi^* = \psi \quad \text{with} \quad C = -\mathbb{1} \otimes \sigma^2 \tag{2.54}$$

translates to

$$\psi_1 = -i\sigma^2\psi_2^*. \tag{2.55}$$

The unfamiliar form is no surprise — a three-dimensional Euclidean space does not allow for Majorana spinors. As an intermediate result, we find

$$\begin{aligned}
\mathcal{L}_3 =& \partial_j\phi\partial^j\phi^\dagger - \frac{i}{2}(\bar{\psi}_1\sigma^j\partial_j\psi_2 + \bar{\psi}_2\sigma^j\partial_j\psi_1) + ff^\dagger \\
&+ \left\{ \frac{\partial W(\phi)}{\partial\phi} f + \frac{i}{4}\left(\bar{\psi}_1 \frac{\partial^2 W(\phi)}{\partial\phi^2}\psi_2 - \bar{\psi}_2\frac{\partial^2 W(\phi)}{\partial\phi^2}\psi_1\right)\right. \\
&\left. + \frac{i}{4}\left(\bar{\psi}_1\frac{\partial^2 W(\phi)}{\partial\phi^2}\psi_1 - \bar{\psi}_2\frac{\partial^2 W(\phi)}{\partial\phi^2}\psi_2\right) + \text{h. c.}\right\}.
\end{aligned} \tag{2.56}$$

Here $\bar{\psi}_i = \psi_i^\dagger$ because in Euclidean spacetimes the conjugating matrix A defined in section 2.1 becomes the identity. To derive the corresponding infinitesimal symmetry transformations, the anticommuting parameter α is split into α_1, α_2 in the same way as ψ.

Now, let us define two-dimensional Dirac spinors $\psi = (\psi_1 + \psi_2)/\sqrt{2}$ and α analogously. This allows us to abandon the Majorana condition altogether. Apart from counting degrees of freedom one can see this, setting

$$\begin{aligned}
\psi_1 &= \frac{1}{\sqrt{2}}\Big((1 - i\sigma^2)\Re(\psi) + i(1 + i\sigma^2)\Im(\psi)\Big), \\
\psi_2 &= \frac{1}{\sqrt{2}}\Big((1 + i\sigma^2)\Re(\psi) + i(1 - i\sigma^2)\Im(\psi)\Big).
\end{aligned} \tag{2.57}$$

This implies

$$\psi_1 = -i\sigma^2\psi_2^* \quad \text{and} \quad \psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2). \tag{2.58}$$

Thus, any Dirac spinor ψ is equivalent to some “Majorana” ψ_1 and ψ_2.

Reformulated by means of ψ and α our Lagrangian and the infinitesimal supersymmetry transformations take the following form:

$$\mathcal{L}_3 = \partial_j\phi\partial^j\phi^\dagger - i\bar{\psi}\sigma^j\partial_j\psi + ff^\dagger + \left\{\frac{\partial W}{\partial\phi}f - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi^T\sigma^2\psi + \text{h. c.}\right\} \tag{2.59}$$

$$\begin{aligned} \delta_\alpha\phi &= \sqrt{2}\alpha^T\sigma^2\psi \\ \delta_\alpha f &= i\sqrt{2}\bar{\alpha}\sigma^j\partial_j\psi \\ \delta_\alpha\psi &= \sqrt{2}(f\alpha - i\sigma^j\partial_j\phi\sigma^2\alpha^*) \end{aligned} \tag{2.60}$$

An explicit computation reveals that $\mathcal{L}_3$ is, indeed, symmetric under the named transformations.

Remark: With the chosen γ^j, σ^2 is a charge conjugation matrix.

As we have started from $\eta = \text{diag}(1,-1,-1,-1)$, we have used a negative signature for the three-dimensional Euclidean metric up to now. Exchanging the sign of the metric simultaneously with the overall sign of the Lagrangian gives

$$\mathcal{L}_3 = \partial_j\phi\partial^j\phi^\dagger + i\bar{\psi}\sigma^j\partial_j\psi - ff^\dagger - \left\{\frac{\partial W}{\partial\phi}f - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi^T\sigma^2\psi + \text{h. c.}\right\}. \tag{2.61}$$

The symmetry transformations remain unaltered. This Lagrangian agrees, under mild assumptions concerning a different notation, with the literature, see e. g. [11].

Let us have a look at the symmetry algebra of our three-dimensional theory. The coordinate transformations leaving the Lagrangian invariant are confined to the special Euclidean group $\mathbb{R} \rtimes SO(3)$. They are generated by the familiar $r(P^m)$ and $r(J^{rs})$, where the indexes m, r, and s are restricted to $\{1,2,3\}$, excluding zero.

As for the supercharge, we split

$$r(Q) = e_1 \otimes r(Q_1) + e_2 \otimes r(Q_2) \tag{2.62}$$

thus getting

$$\delta_\alpha = -i\bar{\alpha}_1 r(Q_2) + i\bar{\alpha}_2 r(Q_1) = \bar{\alpha}_1 r(-iQ_2) + \bar{\alpha}_2 r(iQ_1). \tag{2.63}$$

Hence, we can choose the odd generators to be

$$q_1 := -iQ_2 \quad \text{and} \quad q_2 := iQ_1. \tag{2.64}$$

The Majorana condition implies $r(q_1) = i\sigma^2 r(\bar{q}_2)$ and $q_1 = -i\sigma^2\bar{q}_2$.

The anticommutators are derived by rewriting

$$[\bar{\beta}r(Q), \bar{\alpha}r(Q)] = -2i\bar{\beta}\Gamma^m\alpha\partial_m = -2\bar{\beta}\Gamma_m\alpha P^m \tag{2.65}$$

in the two-dimensional quantities. This yields

$$\{q_i{}^m, \bar{q}_j{}^n\} = 2(\sigma^1)_{ij}(\sigma_k)^{mn}P^k. \tag{2.66}$$

The other two commutators can be reduced directly, giving

$$[P^m, q_i{}^n] = 0, \quad [J^{rs}, q_i] = -\sigma^{rs}q_i \tag{2.67}$$

with

$$\sigma^{rs} = -\frac{i}{4}[\gamma^r, \gamma^s] = \frac{i}{4}[\sigma^r, \sigma^s], \quad \Sigma^{rs} = \mathbb{1} \otimes \sigma^{rs}. \tag{2.68}$$

Now, let us introduce $q = (q_1 + q_2)/\sqrt{2}$ in analogy to the Dirac spinors ψ and α. Then the infinitesimal supersymmetry transformation becomes

$$\delta_\alpha = \bar{\alpha}r(q) + r(\bar{q})\alpha \tag{2.69}$$

and the algebraic supercharge relations can be rewritten as

$$\begin{aligned}
\{q^m, \bar{q}^n\} &= 2(\sigma_k)^{mn}P^k, \\
\{q^m, q^n\} &= 0, \\
[P^m, q^n] &= 0, \\
[J^{rs}, q] &= -\sigma^{rs}q.
\end{aligned} \tag{2.70}$$

A discussion of the three-dimensional Euclidean algebra of supersymmetry can be found in [29]. A probably different, not completely specified notation obstructs a detailed comparison. Hence, we are not able to decide whether all signs and prefactors coincide. However, apart from that, our results agree.

2.5.1. Superfield Formulation

The superfield formalism offers an elegant way of dealing with supersymmetric theories. Superfields are defined over an enlarged set of coordinates containing, in addition to the usual spacetime x, the anticommuting parameters responsible for supersymmetry transformations. The resulting coordinate space is the coset space of the entire symmetry group with respect to the subgroup $SO^{\uparrow}(d,t)$ [2].

The superfield corresponding to our three-dimensional $\mathcal{N}=2$ Wess-Zumino model is given by

$$\begin{aligned}
\Phi(x, \theta, \bar{\theta}) &= \mathrm{e}^{\delta_\theta - \delta_x}\,\phi(0) = \mathrm{e}^{\delta_\theta}\,\phi(x) \\
&= \exp(-ixP + i\bar{\theta}q + i\bar{q}\theta)\phi(0)\exp(ixP - i\bar{\theta}q - i\bar{q}\theta) \\
&= \exp(i\bar{\theta}q + i\bar{q}\theta)\phi(x)\exp(-i\bar{\theta}q - i\bar{q}\theta).
\end{aligned} \tag{2.71}$$

Applying the known δ_θ to $\phi(x)$, we can derive, employing the Fierz identity (2.14), the explicit formulation

$$\begin{aligned}\Phi =&\phi + \sqrt{2}\theta^T\sigma^2\psi + \theta^T\sigma^2\theta f \\ &- i(\bar\theta\sigma^j\theta\partial_j)\phi - i\sqrt{2}(\bar\theta\sigma^j\theta\partial_j)\theta^T\sigma^2\psi - \frac{1}{2}(\bar\theta\sigma^j\theta\partial_j)(\bar\theta\sigma^k\theta\partial_k)\phi \\ =&\,\mathrm{e}^{-i\bar\theta\sigma^j\theta\partial_j}(\phi + \sqrt{2}\theta^T\sigma^2\psi + \theta^T\sigma^2\theta f).\end{aligned} \tag{2.72}$$

Because $\bar\theta$ enters only as a shift in x, Φ may be called chiral. This designation is due to superfields in even-dimensional spacetimes, to which the concept of chirality belongs.

Now, recall the Campbell-Baker-Hausdorff formula

$$\mathrm{e}^X\,\mathrm{e}^Y = \mathrm{e}^Z, \;\; Z = X + Y + \frac{1}{2}[X,Y] + \text{ higher order commutators} \tag{2.73}$$

and the algebraic relations concerning q, see Eq. (2.70). Then, starting from Eq. (2.71), we can compute

$$\begin{aligned}\mathrm{e}^{\delta_\alpha}\,\Phi(x,\theta,\bar\theta) =&\exp\Big(-i(x^k - i\bar\alpha\sigma^k\theta + i\bar\theta\sigma^k\alpha)P_k + i(\bar\theta+\bar\alpha)q + i\bar q(\theta+\alpha)\Big)\,\phi(0) \\ &\exp\Big(i(x^k - i\bar\alpha\sigma^k\theta + i\bar\theta\sigma^k\alpha)P_k - i(\bar\theta+\bar\alpha)q - i\bar q(\theta+\alpha)\Big) \\ =&\Phi(x^k - i\bar\alpha\sigma^k\theta + i\bar\theta\sigma^k\alpha, \theta+\alpha, \bar\theta+\bar\alpha)\end{aligned} \tag{2.74}$$

or, infinitesimally,

$$\delta_\alpha\Phi(x,\theta,\bar\theta) = \Big((-i\bar\alpha\sigma^k\theta + i\bar\theta\sigma^k\alpha)\partial_k - \partial_\theta\alpha + \bar\alpha\partial_{\bar\theta}\Big)\,\Phi(x,\theta,\bar\theta). \tag{2.75}$$

On superspace, the supersymmetry transformation becomes a coordinate translation. This makes it particularly simple to construct invariant actions. Remember that ordinary translational invariance is ensured by a spacetime integration. The same principle can be adopted here. To this end, the usual integral is complemented by a Berezin one, which for each component of the anticommuting θ and $\bar\theta$ is defined by

$$\int \mathrm{d}\zeta = 0, \;\; \int \mathrm{d}\zeta\zeta = 1. \tag{2.76}$$

Hence, any superfield, if integrated over the superspace, gives an action invariant under supersymmetrical transformations. Obviously, as long as Φ is a scalar field, ordinary coordinate transformations leave such an action equally unaffected.

A chiral superfield looses any $\bar\theta$-dependence when integrated over the spacetime. Consequently, it gives zero under an integral involving all superspace coordinates. However, as any possible shift of $\bar\theta$ vanishes along with $\bar\theta$ itself, the integration over $\bar\theta$ can be omitted without destroying the invariance of the result.

The adjoint of a superfield as well as the product of superfields is a superfield itself; furthermore, the multiplication of superfields preserves chirality. Finally, any constant vanishes under a Berezin integral. Hence, the following Lagrangian induces an invariant action:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\Phi\Phi^\dagger - \frac{1}{2i}\left(\int \mathrm{d}^2\theta\; W - \int \mathrm{d}^2\bar{\theta}\; W^\dagger\right)\\ &= -\frac{1}{4}\int \mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\Phi\Phi^\dagger - \left\{\frac{1}{2i}\int \mathrm{d}^2\theta\; W + \text{h. c.}\right\}\end{aligned} \tag{2.77}$$

W is a holomorphic function of Φ and

$$\mathrm{d}^2\theta \equiv \mathrm{d}\theta_1\,\mathrm{d}\theta_2,\;\; \mathrm{d}^2\bar{\theta} \equiv \mathrm{d}\bar{\theta}_2\,\mathrm{d}\bar{\theta}_1. \tag{2.78}$$

An explicit calculation shows that this Lagrangian is, up to a surface term, identical to $\mathcal{L}_3$ from Eq. (2.61) if we, as before, require the coupling constants in W to be real.

On superfields, we have found

$$\delta_\alpha = (-i\bar{\alpha}\sigma^k\theta + i\bar{\theta}\sigma^k\alpha)\partial_k - \partial_\theta\alpha + \bar{\alpha}\partial_{\bar{\theta}}. \tag{2.79}$$

With

$$\delta_\alpha = \bar{\alpha}r(q) + r(\bar{q})\alpha \tag{2.80}$$

this provides us with the representation

$$\begin{aligned} r(q) &= -i\sigma^k\theta\partial_k + \partial_{\bar{\theta}},\\ r(\bar{q}) &= i\bar{\theta}\sigma^k\partial_k - \partial_\theta. \end{aligned} \tag{2.81}$$

Indeed, computing $\{r(q)^m, r(\bar{q})^n\}$ and $\{r(q)^m, r(q)^n\}$ recovers

$$\begin{aligned} \{q^m, \bar{q}^n\} &= 2(\sigma_k)^{mn}P^k,\\ \{q^m, q^n\} &= 0. \end{aligned} \tag{2.82}$$

There is a counterpart to the gained representation of the supercharges, the covariant derivatives D and $\bar{D}$. To derive $r(q)$ and $r(\bar{q})$ we have considered

$$\mathrm{e}^{\delta_\alpha}\,\mathrm{e}^{\delta_\theta-\delta_x}\,\phi(0) = \exp(\bar{\alpha}r(q) + r(\bar{q})\alpha)\,\mathrm{e}^{\delta_\theta-\delta_x}\,\phi(0) \tag{2.83}$$

Now, we take a look at

$$\mathrm{e}^{\delta_\theta-\delta_x}\,\mathrm{e}^{\delta_\alpha}\,\phi(0) = \exp(\bar{\alpha}D + \bar{D}\alpha)\,\mathrm{e}^{\delta_\theta-\delta_x}\,\phi(0). \tag{2.84}$$

Evaluating this in the above manner, we get

$$\begin{aligned} D &= i\sigma^k\theta\partial_k + \partial_{\bar{\theta}},\\ \bar{D} &= -i\bar{\theta}\sigma^k\partial_k - \partial_\theta \end{aligned} \tag{2.85}$$

going along with

$$\begin{aligned} \{D^m, \bar{D}^n\} &= -2(\sigma_k)^{mn} P^k, \\ \{D^m, D^n\} &= \{\bar{D}^m, \bar{D}^n\} = 0, \\ \{r(q)^m, D^n\} &= \{r(\bar{q})^m, D^n\} = \{r(q)^m, \bar{D}^n\} = \{r(\bar{q})^m, \bar{D}^n\} = 0, \\ \{r(P)^m, D^n\} &= \{r(P)^m, \bar{D}^n\} = 0. \end{aligned} \tag{2.86}$$

The term "covariant derivative" is due to the last two lines: they ensure that superfields remain so under the action of D and $\bar{D}$. Thus, covariant derivatives are a suitable tool for the construction of superfields and invariant actions. For our chiral field, the affinity between D and the exponential term in Φ leads to $D^m \Phi = 0$. This is reminiscent of the chirality of superfields in even dimensions.

Now, we are ready to drop our restriction to a standard kinetic term stated in section 2.3. To completely generalize the superfield Lagrangian from Eq. (2.77), we have to replace $\Phi\Phi^\dagger$ by an unconstrained real, scalar, analytical function of Φ, $\Phi^\dagger$, and covariant derivatives acting on the fields, see [14]:

$$\mathcal{L} = -\frac{1}{4} \int \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \; \mathcal{K}(D, \bar{D}, \Phi, \Phi^\dagger) - \left\{ \frac{1}{2i} \int \mathrm{d}^2\theta \; W + \mathrm{h.\,c.} \right\} \tag{2.87}$$

Sometimes, $\mathcal{K}$ is called a Kähler potential. We reserve this designation for $K(\Phi, \Phi^\dagger)$ containing only the $D/\bar{D}$-independent contributions to $\mathcal{K}$.

2.6. The Subsequent Relative: $\mathrm{D} = 2$ and $\mathcal{N} = 2$

Another step of dimensional reduction reveals the two-dimensional $\mathcal{N} = 2$ Wess-Zumino model. With $\gamma^1 = \sigma^1$ and $\gamma^2 = \sigma^2$ neither its Lagrangian density nor its infinitesimal supersymmetry transformations differ visibly from the discussed three-dimensional case. However, as $D = 2$ — in contrast to $D = 3$ — allows for both Weyl and Majorana spinors it may be advantageous to reformulate the equations by an appropriate redefinition of ψ. There are several works considering this model, see e. g. [30] and [31]. In [25] it is treated from the same FRG perspective as is adopted in the present thesis.

3. Functional Renormalization Group

3.1. Critical Phenomena

In thermodynamics, a phase transition occurs when the partition function of a system is not smooth. If it is still continuously differentiable, the phase transition is called continuous or second order. We will say that the thermodynamic state variables corresponding to such a phase transition define a critical point. Landau [32] has tied continuous phase transitions to the reduction of the symmetries of the considered matter to a subgroup. Experiments show that critical points come with a divergent correlation length ξ.

Close to a critical point, most measurable quantities exhibit a power law dependence on the deviation of thermodynamic variables or ξ from their critical values. These powers, their negatives, or inverses are called critical exponents. The same name applies to some exponents governing the physics exactly at the critical point. Let us for example consider a simple uniaxial ferromagnet. The two-point correlation function $C(r)$ relevant for the Curie phase transition concerns the local magnetization. At the critical point, for large distances r in some fixed direction this correlation function asymptotically behaves as [18]

$$C(r) \propto \left(\frac{1}{r}\right)^{d-2+\eta} . \tag{3.1}$$

This η is numbered among the critical exponents; d is, as before, the spatial dimension. The numerical values of critical exponents are independent of most of the features defining a particular system. There are only few different classes characterized by the symmetries of the action, d and the like. This far reaching property is referred to as universality.

Quantum phase transitions are, both by terminology and mathematical description, closely related to the thermodynamical or classical ones. Consider a quantum field theory depending on some free coupling constants. Let the temperature be equal to zero; then the system stays in its ground state. Quantum phase transitions arise at non-analytic points of the ground state energy as a function of the couplings [33]. Again, a classification as first or second order transitions is useful. We call a transition second order or, as before, continuous, whenever an energy scale "characterizing some significant spectral density of fluctuations at zero temperature" [33] (see source for details) vanishes on either side of the critical point. In addition, there is always some length scale, e. g., as above, a correlation length, diverging at the critical point. Landau's symmetry-based interpretation of second order transitions remains valid also in the context of quantum fields. For example, in [22] the breaking of supersymmetry across a quantum critical point is discussed.

As in thermodynamics, universal critical exponents emerge, when observables are considered at or near a critical point. For large distances x (and, again, a fixed direction), the asymptotic behavior of two-point correlation functions $C(x)$ of an operator O with classical dimension Δ becomes

$$C(x) \propto \left(\frac{1}{x}\right)^{2\Delta+\eta}. \tag{3.2}$$

The universal η is called the anomalous dimension of O.

3.2. The Renormalization Group Approach

Imagine, that we are considering some system or, to be precise, a Lagrangian density $\tilde{\mathcal{L}}$ which — via its coupling constants $\tilde{g}_i$ — depends on some variables of state. We want to identify the critical point of this system and study the physics close to it. For future use, we assume the set $\{\tilde{g}_i\}$ to include also all vanishing couplings which would not break the symmetry of $\tilde{\mathcal{L}}$. In contrast, factors Z_{0j} preceding the kinetic terms of the fields $\tilde{F}_j$ are regarded separately because they can be arbitrarily chosen.

Now, let us introduce some "scale" t. If we succeed in defining a t-dependence of the fields and couplings such that some quantity we are interested in remains unaffected, all expressions of the form $\partial_t X = \beta_X$ will be called flow equations and β_X beta functions of a continuous renormalization group [18]. We will potter at transformations along $t = \ln(k/k_0)$, where k is the momentum, which leave the infrared behavior invariant. Particularly, the partition function acquires at most a field-independent factor with t. In addition, it is assumed that for any t short-range interactions dominate the Lagrangian density [18]. For $t \to -\infty$, one can think of the considered transformations as "coarse graining", the averaging over the details of a system.

By means of k, we can rewrite the Lagrangian density in dimensionless fields and coupling constants. If a field $\tilde{F}_j$ explicitly depends on t, we choose Z_{0j} to compensate for this scaling. Otherwise, we set $Z_{0j} = 1$. Absorbing the non-trivial "wave function renormalizations" Z_{0j} by the corresponding fields and couplings, we end up with dimensionless and renormalized fields F_j and coupling constants g_i. Let us introduce $\eta_j = -\partial_t \ln(Z_{0j})$. A solution $\{g_i^*, \eta_j^*\}$ of $\partial_t g_i = \beta_i$, $\partial_t g_i^* = 0$ and $-\eta_j Z_{0j} \equiv \partial_t Z_{0j} = \beta_j$, $\partial_t \eta_j^* = 0$ defines a fixed point of the flow and renormalization group. The fixed point model $\mathcal{L}_*$ is scale invariant; the dimensionless, renormalized fields have scaling dimension zero. The original ones, in contrast, come with a scaling dimension of $\Delta_j + \eta_j^*/2$. Thus, at the fixed point, the correlation function of each field $\tilde{F}_j$ obeys

$$C_j^*(x) \propto \left(\frac{1}{x}\right)^{2\Delta_j+\eta_j^*}. \tag{3.3}$$

Because of the scale invariance of $\mathcal{L}_*$, this is true for any $|x|$. The absence of an exponential decay shows that the correlation length is infinite. The similarity to both Eq. (3.1) and (3.2)

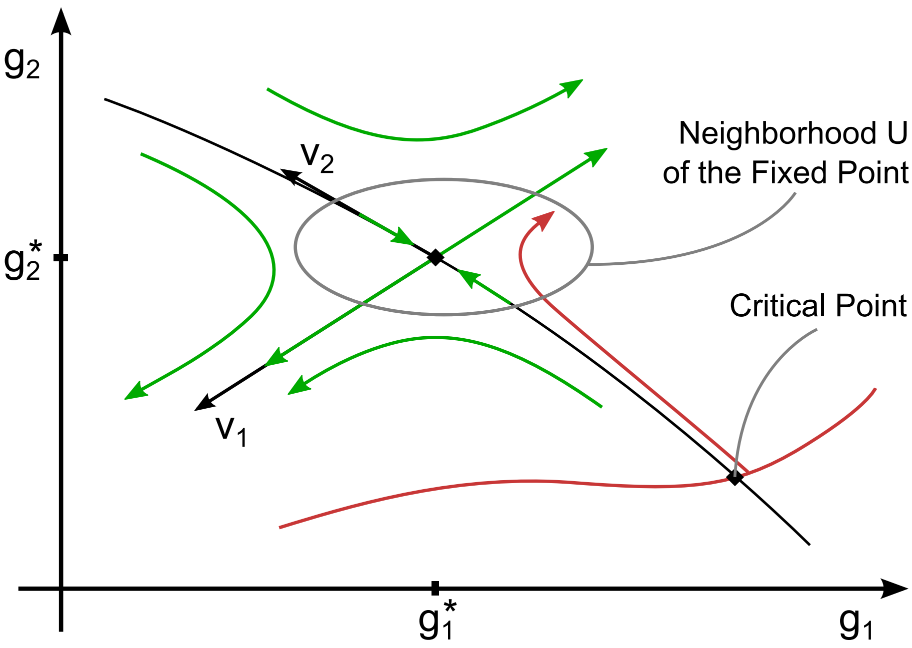


Figure 3.1.: Fixed Point and Critical Point. For clarity, only two couplings g_i are considered. The vectors v_i are eigenvectors of the stability matrix; v_1 corresponds to a relevant, v_2 to an irrelevant scaling variable. The continuation of v_2 is the critical surface. The green arrows depict the structure of RG flows close to the fixed point. The red curve represents the model of interest as a function of some variable of state. The corresponding arrow indicates the flow which is considered in the text to identify the critical exponents introduced in section 3.1. Following [18].

immediately suggest to call η_j an anomalous dimension. Some justification concerning the relation of RG fixed points to critical phenomena will follow below.

Let us linearize the flow equations around a fixed point, assuming that all $\partial\beta_i/\partial g_j|_*$ are well-defined. If we denote the vector of infinitesimal deviations from the fixed point δg, we get

$$\partial_t \delta g = B \delta g \quad \text{with} \quad B_{ij} = \left. \frac{\partial \beta_i}{\partial g_j} \right|_* . \tag{3.4}$$

B is called the stability matrix. To solve this differential equation, we need the eigenvectors v_i and eigenvalues λ_i of B. Though the correspondence is not a direct one, $(-\lambda_i)$ are commonly known as critical exponents. It is useful to define the scaling variables $s_i = v_i \delta g$ satisfying $s_i \propto \mathrm{e}^{\lambda_i t}$. We see that for $t \to -\infty$ a positive λ_i makes s_i vanish, while a negative one provides an amplification. Thus, in a neighborhood U of the fixed point where the linear approximation is applicable, the fixed point is reached, whenever all n "relevant" s_i corresponding to negative λ_i are zero from the start. Variables s_i coming with positive λ_i are called irrelevant, and s_i corresponding to $\lambda_i = 0$ are "marginal".

The vanishing of all relevant s_i just off the fixed point defines a hyperplane with codimension

n. We expect that, beyond the immediate vicinity of the fixed point, this hyperplane can be continuously deformed to contain all points that flow into the fixed point [18]. The resulting n-hypersurface is referred to as critical surface. The existence of further fixed points can limit its extent. Now, return to the model $\tilde{\mathcal{L}}$, whose critical properties we were interested in. Adjusting n variables of state, we can position it on a critical surface. We postulate that the obtained point in coupling space is a critical point of the theory [18]. This proposition is strongly supported by its consequences. For example, as we will argue next, it provides universality, grouping all continuous phase transitions belonging to one critical surface.

The following paragraph is condensed from [18]. Figure 3.1 illustrates the argumentation. To analyze the physics close to the intersection of $\mathcal{L}$ with the critical surface, one can start by infinitesimally distorting the variables of state u_i. Let the change δu_i be chosen tiny enough to ensure that the "misaligned" model flows into the neighborhood U of the fixed point. Remember that the flow equations leave the field-dependent part Z_f of the partition function invariant. Thus, the critical properties of the system are preserved. Inside U, Z_f can be expressed in terms of the scaling variables, which, in turn, depend on the δu_i. This can be used to derive, considering the flow invariance of Z_f in U, the scaling behavior of Z_f. That is, we learn that, and sometimes how, Z_f depends on particular dimensionless combinations of the $\delta u_i/u_0$. For sufficiently small δu, this scaling form of Z_f becomes independent of the critical values of the irrelevant scaling variables. This establishes the universality of all critical exponents in the sense of the previous section which are related to the partition function. The scaling behavior of Z_f fixes these critical exponents as simple combinations of the dimension and $(-\lambda_i)$, the "critical exponents" of the current section. Similar arguments show that the correlation length diverges on the whole critical surface, just as it does at the fixed point. Furthermore, it follows that the long distance behavior of the correlation functions is universal.

3.3. The Wetterich Equation

The Wetterich equation [34] constitutes one of many particular implementations of the renormalization group idea. The object under primary consideration is an average effective action Γ_k as a functional of k-dependent expectation values $\langle F_i \rangle_k$, where F_i are the fields contributing to the model. Γ_k interpolates between the action S and the effective action Γ. Here, Γ is the limit for $k \to 0$; simultaneously, $\langle F_i \rangle_k \to \langle F_i \rangle$. At $k \neq 0$, the path integrals defining Γ_k and $\langle F_i \rangle_k$ become infrared regulated. Low-momentum contributions are more and more suppressed when k increases.

Remark: This is slightly counterintuitive. With coarse graining in mind, one would rather expect that $k \to 0$ removes high frequencies than that low frequencies are added. The first procedure corresponds to a UV-, the second one to an IR-cutoff. However, this issue is irrelevant for the interpretation of the RG results.

The beta functions of all coupling constants and wave function renormalizations can be derived from the flow of Γ_k by appropriate projections. It is not immediately discernible, in which sense the Wetterich flow lets the infrared behavior of a system unaffected. One (formal) way to figure this out makes use of Polchinski's formulation [35] of the renormalization group. It reflects on the flow of a functional $S_\Lambda[F_i]$ along an ultraviolet cutoff Λ. For $\Lambda \nearrow \Lambda_0$, S_Λ is set to approach the initial action S. Furthermore, the flow is defined to ensure

$$\Lambda\partial_\Lambda \int \mathcal{D}F_i \;\, \mathrm{e}^{S_\Lambda|_{\text{field-dependent}}} = 0. \tag{3.5}$$

Apart from a field-independent factor, the partition function of the system stays invariant. Now, one can show that Wetterich's and Polchinski's flow equations are equivalent. To derive one from the other, one has to perform a Legendre transformation, redefine the fields and identify Λ with k. [20]

As we have seen, Polchinski and Wetterich have formulated flow equations concerning functionals. Therefore, both formalisms are classified as "functional" (former "exact") renormalization group approaches.

Let us explicitly discuss the case of a single real scalar field ϕ. The common generating functional related to the partition function is

$$Z[J] = \int \mathcal{D}\phi \;\, \mathrm{e}^{-S[\phi]+\int \mathrm{d}^D q \; J(-q)\phi(q)}\,. \tag{3.6}$$

We define

$$Z_k[J] = \int \mathcal{D}\phi \;\, \mathrm{e}^{-S[\phi]-\Delta S_k[\phi]+\int \mathrm{d}^D q \; J(-q)\phi(q)} \tag{3.7}$$

with

$$\Delta S_k[\phi] = \frac{1}{2}\int \mathrm{d}^D q \; \phi(-q) R_k(q) \phi(q) \tag{3.8}$$

and $R_k(q)$ a positive infrared regulator depending only on q^2 and obeying

$$\begin{aligned} R_k(q) &\to 0 \quad \text{for } k \to 0 \\ R_k(q) &\to \infty \;\; \text{for } k \to \infty. \end{aligned} \tag{3.9}$$

The limits are crucial for the interpolating nature of Γ_k [34]. Concerning the details of R_k, it is assumed that they can be specified in such a way that the final Wetterich equation is non-divergent.

Next, we introduce $W_k = \ln(Z_k)$ and its Legendre transform

$$\tilde{\Gamma}_k[\chi_k] = -W_k[J_\chi] + \int \mathrm{d}^D q \; J_\chi(-q)\chi_k(q) \tag{3.10}$$

with J_χ the inverse of

$$\chi_k(q) = \langle \phi(q) \rangle_{J,k} = \frac{\delta W_k[J]}{\delta J(-q)}. \tag{3.11}$$

Thus,

$$J_\chi(q) = \frac{\delta \tilde{\Gamma}_k[\chi_k]}{\delta \chi_k(-q)}. \tag{3.12}$$

For comparison, the effective action is defined as

$$\Gamma[\chi] = -W[J_\chi] + \int \mathrm{d}^D q \; J_\chi(-q)\chi(q) \tag{3.13}$$

with $W = \ln(Z)$ and

$$\chi(q) = \langle \phi(q) \rangle_J = \frac{\delta W[J]}{\delta J(-q)}. \tag{3.14}$$

Let us consider $\partial_t \tilde{\Gamma}_k[\chi]$, assuming that the partial derivative does not act on χ. We find

$$\begin{aligned} \partial_t \tilde{\Gamma}_k[\chi] &= -\partial_t W_k[J_k] + \partial_t \int \mathrm{d}^D q \; J_k(-q)\chi(q) \\ &= -\partial_t W_k[J_\chi] - \int \mathrm{d}^D q \; \chi(-q)\partial_t J_k(q) + \int \mathrm{d}^D q \; (\partial_t J_k(-q))\chi(q) \\ &= -\partial_t W_k[J_\chi]. \end{aligned} \tag{3.15}$$

Here, J_k is nothing else than J_χ. However, in writing J_k we emphasize that ∂_t has to be applied to J_χ, while J_χ is used only if the opposite is true. Now,

$$\begin{aligned} \partial_t W_k[J_\chi] &= -\frac{1}{2Z_k[J_\chi]} \int \mathrm{d}^D q \; \partial_t R_k(q) \frac{\delta^2 Z_k[J_\chi]}{\delta J(-q)\delta J(q)} \\ &= -\frac{1}{2} \int \mathrm{d}^D q \; \partial_t R_k(q) \left(\frac{\delta W_k[J_\chi]}{\delta J(-q)} \frac{\delta W_k[J_\chi]}{\delta J(q)} + \frac{\delta^2 W_k[J_\chi]}{\delta J(-q)\delta J(q)} \right) \\ &= -\partial_t \Delta S_k[\chi] - \frac{1}{2} \int \mathrm{d}^D q \; \partial_t R_k(q) \frac{\delta^2 W_k[J_\chi]}{\delta J(-q)\delta J(q)}. \end{aligned} \tag{3.16}$$

Thus, for

$$\Gamma_k[\chi] := \tilde{\Gamma}_k[\chi] - \Delta S_k[\chi] \tag{3.17}$$

we get

$$\partial_t \Gamma_k[\chi] = \frac{1}{2} \int \mathrm{d}^D q \; \partial_t R_k(q) \frac{\delta^2 W_k[J_\chi]}{\delta J(-q)\delta J(q)}. \tag{3.18}$$

Only two more steps are needed to arrive at a functional differential equation for $\Gamma_k[\chi]$. First, consider

$$\begin{aligned} \frac{\delta \Gamma_k[\chi]}{\delta \chi(p)} &= -\int \mathrm{d}^D q \; \frac{\delta W_k[J_\chi]}{\delta J(q)} \frac{\delta J_\chi(q)}{\delta \chi(p)} + \int \mathrm{d}^D q \; \frac{\delta J_\chi(-q)}{\delta \chi(p)} \chi(q) + J_\chi(-p) - \chi(-p) R_k(\chi) \\ &= J_\chi(-p) - \chi(-p) R_k(p) \end{aligned} \tag{3.19}$$

yielding

$$\frac{\delta J_\chi(p)}{\delta \chi(q)} = \frac{\delta^2 \Gamma_k[\chi]}{\delta \chi(-p)\delta \chi(q)} + R_k(p)\delta(p-q). \tag{3.20}$$

Then

$$\begin{aligned}\delta(q-q') = \frac{\delta\chi(q)}{\delta\chi(q')} = \frac{\delta^2 W_k[J_\chi]}{\delta\chi(q')\delta J(-q)} &= \int \mathrm{d}^D q'' \, \frac{\delta^2 W_k[J_\chi]}{\delta J(-q)\delta J(q'')} \frac{\delta J_\chi(q'')}{\delta\chi(q')} \\ &= \int \mathrm{d}^D q'' \, \frac{\delta^2 W_k[J_\chi]}{\delta J(-q)\delta J(q'')} \left(\frac{\delta^2 \Gamma_k[\chi]}{\delta\chi(-q'')\delta\chi(q)} + R_k(q'')\delta(q''-q) \right).\end{aligned} \tag{3.21}$$

In this sense,

$$\frac{\delta^2 W_k[J_\chi]}{\delta J(-q)\delta J(q'')} = \left(\frac{\delta^2 \Gamma_k[\chi]}{\delta\chi(-q)\delta\chi(q'')} + R_k(q)\delta(q-q'') \right)^{-1} \tag{3.22}$$

and the Wetterich equation reads

$$\begin{aligned}\partial_t \Gamma_k[\chi] &= \frac{1}{2} \int \mathrm{d}^D q \; \partial_t R_k(q) \left(\frac{\delta^2 \Gamma_k[\chi]}{\delta\chi(-q)\delta\chi(q)} + R_k(q)\delta(0) \right)^{-1} \\ &= \frac{1}{2} \int \mathrm{d}^D p \, \mathrm{d}^D q \; \partial_t R_k(q)\delta(p-q) \left(\frac{\delta^2 \Gamma_k[\chi]}{\delta\chi(-q)\delta\chi(p)} + R_k(q)\delta(q-p) \right)^{-1} \\ &= \frac{1}{2} \int \mathrm{d}^D p \, \mathrm{d}^D q \; \partial_t R_k(q)\delta(p-q) \left(\frac{\delta^2 \Gamma_k[\chi]}{\delta\chi^*(q)\delta\chi(p)} + R_k(q)\delta(q-p) \right)^{-1} \\ &\equiv \frac{1}{2} \operatorname{Tr} \left(\partial_t R_k (\Gamma_k^{(2)} + R_k)^{-1} \right).\end{aligned} \tag{3.23}$$

To make the relationship of Γ_k to Γ a bit more obvious, we can follow [20]: exponentiating Γ_k (Eq. (3.17)) and employing the definitions of $\tilde{\Gamma}_k$ (Eq. (3.10)), W_k, and Z_k (Eq. (3.7)) we arrive, after a change of the integration variable, at

$$\mathrm{e}^{-\Gamma_k[\chi]} = \int \mathcal{D}\phi' \, \mathrm{e}^{-S[\phi'+\chi] - \Delta S_k[\phi'] + \int \mathrm{d}^D q \frac{\delta \Gamma_k}{\delta\chi(q)} \phi'(q)}. \tag{3.24}$$

Apart from the regulator term ΔS_k, this is identical to the corresponding expression for Γ.

The Wetterich formulation of the RG can be applied to theories with different field content. We are interested in systems composed of complex scalar fields ϕ_i and spin-$\frac{1}{2}$ fermions ψ_j. The corresponding generalizations of our above derivation can be found e. g. in [34] and [36]. Let us introduce

$$\begin{aligned}\Psi_c(q) &= \left(\phi_i(q), \phi_i^*(-q), \psi_j^T(q), \psi_j^\dagger(-q) \right)^T \\ \text{and} \quad \Psi_r(q) &= \left(\Re[\phi_i](q), \Im[\phi_i](q), \psi_j^T(q), \psi_j^\dagger(-q) \right)^T.\end{aligned} \tag{3.25}$$

There are several possibilities to define a field component vector Ψ to start with. However, the present choices make the subsequent formalism particularly simple. If we set

$$\begin{aligned}\Gamma_k^{(2)}(p,q) &= \frac{\overrightarrow{\delta}}{\delta\Psi_{c/r}^\dagger(p)} \Gamma_k \frac{\overleftarrow{\delta}}{\delta\Psi_{c/r}(q)} \\ \text{and} \quad \Delta S_k &= \frac{1}{2} \int \mathrm{d}^D q \, \sum_{l,s} [\Psi_{c/r}^\dagger(q)]_l \, [R_k(q)]_{ls} \, [\Psi_{c/r}(q)]_s,\end{aligned} \tag{3.26}$$

the flow of Γ_k becomes

$$\begin{aligned}\partial_t \Gamma_k =& \frac{1}{2} \int \mathrm{d}^D p \, \mathrm{d}^D q \\ & \Big(\sum_{l \in \{i\}} - \sum_{l \in \{j\}} \Big) \sum_s [\partial_t R_k(q) \delta(p-q)]_{ls} \left[\left(\Gamma_k^{(2)}(q,p) + R_k(q)\delta(q-p) \right)^{-1} \right]_{sl} \\ \equiv & \frac{1}{2} \, \mathrm{STr} \left(\partial_t R_k (\Gamma_k^{(2)} + R_k)^{-1} \right).\end{aligned} \tag{3.27}$$

Here, $\{i\}$ contains the indexes corresponding to bosonic components of $\Psi_{c/r}$, while $\{j\}$ covers the remaining fermionic ones. The S in front of the trace symbol refers to the notion of a supertrace. It reflects the $\{i\}/\{j\}$ splitting in the line before.

As before, the regulation is supposed to ensure that the Wetterich flow is well-defined. In the previous section we have restricted our discussion to renormalization groups which preserve the symmetries of the action along its flow. To implement this in the Wetterich equation, we have to require that ΔS_k does not break any symmetry of S. This particularly entails that $[R_k]_{ls}$ needs to be block diagonal.

3.4. Truncation and Regulators

The Wetterich equation can be understood as a partial differential equation involving infinitely many variables t and $F_i(q)$. Thus, it can be scarcely hoped to find an exact solution. Instead, one has to decide on some approximations. Besides treating the flow equation perturbatively [19], the FRG approach allows for systematic truncation schemes beyond the weak interaction regime. Two particularly well known examples are the derivative and the vertex expansion. In the context of supersymmetry, intrinsically supersymmetric truncations based on the superfield formulation, see e. g. [37], are applied.

Truncations can introduce a spurious regulator-dependence of the desired results. This immediately brings up the question whether one could optimize the regulator to improve the convergence of the chosen truncation scheme. There are several methods which have been proposed to this end. In [38], we find the advice to minimize the regulator-dependence of the wanted observable. Empirical results reinforce this method. According to [38], the reason for the good performance is that this optimization minimizes the generation of neglected irrelevant operators in the course of the truncated flow.

Litim has established [39] a mathematically explicit, very generic optimization criterion for the regulator $R_k(q)$. It is almost independent of the theory under consideration; particularly, it comes with the great advantage to be applicable before proceeding to the intended computations. The optimization prescription reads that one should choose a regulator which maximizes the q-minimum of the q- and k-dependent inverse effective propagator at vanishing fields. In [39–41], several physical interpretations of this handling are discussed. For example, it maximizes the radius of convergence for amplitude expansions (which we do not consider in the present work). Furthermore, it can again be understood as a minimum sensitivity condition. By [41] we are provided with an explicit, simple example of both a bosonic and a fermionic optimized regulator functions. Writing that a regulator is of Litim type, we will always mean that its structure is identical to the one proposed by Litim for fermions. This does not imply that the corresponding regulator matrix indeed fulfills the Litim criterion. As supersymmetry relates bosonic and fermionic degrees of freedom, the regulation of the two subsets of fields is correlated. Thus, the separate consideration in [41] is insufficient.

4. Critical Exponents of the 3D $\mathcal{N} = 2$ Wess-Zumino Model

Starting from the Wetterich equation (3.27), we apply different truncations and regulators to explore the critical properties of the three-dimensional $\mathcal{N} = 2$ Wess-Zumino model. From [42] we know that the flow of the average action of a Wess-Zumino model does not modify the superpotential, $\partial_t W_k(\phi) = 0$. In [14] this "non-renormalization theorem" is proven for the general four-dimensional $\mathcal{N} = 1$ Wess-Zumino model by means of Polchinski's formulation of the renormalization group. Though we consider the flow of the average effective action, these results cause the expectation that $\partial_t W_k(\phi) = 0$ holds also for us.

Supersymmetry enforces that the fields have a wave function renormalization in common. Thus, only one anomalous dimension η has to be considered. We are most interested in "non-trivial" fixed points with $\eta_* \neq 0$. An ϵ-expansion about four dimensions gives reason to expect the existence of such a fixed point at $W_*(\phi) \propto \phi^3$ [43]. Starting from this assumption, symmetry arguments prove that $\eta_* = 1/3$ [44]. This defines the fixed point model which is considered in [11] and primarily analyzed in the present thesis.

In [11], the values of two critical exponents are conjectured by means of a bootstrapping approach:

$$\frac{1}{\nu_b} = 1.0902(20) \tag{4.1}$$

and thus, because of an exact supersymmetric scaling relation [5],

$$\omega_b = 2 - \frac{1}{\nu_b} = 0.9098(20). \tag{4.2}$$

According to [5], ν describes a change of the bosonic mass at a constant fermionic one. Such a deviation from the fixed point breaks the supersymmetry. As our formalism is from end to end supersymmetric, we are not able to derive ν directly — just as [11] cannot provide us with independent results for ν and ω. As for ω, comparing [11] to [45] suggests that it has to be identified with an eigenvalue λ of the stability matrix generating a leading order correction $\propto \phi^3$ to W.

As will become obvious in the course of the upcoming computations, this does not fix λ unambiguously. Let us eliminate the auxiliary field from our three-dimensional Wess-Zumino Lagrangian (2.61). Then we find that the bosonic potential is given by

$$V(\phi, \phi^\dagger) = -|W'(\phi)|^2. \tag{4.3}$$

Now, we can argue with [46] that we should assign the smallest positive eigenvalue corresponding to $\delta W \propto \phi^3 + \mathcal{O}(\phi^4)$ to ω. Our results support the assumption that this definition is consistent with the one taken as a basis in [11].

In [25], Synatschke-Czerwonka et al. derive the beta-functions for the two-dimensional $\mathcal{N}=2$ Wess-Zumino model employing the same truncation as is discussed in section 4.2 of the present work. Because of the intimate correlation between the $D=3$ and $D=2$ $\mathcal{N}=2$ Wess-Zumino models, we expect that the flow equations in [25] can be recovered by dimensionally reducing our corresponding results.

4.1. LPA′ Truncation

Seizing the nomenclature common in the context of derivative expansions, we call a truncation the "local potential approximation" (LPA) if only the superpotential is assumed to become scale-dependent. Additionally including a (constant) positive wave function renormalization $Z_0 \equiv Z_0(t)$ brings us to the term LPA′. Thus, our first ansatz for Γ_k reads

$$\begin{aligned}\Gamma_k &= \int \mathrm{d}^3x \, \left(Z_0(\partial_j\phi\partial^j\phi^\dagger + i\bar{\psi}\sigma^j\partial_j\psi - ff^\dagger) - \left\{ \frac{\partial W_k(\phi)}{\partial\phi} f - \frac{1}{2}\frac{\partial^2 W_k(\phi)}{\partial\phi^2}\psi^T\sigma^2\psi + \mathrm{h.\,c.} \right\} \right) \\ &= -\frac{1}{4} Z_0 \int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \; \Phi\Phi^\dagger - \left\{ \frac{1}{2i} \int \mathrm{d}^3x \, \mathrm{d}^2\theta \; W_k(\Phi) + \mathrm{h.\,c.} \right\}.\end{aligned} \tag{4.4}$$

Needless to say that this truncation is supersymmetric as long as we demand — and we do demand — W_k to be holomorphic. Sticking to our notation for W, we identify

$$W_k(\phi) \equiv \sum_{n=1}^{\infty} \frac{c_n(k)}{n}\phi^n \tag{4.5}$$

again neglecting the physically irrelevant offset.

A supersymmetric regulator term ΔS_k can be turned into the form

$$\Delta S_k = -\frac{1}{4}\int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \; \Phi^\dagger \rho_2(D,\bar{D})\Phi - \left\{ \frac{1}{4i}\int \mathrm{d}^3x \, \mathrm{d}^2\theta \; \Phi\rho_1(D,\bar{D})\Phi + \mathrm{h.\,c.} \right\}. \tag{4.6}$$

Up to the fact that ρ_2 has to be Hermitian, the t-dependent ρ_i are arbitrary scalar functions of the supercovariant derivatives. For a start, we will additionally assume them to be analytical. Then they can be replaced by some equally unrestricted $r_i(-\partial_x^2)$. To show this, we follow [14].

Proof: First, let us argue that applying any ρ_i-term to Φ is proportional to acting on Φ with an alternating product of $D^T\sigma^2 D$ and $\bar{D}\sigma^2 D^*$ times some power of ∂_x^2. Recall that Φ is chiral. Thus, a D-component preceding the field annihilates it. Consequently, all possible contributions to $\rho_i\Phi$ involving covariant derivatives start from

$$t_1 \propto \cdots \bar{D}_{\alpha_2}\bar{D}_{\alpha_1}\Phi \quad\text{or}\quad t_2 \propto \cdots D_{\alpha_2}\bar{D}_{\alpha_1}\Phi. \tag{4.7}$$

By means of the anticommutation relations (2.86) of D and $\bar{D}$ we find

$$t_1 \propto \cdots \bar{D}\sigma^2 D^*\Phi \quad\text{and}\quad t_2 \propto \cdots \sigma^j_{\alpha_2\;\alpha_1}\partial_j\Phi. \tag{4.8}$$

Iterating this argument, we can turn all covariant derivatives which we assume to make up the ellipses into a corresponding form. In the end, we have to impose a contraction of all the α indexes. If the number of $\sigma^j\partial_j$ terms and thus multiplied (transposed) Pauli matrices is odd, we get zero. If, instead, it is even, say $2n$, depending on the particular contraction we are left with zero, a proportionality to ∂_x^{2n} or to $\partial_1^{2n}-\partial_2^{2n}+\partial_3^{2n}$. Because the last variant contradicts the requirement that the ρ_i are scalars, it is extraneous.

As can be seen by means of partial integration, the chirality of Φ and thus antichirality of $\Phi^\dagger$ implies that a ρ_2-term contributing to ΔS_k gives a non-vanishing result only if the overall number of $D^T\sigma^2 D$ and $\bar{D}\sigma^2 D^*$ factors is even, while for ρ_1 it has to be odd, instead. Again employing the anticommutation relations for the covariant derivatives we find that each pair $(D^T\sigma^2 D)(\bar{D}\sigma^2 D^*)$ can be essentially reduced to ∂_x^2. Discounting the h. c. addend in Eq. (4.6), this leaves us with integrands proportional to

$$\Phi^\dagger\partial_x^{2n}\Phi,\quad \Phi\partial_x^{2n}\Phi,\quad\text{and}\quad \Phi\bar{D}\sigma^2 D^*\partial_x^{2n}\Phi \quad\text{with}\quad n\in\mathbb{N}_0. \tag{4.9}$$

Finally, an explicit calculation reveals

$$\Phi\bar{D}\sigma^2 D^*\partial_x^{2n}\Phi \propto \Phi\partial_x^{2(n+1)}\Phi. \tag{4.10}$$

Thus, we can write

$$\Delta S_k = -\frac{1}{4}\int \mathrm{d}^3x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\;\Phi^\dagger r_2(-\partial_x^2)\Phi - \left\{\frac{1}{4i}\int \mathrm{d}^3x\,\mathrm{d}^2\theta\;\Phi r_1(-\partial_x^2)\Phi + \mathrm{h.\,c.}\right\}. \tag{4.11}$$

□

We believe that this result can be generalized to account also for non-analytical ρ_i.

Let us introduce the real and imaginary parts of the bosonic fields:

$$\phi = \phi_1 + i\phi_2, \quad f = f_1 + if_2. \tag{4.12}$$

Then in Fourier space component notation Γ_k and ΔS_k read

$$\begin{aligned}\Gamma_k = \int \mathrm{d}^3q \, \Bigg(& Z_0 \Big(q^2\phi_1(q)\phi_1(-q) + q^2\phi_2(q)\phi_2(-q) - \bar{\psi}(q)\sigma^j q_j \psi(q) \\ & -f_1(q)f_1(-q) - f_2(q)f_2(-q) \Big) \\ & -\frac{1}{2}\left\{ \frac{\partial W_k}{\partial \phi_1}(q) f_1(-q) + i\frac{\partial W_k}{\partial \phi_1}(q) f_2(-q) - i\frac{\partial W_k}{\partial \phi_2}(q) f_1(-q) + \frac{\partial W_k}{\partial \phi_2}(q) f_2(-q) \right. \\ & \left. -\frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 W_k}{\partial \phi_1^2} - \frac{1}{2}\frac{\partial^2 W_k}{\partial \phi_2^2} - i\frac{\partial^2 W_k}{\partial \phi_1 \partial \phi_2} \right)(q) \frac{1}{\sqrt{2\pi}^3} \int \mathrm{d}^3q' \; \psi^T(q')\sigma^2\psi(-q-q') + \mathrm{h.c.} \right\} \Bigg)\end{aligned} \tag{4.13}$$

and, requiring besides $r_2(q^2)$ also $r_1(q^2)$ to be real,

$$\begin{aligned}\Delta S_k = \int \mathrm{d}^3q \, \Bigg(& r_2(q^2) \Big(q^2\phi_1(q)\phi_1(-q) + q^2\phi_2(q)\phi_2(-q) - \bar{\psi}(q)\sigma^j q_j \psi(q) \\ & -f_1(q)f_1(-q) - f_2(q)f_2(-q) \Big) \\ & -r_1(q^2) \Big(2\phi_1(q)f_1(-q) - 2\phi_2(q)f_2(-q) - \frac{1}{2}\psi^T(-q)\sigma^2\psi(q) - \frac{1}{2}\bar{\psi}(-q)\sigma^2\psi^*(q) \Big) \Bigg).\end{aligned} \tag{4.14}$$

With

$$\begin{aligned}\Psi(q) &= \Big(\phi_1(q), \phi_2(q), f_1(q), f_2(q), \psi^T(q), \bar{\psi}(-q) \Big)^T \\ \text{and} \;\; \Psi^\dagger(q) &= \Big(\phi_1(-q), \phi_2(-q), f_1(-q), f_2(-q), \bar{\psi}(q), \psi^T(-q) \Big)\end{aligned} \tag{4.15}$$

the regulator term becomes

$$\Delta S_k = \int \mathrm{d}^3q \; \Psi^\dagger(q) R_k(q) \Psi(q) \tag{4.16}$$

where R_k is the block diagonal regulator matrix composed of the first, "bosonic" block

$$R_B = 2\begin{pmatrix} q^2 r_2 \mathbb{1} & -r_1\sigma^3 \\ -r_1\sigma^3 & -r_2\mathbb{1} \end{pmatrix} \tag{4.17}$$

and the fermionic one

$$R_F = \begin{pmatrix} -r_2\sigma^j q_j & r_1\sigma^2 \\ r_1\sigma^2 & -r_2\sigma^{jT} q_j \end{pmatrix}. \tag{4.18}$$

Remark: The introduced structure of ΔS_k is independent of the considered truncation. However, the matrix expressions for R_B and R_F look somewhat different in the subsequent sections. The reason is that we change the definition of Ψ, moving from Ψ_r to Ψ_c (see section 3.3). First, Ψ_r

has been chosen because it increased the similarity to some previous works in this field. Yet Ψ_c heavily simplifies the computations.

As we have set W_k to be holomorphic, it satisfies the Cauchy-Riemann differential equations, which read

$$\frac{\partial v}{\partial \phi_1} = -\frac{\partial u}{\partial \phi_2}, \quad \frac{\partial v}{\partial \phi_2} = \frac{\partial u}{\partial \phi_1} \tag{4.19}$$

with u the real and v the imaginary part of W_k. Thus, Γ_k can be rewritten as

$$\begin{aligned}\Gamma_k = \int \mathrm{d}^3 q \, \Bigg(& Z_0 \left(q^2 \phi_1(q)\phi_1(-q) + q^2 \phi_2(q)\phi_2(-q) - \bar{\psi}(q)\sigma^j q_j \psi(q) \right. \\ & \left. - f_1(q) f_1(-q) - f_2(q) f_2(-q) \right) \\ & - \left\{ \frac{\partial u}{\partial \phi_1}(q) f_1(-q) + i \frac{\partial u}{\partial \phi_1}(q) f_2(-q) - i \frac{\partial u}{\partial \phi_2}(q) f_1(-q) + \frac{\partial u}{\partial \phi_2}(q) f_2(-q) \right. \\ & \left. - \left(\frac{\partial^2 u}{\partial \phi_1^2} - i \frac{\partial^2 u}{\partial \phi_1 \partial \phi_2} \right)(q) \frac{1}{\sqrt{2\pi}^3} \int \mathrm{d}^3 q' \; \psi^T(q') \sigma^2 \psi(-q-q') + \mathrm{h.\,c.} \right\} \Bigg).\end{aligned} \tag{4.20}$$

Now, we are ready to derive the truncated flows of W_k and Z_0.

First, let us restrict all fields F_i to real space constants. Then $F_i(q) = \sqrt{2\pi}^3 F_i \delta(q)$. Setting in addition $\psi = 0$ gives

$$\partial_t \Gamma_k = (2\pi)^3 \delta(0) \partial_t \left(-Z_0 (f_1^2 + f_2^2) - 2 \left(\frac{\partial u}{\partial \phi_1} f_1 + \frac{\partial u}{\partial \phi_2} f_2 \right) \right) \tag{4.21}$$

for the left hand side of the Wetterich equation. Obviously, our manipulations have not narrowed the ability to extract the flows we are interested in. Similarly, eliminating f instead of ψ yields

$$\partial_t \Gamma_k = (2\pi)^3 \delta(0) \partial_t \left\{ \left(\frac{\partial^2 u}{\partial \phi_1^2} - i \frac{\partial^2 u}{\partial \phi_1 \partial \phi_2} \right) \psi^T \sigma^2 \psi + \mathrm{h.\,c.} \right\}, \tag{4.22}$$

which, not allowing to derive the flow of Z_0 or the whole superpotential anymore, still preserves all necessary information on the second derivatives of W_k. More precisely, to the order of the LPA$'$ truncation we can get

- $\partial_t W_k$ from the first f_i-derivatives at $f_1 = f_2 = 0$ and
- $\partial_t Z_0$ from the second f_1- or f_2-derivative at $f_1 = f_2 = \phi_1 = \phi_2 = 0$

applied to

$$\text{case I:} \quad \frac{1}{2} \mathrm{STr} \left(\partial_t R_k (\Gamma_k^{(2)} + R_k)^{-1} \right) \quad \text{at constant fields and } \psi = \bar{\psi} = 0. \tag{4.23}$$

For a check, we additionally compute $\partial_t W_k''$ taking the ψ_2-ψ_1-derivative at $\psi = \bar{\psi} = 0$ of

$$\text{case II:} \quad \frac{1}{2} \mathrm{STr} \left(\partial_t R_k (\Gamma_k^{(2)} + R_k)^{-1} \right) \quad \text{at constant fields and } f_1 = f_2 = 0. \tag{4.24}$$

That is why calculating

$$\Gamma_k^{(2)}(p,q) = \frac{\overrightarrow{\delta}}{\delta\Psi^\dagger(p)}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\Psi(q)} \tag{4.25}$$

we immediately confine the result to the case I or II conditions. The obtained matrices are presented in appendix A. Next, $\Gamma_k^{(2)}(p,q) + R_k(q)\delta(p-q)$ has to be inverted. Because $\Gamma_k^{(2)}(p,q)$ turns out to be proportional to $\delta(p-q)$, this reduces to an inversion of the corresponding matrix part; the $\delta(p-q)$ factor remains unaffected.

In case I, $\Gamma_k^{(2)}$ is a block diagonal matrix composed of ordinary commuting elements. Furthermore, only the bosonic block contains an f-dependence. Thus, we can restrict our computations to the first diagonal blocks of all matrices involved. The supertrace becomes a common trace and, when applied to the f_i-derivatives needed for $\partial_t W_k$, evaluates to zero. Consequently,

$$\partial_t W_k = 0. \tag{4.26}$$

The proposed non-renormalization theorem holds at least to the order of the LPA$'$ truncation. The projection onto $\partial_t Z_0$ yields

$$\partial_t Z_0 = -Z_0\eta = -\frac{4g^2}{(2\pi)^3}\int \mathrm{d}^3q\ \frac{h}{v^3}\left(2hM\partial_t r_1 - u\partial_t r_2\right) \tag{4.27}$$

where

$$m \equiv c_2,\quad g \equiv c_3,\quad h = Z_0 + r_2,\quad M = m + r_1,\quad u = M^2 - q^2h^2,\quad \text{and}\quad v = M^2 + q^2h^2. \tag{4.28}$$

Handling case II is a little bit more involved, as $\Gamma_k^{(2)}$, this time, is a full “supermatrix”. The off-diagonal blocks consist of anticommuting elements. This makes the inversion less straightforward. Following [47], we expand the $(\Gamma_k^{(2)} + R_k)$-matrix, let us call it G^{-1}, in its fermionic content:

$$G^{-1} = \Gamma_0 + M_1\psi_1 + M_2\psi_2 + M_3\bar{\psi}_1 + M_4\bar{\psi}_2 + M_5\psi_1\psi_2 + M_6\bar{\psi}_1\bar{\psi}_2. \tag{4.29}$$

Then for regular Γ_0 an explicit computation affirms that

$$\begin{aligned} G =& \Gamma_0^{-1} - \Gamma_0^{-1}(M_1\psi_1 + M_2\psi_2 + M_3\bar{\psi}_1 + M_4\bar{\psi}_2 + M_5\psi_1\psi_2 + M_6\bar{\psi}_1\bar{\psi}_2)\Gamma_0^{-1} \\ &+ \Gamma_0^{-1}\left((M_1\psi_1)\Gamma_0^{-1}(M_2\psi_2) + \text{all further permutations with two } M\right)\Gamma_0^{-1} \\ &- \Gamma_0^{-1}(\text{all permutations with three } M)\Gamma_0^{-1} \\ &+ \Gamma_0^{-1}(\text{all permutations with four } M)\Gamma_0^{-1}. \end{aligned} \tag{4.30}$$

The supertrace of the product of $\partial_t R_k$ with the contribution to G which is proportional to $\psi_1\psi_2$ vanishes. Thus,

$$\partial_t W_k'' = 0 \tag{4.31}$$

in agreement with the above result.

To proceed, we introduce the dimensionless renormalized quantities

$$\begin{aligned} \chi &= Z_0^{1/2} k^{(2-D)/2} \phi &&= \left(\frac{Z_0}{k}\right)^{1/2} \phi, \\ w_k(\chi) &= k^{1-D} W_k(\phi) &&= k^{-2} W_k(\phi), \\ \text{and}\ \ c_n' &= Z_0^{-n/2} k^{1-D+n(D-2)/2} c_n &&= Z_0^{-n/2} k^{-2+n/2} c_n; \end{aligned} \tag{4.32}$$

particularly

$$m' = \frac{1}{Z_0 k} m \ \text{ and }\ g' = \left(\frac{1}{Z_0^3 k}\right)^{1/2} g. \tag{4.33}$$

Thus,

$$w_k(\chi) = \sum_{n=1}^{\infty} \frac{c_n'}{n} \chi^n. \tag{4.34}$$

To render the regulating functions dimensionless, too, we replace r_1 by kr_1. Additionally, we choose the r_i to be proportional to Z_0, $r_i =: Z_0 r_i'$. This eliminates the unphysical wave function renormalization from the flow equations. Finally, we migrate to the dimensionless momentum variable $q' = q/k$ simultaneously defining $r_i''(q^2) = r_i'(k^2 q^2)$. From now on omitting all primes we get

$$\begin{aligned} \partial_t w_k &= (1-D) w_k + \frac{1}{2}(\eta + D - 2)\chi w_k' \\ &= -2 w_k + \frac{1}{2}(\eta + 1)\chi w_k' \\ \text{and}\ \ \eta &= \frac{\Omega_D}{(2\pi)^D} 4 g^2 \int \mathrm{d}q\ q^{D-1} \frac{h}{v^3} \Big(2hM(\partial_t - q\partial_q - \eta + 1) r_1 - u(\partial_t - q\partial_q - \eta) r_2 \Big) \\ &= \frac{2}{\pi^2} g^2 \int \mathrm{d}q\ q^2 \frac{h}{v^3} \Big(2hM(\partial_t - q\partial_q - \eta + 1) r_1 - u(\partial_t - q\partial_q - \eta) r_2 \Big) \\ \text{with}\ \ h &= 1 + r_2,\ \ M = m + r_1,\ \ u = M^2 - q^2 h^2,\ \text{ and }\ v = M^2 + q^2 h^2; \end{aligned} \tag{4.35}$$

Ω_D denotes the surface of a sphere in D dimensions.

Remark: In contrast to the flow equations as formulated in dimensioned and non-renormalized quantities, here ∂_t leaves χ, not ϕ invariant.

The fixed point equation for the superpotential reads

$$w_* = \frac{1}{2(D-1)}(\eta_* + D - 2)\chi w_*' = \frac{1}{4}(\eta_* + 1)\chi w_*'. \tag{4.36}$$

For w_* non-vanishing, this is solved by

$$w_* = C_* \chi^n \ \text{ with }\ n = \frac{2(D-1)}{\eta_* + D - 2} = \frac{4}{\eta_* + 1} \ \text{ and }\ C_* \neq 0. \tag{4.37}$$

Because of the holomorphy of w_*, any χ-derivative of w_* has to be free of divergences. Besides, as we have set c_0 to zero, $w_* \neq 0$ cannot be constant. Therefore n has to be a positive integer.

Hence, we get

$$\eta_* \in \left\{ \frac{1}{n} \Big(2(D-1) - n(D-2) \Big), \;\; n \in \mathbb{N} \right\} = \left\{ \frac{1}{n}(4-n), \;\; n \in \mathbb{N} \right\}. \tag{4.38}$$

For $w_* = 0$, η_* remains unrestricted beyond $\partial_t \eta_* = 0$ so far.

From Eq. (4.35) we see that $\eta_* \propto g_*^2$. Thus, only $w_* = c_3^* \chi^3/3$, $c_3^* \neq 0$ allows for a non-vanishing η_*:

$$\eta_* = \frac{4-D}{3} = \frac{1}{3}. \tag{4.39}$$

Meanwhile, Eq. (4.38) implies that $\eta_* = 0$ can be admitted only if $w_* = 0$ or $w_* = C_* \chi^n$, $C_* \neq 0$ with

$$n = \frac{2(D-1)}{D-2} = 4. \tag{4.40}$$

Thus, we have identified two trivial fixed points and substantiated our expectations concerning the existence of a non-trivial one at $\eta_* = 1/3$. To continue, we have to fix the regulator functions.

Remark: Remember that Eq. (4.35) is due to a truncation. We expect, see e. g. [48] and [49], that with decreasing D the number of admissible fixed point increases. At last, all η_* comprised by Eq. (4.38) should appear in $D = 2$.

4.1.1. Callan-Symanzik Regulator

Let us start by setting

$$r_1 = 1, \;\; r_2 = 0. \tag{4.41}$$

Then the integral in the η-equation (4.35) can be solved for all $0 < D < 6$. Assuming $M > 0$ we are left with

$$\begin{aligned} \eta &= \frac{\Omega_D \, g^2 (D-2)(D-4)}{\Omega_D \, g^2 (D-2)(D-4) + 4(2\pi)^{D-1} \sin(\frac{D\pi}{2}) M^{5-D}} \\ &= \frac{g^2}{g^2 + 4\pi(m+1)^2}. \end{aligned} \tag{4.42}$$

Remark: The restriction to $M = m + 1 > 0$ is legitimate because we need the result to be valid only in an infinitesimal neighborhood of $m_* = 0$.

To specify the superpotential at the non-trivial fixed point we plug in $\eta_* = 1/3$ and $m_* = 0$. This yields

$$g_*^2 = (c_3^*)^2 = \frac{2(2\pi)^{D-1} \sin(\frac{D\pi}{2})}{\Omega_D (D-2)(D-4)} = 2\pi. \tag{4.43}$$

Linearizing the flow of w_k about a fixed point, $w_k = w_* + \delta w_k$, gives

$$\begin{aligned}\partial_t \delta w_k &= (1-D)\delta w_k + \frac{1}{2}(\eta_* + D - 2)\chi\delta w_k' + \frac{1}{2}\chi w_*' \,\mathrm{d}\eta \\ &= -2\delta w_k + \frac{1}{2}(\eta_* + 1)\chi\delta w_k' + \frac{1}{2}\chi w_*' \,\mathrm{d}\eta, \\ \mathrm{d}\eta &= \left.\frac{\partial\eta}{\partial m}\right|_* \mathrm{d}m + \left.\frac{\partial\eta}{\partial g}\right|_* \mathrm{d}g.\end{aligned} \tag{4.44}$$

At both trivial fixed points $\mathrm{d}\eta$ vanishes, leaving

$$\partial_t \delta w_k = (1-D)\delta w_k + \frac{1}{2}(D-2)\chi\delta w_k' = -2\delta w_k + \frac{1}{2}\chi\delta w_k'. \tag{4.45}$$

This differential equation is solved by

$$\delta w_k = \delta w\, \mathrm{e}^{\lambda t} \tag{4.46}$$

with

$$\lambda\delta w = (1-D)\delta w + \frac{1}{2}(D-2)\chi\delta w' = -2\delta w + \frac{1}{2}\chi\delta w'. \tag{4.47}$$

Substituting

$$\delta w = \sum_{n=1}^{\infty} \frac{\mathrm{d}c_n}{n}\chi^n, \tag{4.48}$$

we get

$$\lambda\,\mathrm{d}c_n = \left(1 - D + \frac{n}{2}(D-2)\right)\mathrm{d}c_n = \left(-2 + \frac{n}{2}\right)\mathrm{d}c_n \;\; \forall n. \tag{4.49}$$

The eigenvalues of our stability matrix can be read off immediately:

$$\lambda_n = 1 - D + \frac{n}{2}(D-2) = -2 + \frac{n}{2}. \tag{4.50}$$

The corresponding eigendirections — the eigenvectors in terms of δw — are $\delta w_n \propto \chi^n$. We see that the critical exponents $(-\lambda_n)$ equal the classical dimension of the coupling constants c_n. This is the compulsory result for any trivial fixed point.

At the non-trivial fixed point

$$\left.\frac{\partial\eta}{\partial m}\right|_* = 2\eta_*^2(D-5) = -\frac{4}{9}, \quad \left.\frac{\partial\eta}{\partial g}\right|_* = 2\eta_*(1-\eta_*)\frac{1}{g_*} = \frac{4}{9}\frac{1}{g_*}. \tag{4.51}$$

Thus,

$$\begin{aligned}\lambda\,\mathrm{d}c_n &= \left(1 - D + \frac{n}{2}(\eta_* + D - 2)\right)\mathrm{d}c_n = \left(-2 + \frac{2}{3}n\right)\mathrm{d}c_n \;\; \forall n \neq 3 \\ \text{and} \;\; \lambda\,\mathrm{d}c_3 &= -\frac{2}{3}g_*\,\mathrm{d}c_2 + \left(\frac{D-3}{2} + \frac{2}{3}\right)\mathrm{d}c_3 = -\frac{2}{3}g_*\,\mathrm{d}c_2 + \frac{2}{3}\,\mathrm{d}c_3.\end{aligned} \tag{4.52}$$

For $n \neq 3$ this gives

$$\lambda_n = 1 - D + \frac{n}{2}(\eta_* + D - 2) = -2 + \frac{n}{2}(\eta_* + 1) = -2 + \frac{2}{3}n \tag{4.53}$$

and the third eigenvalue becomes

$$\lambda_3 = \frac{D-3}{2} + \frac{2}{3} = \frac{2}{3}. \tag{4.54}$$

The corresponding eigendirections are $\delta w_{n\neq 2} \propto \chi^n$ and $\delta w_2 \propto 2\chi^2 + g_*\chi^3$.

Hence, according to the definition of ω at the beginning of the present chapter we have found

$$\omega = \lambda_3 = \frac{D-3}{2} + \frac{2}{3} = \frac{2}{3}. \tag{4.55}$$

Our deviation from [11] is quite large,

$$e_\omega := \frac{|\omega - \omega_b|}{\omega_b} \approx 27\%. \tag{4.56}$$

However, because of the rough truncation employed this is not very surprising.

4.1.2. Litim-Type Regulator

To get a brief insight into the regulator-dependence of the LPA′ results, we turn to

$$r_1 = 0, \quad r_2 = \left(\frac{1}{q} - 1\right)\Theta(1-q^2). \tag{4.57}$$

This gives

$$\eta = 4g^2\Omega_D(D-1)\frac{m^2-1}{4g^2\Omega_D(m^2-1) - (2\pi)^D(D-2)(D-1)(m^2+1)^3} \tag{4.58}$$

$$= 2g^2\frac{m^2-1}{g^2(m^2-1) - \pi^2(m^2+1)^3}. \tag{4.59}$$

The linearization of $\partial_t w$ at w_* remains

$$\begin{aligned}\partial_t\delta w_k &= (1-D)\delta w_k + \frac{1}{2}(\eta_* + D - 2)\chi\delta w_k' + \frac{1}{2}\chi w_*'\,\mathrm{d}\eta \\ &= -2\delta w_k + \frac{1}{2}(\eta_*+1)\chi\delta w_k' + \frac{1}{2}\chi w_*'\,\mathrm{d}\eta, \\ \mathrm{d}\eta &= \left.\frac{\partial\eta}{\partial m}\right|_*\mathrm{d}m + \left.\frac{\partial\eta}{\partial g}\right|_*\mathrm{d}g\end{aligned} \tag{4.44}$$

and for the trivial fixed points we, once more, find $\mathrm{d}\eta = 0$. Thus, the $\eta_* = 0$ results are not affected by the change of regulator.

At the non-trivial fixed point with $\eta_* = 1/3$, by contrast,

$$\frac{\partial\eta}{\partial m}\bigg|_* = 0 \quad\text{and}\quad \frac{\partial\eta}{\partial g}\bigg|_* = \eta_*\left(2 - \frac{2}{D-2}\eta_*\right)\frac{1}{g_*} = \frac{5}{9}\frac{1}{g_*} \tag{4.60}$$

differs from above. Now,

$$\begin{aligned}\lambda\,\mathrm{d}c_n &= \left(1 - D + \frac{n}{2}(\eta_* + D - 2)\right)\mathrm{d}c_n = \left(-2 + \frac{2}{3}n\right)\mathrm{d}c_n \quad \forall n \neq 3\\ \text{and}\quad \lambda\,\mathrm{d}c_3 &= \left(\frac{D-3}{2} + \frac{5}{6}\right)\mathrm{d}c_3 = \frac{5}{6}\,\mathrm{d}c_3;\end{aligned} \tag{4.61}$$

the $n = 2$ eigendirection gets simplified to $\delta w_2 \propto \chi^2$ and λ_3 becomes

$$\lambda_3 = \left(\frac{D-3}{2} + \frac{5}{6}\right) = \frac{5}{6}. \tag{4.62}$$

Hence,

$$\omega = \lambda_3 = \frac{5}{6} \tag{4.63}$$

gets closer to ω_b:

$$e_\omega \approx 8\,\% \tag{4.64}$$

Remark: As long as the non-renormalization theorem $\partial_t W_k = 0$ holds, the spectrum of the stability matrix inevitably contains the eigenvalues

$$\lambda_n = 1 - D + \frac{n}{2}(\eta_* + D - 2) = -2 + \frac{2}{3}n, \quad n \neq 3. \tag{4.65}$$

4.2. Beyond LPA′: Adding Momentum-Dependence

Hoping to improve our results we move on to the — still supersymmetric — truncation

$$\Gamma_k = -\frac{1}{4}\int \mathrm{d}^3x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar\theta\;\Phi^\dagger z_k(D,\bar D)\Phi - \left\{\frac{1}{2i}\int \mathrm{d}^3x\,\mathrm{d}^2\theta\;W_k(\Phi) + \text{h. c.}\right\} \tag{4.66}$$

where z_k is another arbitrary scalar, Hermitian function of the supercovariant derivatives. Just as ρ_2 in the generic formulation (4.6) of ΔS_k, $z_k(D,\bar D)$ can be substituted by some $Z_k(-\partial_x^2)$. Thus, we start from

$$\begin{aligned}\Gamma_k = \int \mathrm{d}^3q\;\Bigg(&Z_k(q^2)(q^2\phi(q)\phi^\dagger(q) + i\bar\psi(q)\sigma^j q_j\psi(q) - f(q)f^\dagger(q))\\ &-\left\{W_k'(q)f(-q) - \frac{1}{2}W_k''(q)\frac{1}{\sqrt{2\pi}^3}\int \mathrm{d}^3q'\;\psi^T(q')\sigma^2\psi(-q-q') + \text{h. c.}\right\}\Bigg).\end{aligned} \tag{4.67}$$

$Z_k(0)$ is the wave function renormalization. Therefore we introduce the notation

$$Z_k(0) = Z_0 \quad\text{and}\quad Z_k(q^2) = Z_0\zeta_k(q^2). \tag{4.68}$$

As announced, we turn to

$$\begin{aligned} \Psi(q) &= \Big(\phi(q), \phi^\dagger(-q), f(q), f^\dagger(-q), \psi^T(q), \bar{\psi}(-q)\Big)^T \\ \text{and}\ \ \Psi^\dagger(q) &= \Big(\phi^\dagger(q), \phi(-q), f^\dagger(q), f(-q), \bar{\psi}(q), \psi^T(-q)\Big) \end{aligned} \tag{4.69}$$

with the result that the diagonal blocks of the regulator matrix defining ΔS_k as presented in the last section, see Eq. (4.14)ff, become

$$R_B = \begin{pmatrix} q^2 r_2 \mathbb{1} & -r_1\sigma^1 \\ -r_1\sigma^1 & -r_2\mathbb{1} \end{pmatrix} \tag{4.70}$$

and

$$R_F = \begin{pmatrix} r_2\sigma^j q_j & r_1\sigma^2 \\ r_1\sigma^2 & r_2\sigma^{jT} q_j \end{pmatrix}. \tag{4.71}$$

Remark: Though the fermionic branch of Ψ has not changed, the signs on the diagonal of R_F differ from above. This is due to a sign reversal in the definition of Fourier transformations.

If we restricted all fields to real space constants in the present ansatz for Γ_k, we would lose all information on the momentum-dependence of ζ_k. Therefore, we make use of the substitution $F_i(q) = \sqrt{2\pi}^3 F_i \delta(q)$ only with regard to f, ψ, and their Hermitian conjugates, so far. This gives

$$\begin{aligned} \Gamma_k = \int \mathrm{d}^3 q\ Z_k(q^2) q^2 \phi(q)\phi^\dagger(q) - (2\pi)^3\delta(0) Z_k(0) f f^\dagger \\ - \sqrt{2\pi}^3 \left\{ W_k'(q=0) f - \frac{1}{2} W_k''(q=0)\psi^T\sigma^2\psi + \text{h. c.} \right\}. \end{aligned} \tag{4.72}$$

Setting additionally $\psi = \bar{\psi} = 0$ leaves

$$\Gamma_k = \int \mathrm{d}^3 q\ Z_k(q^2) q^2 \phi(q)\phi^\dagger(q) - (2\pi)^3\delta(0) Z_k(0) f f^\dagger - \sqrt{2\pi}^3 \left\{ W_k'(q=0) f + \text{h. c.} \right\}. \tag{4.73}$$

Hence, applying this projection to the Wetterich equation we simplify it without diminishing our ability to derive, to the order of the present truncation, the flows of Z_k and W_k. Subsequently, the beta-function of Z_k can be obtained setting $f = f^\dagger = 0$ and taking the functional $\phi^\dagger$-ϕ-derivative at vanishing ϕ and $\phi^\dagger$. To extract the flow of the superpotential we can return to constant ϕ and $\phi^\dagger$ and compute the f-derivative at $\phi^\dagger = f = f^\dagger = 0$.

Our result for $\Gamma_k^{(2)}$ at constant f, $f^\dagger$ and vanishing ψ, $\bar{\psi}$ can be looked up in appendix B. The obtained $\Gamma_k^{(2)}(p,q)$ is block diagonal but not proportional to $\delta(p-q)$ anymore. However, at constant ϕ and $\phi^\dagger$ this virtue is regained. Thus, the derivation of $\partial_t W_k$ can be conducted in the same manner as with the LPA$'$ truncation. The computation of $\partial_t W_k$ is even once more simplified by the fact that only the bosonic block of $\Gamma_k^{(2)}$ contains an f-dependence. As before, we find that W_k does not renormalize,

$$\partial_t W_k = 0. \tag{4.74}$$

In the context of $\partial_t Z_k$ there is nothing that could be done about the non-trivial momentum-dependence of $\Gamma_k^{(2)}$. Therefore, it is impossible to reduce the inversion of $\Gamma_k^{(2)} + R_k$ to a simple matrix operation. Instead, we follow [50], first rewriting the right hand side of the Wetterich equation as

$$\frac{1}{2}\,\mathrm{STr}\left(\partial_t R_k(\Gamma_k^{(2)} + R_k)^{-1}\right) = \frac{1}{2}\,\mathrm{STr}\left(\tilde{\partial}_t \ln(\Gamma_k^{(2)} + R_k)\right) \tag{4.75}$$

where $\tilde{\partial}_t$ is assumed to apply exclusively to the t-dependence of R_k. Splitting $\Gamma_k^{(2)} + R_k$ into a field-independent constituent $\Gamma_0(p,q) \propto \delta(p-q)$ and the field-dependent part $\Delta\Gamma$, we can expand

$$\frac{1}{2}\,\mathrm{STr}\left(\tilde{\partial}_t \ln(\Gamma_k^{(2)} + R_k)\right) = \frac{1}{2}\,\mathrm{STr}\left(\tilde{\partial}_t\left(\ln(\Gamma_0) + \Gamma_0^{-1}\Delta\Gamma - \frac{1}{2}(\Gamma_0^{-1}\Delta\Gamma)^2 + \ldots\right)\right). \tag{4.76}$$

Because every non-vanishing element of $\Delta\Gamma$ contains at least one ϕ or $\phi^\dagger$ factor and we are going to compute a ϕ-$\phi^\dagger$-derivative at vanishing fields, we are fine with the terms

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{STr}\left(\tilde{\partial}_t\left(\Gamma_0^{-1}\Delta\Gamma - \frac{1}{2}(\Gamma_0^{-1}\Delta\Gamma)^2\right)\right)\\
&= -\frac{1}{2}\,\mathrm{STr}\left(\Gamma_0^{-1}\partial_t R_k \Gamma_0^{-1}\Delta\Gamma\right)\\
&\quad + \frac{1}{4}\,\mathrm{STr}\left(\Gamma_0^{-1}\partial_t R_k\Gamma_0^{-1}\Delta\Gamma\Gamma_0^{-1}\Delta\Gamma\right) + \frac{1}{4}\,\mathrm{STr}\left(\Gamma_0^{-1}\Delta\Gamma\partial_t\Gamma_0^{-1}R_k\Gamma_0^{-1}\Delta\Gamma\right).
\end{aligned} \tag{4.77}$$

Evaluating them at $f = f^\dagger = 0$ and taking the ϕ-derivatives we finally find

$$\begin{aligned}
&\partial_t Z_k(p^2) = -\frac{4g^2}{(2\pi)^D}\int \mathrm{d}^D q\,\frac{h(q-p)}{v^2(q)v(q-p)}\left(2h(q)M(q)\partial_t r_1(q^2) - u(q)\partial_t r_2(q^2)\right)\\
&\text{with}\ \ h = Z_k + r_2,\ \ M = m + r_1,\ \ u = M^2 - \left(\frac{q^2}{k^2}\right)h^2,\ \ \text{and}\ \ v = M^2 + \left(\frac{q^2}{k^2}\right)h^2.
\end{aligned} \tag{4.78}$$

Assuming $Z_k(p^2) = Z_0$ recovers the LPA$'$ flow of Z_0.

In dimensionless, renormalized quantities including $p \to kp$ — see section 4.1 for details concerning the other redefinitions — the flow equation of $Z_k(p^2)$ or rather $\zeta_k(p^2)$ reads

$$\begin{aligned}
(\partial_t - p\partial_p - \eta)\zeta_k(p^2) &= -\frac{4g^2}{(2\pi)^D}\int \mathrm{d}^D q\,\frac{h(q-p)}{v^2(q)v(q-p)}\\
&\quad\left(2h(q)M(q)(\partial_t - q\partial_q - \eta + 1)r_1(q^2) - u(q)(\partial_t - q\partial_q - \eta)r_2(q^2)\right)\\
&= -\frac{4g^2}{(2\pi)^3}\int \mathrm{d}^3 q\,\frac{h(q-p)}{v^2(q)v(q-p)}\\
&\quad\left(2h(q)M(q)(\partial_t - q\partial_q - \eta + 1)r_1(q^2) - u(q)(\partial_t - q\partial_q - \eta)r_2(q^2)\right)
\end{aligned} \tag{4.79}$$

with

$$h=\zeta_k+r_2,\;\; M=m+r_1,\;\; u=M^2-q^2h^2,\;\; \text{and}\;\; v=M^2+q^2h^2. \tag{4.80}$$

The beta-function of the superpotential remains as stated in Eq. (4.35). Eq. (4.79) must particularly hold at $p=0$, where it gives the flow of Z_0

$$\begin{aligned}\eta=&\frac{4\Omega_D}{(2\pi)^D}g^2\int\mathrm{d}q\;q^{D-1}\frac{h}{v^3}\left(2hM(\partial_t-q\partial_q-\eta+1)r_1-u(\partial_t-q\partial_q-\eta)r_2\right)\\ =&\frac{2}{\pi^2}g^2\int\mathrm{d}q\;q^2\frac{h}{v^3}\left(2hM(\partial_t-q\partial_q-\eta+1)r_1-u(\partial_t-q\partial_q-\eta)r_2\right).\end{aligned} \tag{4.81}$$

Again, η_* vanishes if $g_*=0$. Thus, as before, only the three fixed points $\eta_*=0$ with $w_*=0$ or $w_*=C_*\chi^4$, $C_*\neq 0$ and $\eta_*=1/3$ with $w_*=c_3^*\chi^3/3$, $c_3^*\neq 0$ are admissible. This time, we immediately address the non-trivial one.

Before doing so, let us heuristically continue the obtained flow equations to $D=2$ by merely reducing the dimension of the coordinate space. Then the non-renormalization property

$$\partial_t W_k(\phi)=0 \tag{4.82}$$

is preserved and the flow of Z_k is conveyed by Eq. (4.78) evaluated at $D=2$. It turns out that this, indeed, gives almost the same as has been found in [25] for the $D=2$ $\mathcal{N}=2$ Wess-Zumino model. The only difference is an apparently missing factor of two on the right hand side of our $Z_k(p^2)$-flow. However, this traces back to the fact that we have started from a doubled Γ_k-ansatz and ΔS_k as compared to [25].

4.2.1. Litim-Type Regulator I

First, we try out the so far more successful regulator from subsection 4.1.2,

$$r_1=0,\;\; r_2=\left(\frac{1}{q}-1\right)\Theta(1-q^2), \tag{4.57}$$

and polynomially truncate ζ_k in the simplest possible way not coinciding with LPA′,

$$\zeta_k(p^2)=1+p^2\zeta_1. \tag{4.83}$$

Then the flow equation for Z_k is satisfied to the order of the extended truncation if the equation itself and its second p-derivative are fulfilled at $p=0$.

Remark: Projected onto odd powers of p, the flow equation of Z_k vanishes identically. This can be seen by an n-induction concerning

$$\int\mathrm{d}^3q\;\partial_p^n F(|\vec{q}-\vec{p}|)=\int\mathrm{d}q\;q^2\int_{-1}^{1}\mathrm{d}(\cos\theta)\;\partial_p^n F(x),\;\; x=\sqrt{p^2+q^2-2pq\cos\theta} \tag{4.84}$$

as expressed in terms of $\partial_p = (p - q\cos\theta)/x\ \partial_x$. Recall that

$$\int_{-1}^{1} \mathrm{d}(\cos\theta)\ (\cos\theta)^{2n+1} = 0. \tag{4.85}$$

Plugging the regulator definition into the η-equation (4.81) we get

$$\eta = -\frac{2}{\pi^2} g^2 \int_0^1 \mathrm{d}q\ \frac{y(m^2 - y^2)}{(m^2 + y^2)^3}(1 - \eta(1-q)) \quad \text{with} \quad y = 1 + q^3\zeta_1. \tag{4.86}$$

At the non-trivial fixed point, where $\eta_* = 1/3$ and $m_* = 0$, this becomes

$$\frac{1}{3} = \frac{2}{\pi^2} g_*^2 \int_0^1 \mathrm{d}q\ \frac{2+q}{3y_*^3}. \tag{4.87}$$

To compute the second derivative of the Z_k-flow at $p = 0$, we have to evaluate an integral I of the form

$$I = \int \mathrm{d}q\ f(q)\delta(1-q)\Theta(1-q^2) \tag{4.88}$$

where $f(q)$ itself contains Heaviside functions, whose prefactors, however, vanish at $q = 1$. Adopting the definition $\Theta(0) = 1/2$ leads to $I = f(1)/2$. At $m = m_* = 0$, we thus get

$$\begin{aligned} \partial_t\zeta_1|_{m=0} &= \frac{2}{3\pi^2} g^2 \int_0^1 \mathrm{d}q\ \frac{\tilde{y}}{y^5}(2+q)q\zeta_1 + \frac{1}{6\pi^2} g^2 \frac{1}{(1+\zeta_1)^4} + (2+\eta)\zeta_1 \\ &\text{with} \quad \tilde{y} = 1 - 2q^3\zeta_1. \end{aligned} \tag{4.89}$$

Now, we have to specify the fixed point values g_* and ζ_1^*, $\partial_t\zeta_1^* = 0$ from Eq. (4.87) and (4.89). Let us solve Eq. (4.87) for $1/g_*^2$ and multiply it by ζ_1^*. In the fixed point form of Eq. (4.89), we bring $(2 + \eta_*)\zeta_1^*$ to the left hand side and divide the result by $-(2 + \eta_*)g_*^2$. Subtracting the two results from each other gives

$$0 = \left(7\int_0^1 \mathrm{d}q\ \frac{1}{y_*^3}(2+q) + \int_0^1 \mathrm{d}q\ \frac{\tilde{y}_*}{y_*^5}(2+q)q \right)\zeta_1^* + \frac{1}{4(1+\zeta_1^*)^4} =: F(\zeta_1^*) \tag{4.90}$$

independent of g_*. The roots of $F(\zeta_1)$ yielding a finite value of g_*^2 upon insertion into one of the fixed point equations are exactly the approximations of ζ_1^* available from the current truncation. The integrals can be solved analytically for any $\zeta_1 > -1$. Subsequently, one can numerically identify the roots ζ_{1R} of $F(\zeta_1)$. The FindRoot routine of Mathematica returns $\zeta_{1R} \approx -0.0138$, this result being supported by the evidence of the graph of $F(\zeta_1)$. As this corresponds to $g_*^2 \approx 1.9509$ finite, $\zeta_{1R} = \zeta_1^*$.

The next step on our way towards critical exponents is to linearize the flow equations of both the superpotential and ζ_1 about the fixed point. To this end, we particularly need the m-derivative of $\partial_t\zeta_1$ at $m = m_* = 0$. However, $\partial_t\zeta_1 - (2+\eta)\zeta_1$ can be expressed as an integral over a sum where the m-dependence of each addend is confined to a factor of the form

$$\frac{m^2 - y^2}{(m^2+y^2)^s}, \quad s \geq 3. \tag{4.91}$$

Thus,

$$\left.\frac{\partial(\partial_t\zeta_1)}{\partial m}\right|_* = 0. \tag{4.92}$$

All other derivatives of $\partial_t\zeta_1$ can be computed directly from the $m = m_* = 0$ formulation of the ζ_1-flow as given by Eq. (4.89). Finally, we get

$$\begin{aligned} \partial_t \delta w_k &= -2\delta w_k + \frac{2}{3}\chi\delta w_k' + \chi^3(a\,\mathrm{d}g + b\,\mathrm{d}\zeta_1) \\ \text{and} \quad \partial_t\,\mathrm{d}\zeta_1 &= \tilde{a}\,\mathrm{d}g + \tilde{b}\,\mathrm{d}\zeta_1, \end{aligned} \tag{4.93}$$

where a, b, $\tilde{a}$ and $\tilde{b}$ are numbers, whose computation again involves only analytically solvable integrals.

As before, we turn to the exponential ansatz $\delta w_k = \delta w\,\mathrm{e}^{\lambda t}$ however adding

$$d\zeta_1 \to d\zeta_1\,\mathrm{e}^{\lambda t}\,. \tag{4.94}$$

Hence, we have to deal with the following set of equations:

$$\begin{aligned} \lambda\,\mathrm{d}c_n &= \left(-2 + \frac{2}{3}n\right)\mathrm{d}c_n \quad \forall n \neq 3, \\ \lambda\,\mathrm{d}c_3 &= a\,\mathrm{d}c_3 + b\,\mathrm{d}\zeta_1, \\ \lambda\,\mathrm{d}\zeta_1 &= \tilde{a}\,\mathrm{d}c_3 + \tilde{b}\,\mathrm{d}\zeta_1. \end{aligned} \tag{4.95}$$

One immediately recognizes the familiar

$$\lambda_n = -2 + \frac{2}{3}n \quad \forall n \neq 3 \tag{4.96}$$

going along with $\delta w_n \propto \chi^n$ and $\mathrm{d}\zeta_1 = 0$. Recall that, as the non-renormalization theorem has not failed, these eigenvalues necessarily had to reappear. The non-diagonal part of the stability matrix reads

$$\begin{pmatrix} a & b \\ \tilde{a} & \tilde{b} \end{pmatrix}. \tag{4.97}$$

Thus, the remaining eigenvalues are determined by

$$(a-\lambda)(\tilde{b}-\lambda) - \tilde{a}b = 0 \tag{4.98}$$

yielding

$$\lambda_+ = 2.5301, \quad \lambda_- = 0.8317. \tag{4.99}$$

The corresponding eigendirections are

$$(\delta w_+, \mathrm{d}\zeta_{1+}) \propto (\chi^3, \mp 14.4193) \ \text{ and } \ (\delta w_-, \mathrm{d}\zeta_{1-}) \propto (\chi^3, \pm 0.0228) \tag{4.100}$$

where the upper signs are valid for $g_* > 0$ while the lower ones apply to $g_* < 0$. As

$$\frac{3}{2}(\lambda_+ + 2) = 6.7952, \ \frac{3}{2}(\lambda_- + 2) = 4.2475 \tag{4.101}$$

are far away from integer numbers, the spectrum of the stability matrix is non-degenerate.

According to our definition from the beginning of this chapter

$$\omega = \lambda_- = 0.8317. \tag{4.102}$$

This is very close to our previous result, $\omega = 5/6$, which has been obtained from the LPA$'$ truncation in conjunction with the present regulator. As compared to [11], we have become slightly worse; the updated deviation is $e_\omega \approx 9\%$.

The next natural step would be to proceed to higher orders of ζ_k-truncation. However, the present regulator is not very appropriate for this task. For $\zeta_k = 1 + q^2\zeta_1$, it ensures that the denominators of the integrands not involving δ-distributions are of a fairly simple form, namely $(1 + q^3\zeta_1)^n$ with $n \in \mathbb{N}$. Meanwhile, including ζ_2 we would already find $(1 + q^3\zeta_1 + q^5\zeta_2)^n$ instead. Therefore, we have decided to modify the regulator function r_2 as discussed in the next section.

4.2.2. Litim-Type Regulator II

To simplify the integrals contributing to higher order polynomial truncations of ζ_k, we choose

$$r_1 = 0, \ \ r_2 = \zeta_k(q^2)\left(\frac{1}{q}\frac{\zeta_k(1)}{\zeta_k(q^2)} - 1\right)\Theta(1 - q^2). \tag{4.103}$$

Setting $\zeta_k = 1$ recovers our former Litim-type regulator. The inclusion of $\zeta_k(q^2)$ relieves us, for arbitrary order of ζ_k-truncation, of any q-dependence in the relevant denominators: for any q with $\Theta(1 - q^2) = 1$ we get

$$v(q) = m^2 + q^2\left(\zeta_k(q^2) + \zeta_k(q^2)\left(\frac{1}{q}\frac{\zeta_k(1)}{\zeta_k(q^2)} - 1\right)\right)^2 = m^2 + \zeta_k^2(1) = \text{const}\,. \tag{4.104}$$

Hence, all well-defined integrals can be easily solved by hand.

The flow equation for ζ_k becomes

$$(\partial_t - p\partial_p - \eta)\zeta_k(p^2) = \frac{4g^2}{(2\pi)^3}\int \mathrm{d}^3q\, \frac{h(p-q)u(q)}{v(p-q)v^2(q)}\Theta(1-q^2)(\partial_t - q\partial_q - \eta)\left(\frac{1}{q}C - \zeta_k(q^2)\right) \tag{4.105}$$

where C is used as a shortcut for $\zeta_k(1)$. At $p=0$ this gives

$$\eta = -\frac{2}{\pi^2}g^2C\frac{m^2-C^2}{(m^2+C^2)^3}\left((\partial_t + 1 - \eta)C + \int_0^1 \mathrm{d}q\; q(q\partial_q + \eta - \partial_t)\zeta_k(q^2)\right) \tag{4.106}$$

which at the non-trivial fixed point amounts to

$$\frac{2}{\pi^2}g_*^2\frac{1}{C_*^3}\left(2C_* + \int_0^1 \mathrm{d}q\; q(3q\partial_q + 1)\zeta_k^*(q^2)\right) - 1 = 0. \tag{4.107}$$

The p^2-projection of the ζ_k-flow yields

$$\partial_t\zeta_1|_{m=0} = \frac{1}{6\pi^2}g^2\frac{(C+\partial_q\zeta_k(1))^2}{C^4} + (2+\eta)\zeta_1. \tag{4.108}$$

The same reasons as in the previous section allow us to proceed without ever considering $\partial_t\zeta_1|_{m\neq 0}$, again. At the $\eta_* = 1/3$ fixed point, we get

$$\frac{1}{2\pi^2}g_*^2\frac{(C_* + \partial_q\zeta_k^*(1))^2}{C_*^4} + 7\zeta_1^* = 0. \tag{4.109}$$

Hence, if we terminate the truncation at ζ_1, the fixed point values ζ_1^* become the roots of

$$F(\zeta_1) = 35\zeta_1(2+3\zeta_1)(1+\zeta_1) + (1+3\zeta_1)^2 \tag{4.110}$$

which correspond to finite values of g_*^2. $F(\zeta_1)$ is a third order polynomial. The analytical solutions of $F=0$ give, when rounded:

ζ_1^*	-0.0136	-0.6334	-1.1054
g_*^2	1.9339	1.9500	0.0035

For the first time, we find several fixed point values of g^2. However, it is a common feature of polynomial truncations to generate spurious solutions. There are at least two ways one can think of to distinguish them from the relevant ones. First, unphysical results should not converge with increasing order of truncation. Secondly, the actual fixed point is expected to evolve into one with $g_* = 0$, when the fixed point equations are continued via $\mathbb{R}$ to $D=4$ in the same manner as they have been compared to the two-dimensional results in [25] (see section 4.2). In four dimensions, the Wess-Zumino model cannot feature a fixed point with g_* non-vanishing [14].

The linearization of the flow equations results in

$$\begin{aligned} \lambda\,\mathrm{d}c_n &= \left(-2+\frac{2}{3}n\right)\mathrm{d}c_n \;\;\forall n \neq 3, \\ \lambda\,\mathrm{d}c_3 + x\lambda\,\mathrm{d}\zeta_1 &= a\,\mathrm{d}c_3 + b\,\mathrm{d}\zeta_1, \\ \tilde{x}\lambda\,\mathrm{d}\zeta_1 &= \tilde{a}\,\mathrm{d}c_3 + \tilde{b}\,\mathrm{d}\zeta_1 \end{aligned} \tag{4.111}$$

with some numbers a, b, x and tilded for each fixed point solution (ζ_1^*, g_*^2). Remarkably, in contrast to our former computation, there are $x \neq 0$ and $\tilde{x} \neq 1$. This is due to the fact that expression (4.106) for η contains $\partial_t C$ and thus $\partial_t \zeta_1$. If we wished, we could have eliminated this dependence before linearizing the equations, plugging in the ζ_1-flow from Eq. (4.108). However, as we have not done this, the non-diagonal part of the stability matrix this time reads

$$\begin{pmatrix} a - x\tilde{a}/\tilde{x} & b - x\tilde{b}/\tilde{x} \\ \tilde{a}/\tilde{x} & \tilde{b}/\tilde{x} \end{pmatrix}. \tag{4.112}$$

The diagonal sector, as before, leaves us with $\lambda_n = -2 + (2/3)n$ and $(\delta w_n, \mathrm{d}\,\zeta_1) \propto (\chi^n, 0)$ for $n \neq 3$, while the non-diagonal one evaluates to:

	ζ_1^*	g_*^2	λ_+	λ_-
I	-0.0136	1.9339	2.4109	0.8443
II	-0.6334	1.9500	2.8543	-3.9161
III	-1.1054	0.0035	143.5630	0.2603

As none of the fixed point solutions gives $b/x = \tilde{b}/\tilde{x}$, all corresponding eigendirections have non-vanishing χ^3 contributions. The spectrum of the stability matrix is, once more, non-degenerate. For case I and III, ω_i is determined by λ_-^i; case II implies $\omega_{II} = \lambda_+$.

Let us compare the revealed eigenvalues to our previous results. Case II stands out by exhibiting a changed number of relevant directions. Of the other two solutions I looks by far more familiar. Not only is ω_I particularly close to the LPA$'$ result obtained with the Litim-type regulator; λ_+^I resembles the corresponding eigenvalue derived in the previous section. Thus, we have good reason to anticipate that case I belongs to the convergent branch of fixed point solutions as evolving with increasing order of ζ_k-truncation. Hence, we identify

$$\omega = \lambda_-^I = 0.8443 \;\Rightarrow\;\; e_\omega \approx 7\,\% \tag{4.113}$$

gaining a slight shift towards [11].

When moving on to a truncation including ζ_2, we have to consider also the p^4 projection of the flow equation for ζ_k (4.79). The left hand side is easily evaluated to $4!(\partial_t - 4 - \eta)\zeta_2$. On the right hand side we are left with a sum of integrands having the form $f(q)\Theta(1-q^2)$ times $\delta(1-q)$, $\delta'(1-q)$, $\delta''(1-q)$, or $\delta^2(1-q)$ where $f(q)$ contains some more Heaviside functions but no additional δ-distributions or their derivatives. Integrating by parts, we can convert the $\Theta\delta''$-term to $\Theta\delta'$ and $\delta\delta'$, and $\delta\delta'$ to δ^2. Furthermore, $\Theta\delta'$ splits into $\Theta\delta$, δ^2 and $\Theta\delta^2$. The last product arises because the derivative of the f-factor in front of $\Theta\delta'$ contributes a term proportional to an additional δ. Thus, we end up with integrands of the form $f\Theta\delta$, $f\Theta\delta^2$, and $f\delta^2$.

While it is not the first time we have to compute a $\Theta\delta$-integral, a straightforward treatment of the terms involving squared δ-distributions fails: One could hope that after the evaluation of all such integrals writing

$$\int \mathrm{d}q\; f(q)\delta^2(1-q) = f(1)\delta(0) \;\text{ with }\; \Theta(0)=\frac{1}{2} \tag{4.114}$$

the divergent results cancel each other. However, this is not even true for $\zeta_k = 1$ and thus cannot hold for a generic $\zeta_k = 1 + q^2\zeta_1 + q^4\zeta_2 + \ldots$ Maybe, carefully treating the distributions as limits of some function sequences could resolve this problem. We have decided to resign in view of the difficulties and to try out yet another regulator.

4.2.3. Callan-Symanzik Regulator

As at the very beginning of our computations, we turn to the regulator

$$r_1 = 1, \quad r_2 = 0. \tag{4.115}$$

Thus, we get

$$\begin{aligned}(\partial_t - p\partial_p - \eta)\zeta_k(p^2) = -\frac{8g^2}{(2\pi)^3}\int \mathrm{d}^3q\; \frac{\zeta_k((p-q)^2)\zeta_k(q^2)M(q)}{v^2(q)v(p-q)}(1-\eta)\\ \text{with }\; M=m+1,\quad v=M^2+q^2\zeta_k^2.\end{aligned} \tag{4.116}$$

At $p=0$ this yields

$$\eta = \frac{4}{\pi^2}g^2M(1-\eta)\int \mathrm{d}q\; q^2\frac{\zeta_k^2(q^2)}{v^3(q)}, \tag{4.117}$$

and applying $\partial_p^2\big|_0$ to the full flow equation gives

$$\begin{aligned}\partial_t\zeta_1 =& (2+\eta)\zeta_1 - \frac{2}{3\pi^2}g^2M(1-\eta)\int \mathrm{d}q\; q^2\frac{\zeta_k}{v^3}\\ &\left(\frac{2}{q}\zeta_k' + \zeta_k'' - \frac{2\zeta_k}{v}\left(3\zeta_k^2 + 8q\zeta_k\zeta_k' + 3q^2\zeta_k'^2 + q^2\zeta_k\zeta_k''\right) + \frac{8q^2\zeta_k^3}{v^2}\left(\zeta_k^2 + 2q\zeta_k\zeta_k' + q^2\zeta_k'^2\right)\right)\end{aligned} \tag{4.118}$$

where the primes denote partial derivatives with respect to q.

For a start, we again consider the ζ_k-truncation terminating at ζ_1. Let us, first, rule out $\zeta_1^* = 0$ by plugging it into the fixed point equations arising from (4.117) and (4.118). Next, we solve both fixed point equations for $1/g_*^2$, multiply them by ζ_{1*}^6, and call the gained right hand sides $F_1(\zeta_1^*)$ and $F_2(\zeta_1^*)$. Then the non-vanishing roots of $F_1(\zeta_1) - F_2(\zeta_1)$ or $1/F_1 - 1/F_2$ corresponding to finite values of g_*^2 give the fixed point values of ζ_1.

Remark: At $\zeta_1 = 0$, $F_1 - F_2$ has an inevitable, spurious root. The reason is that, when formulating the F_i, we have multiplied both fixed point equations with powers of ζ_1 high enough to cause $F_i \propto \zeta_1$, which leads to $F_i(0) = 0$.

To evaluate $F_i(\zeta_1)$, one has to solve several integrals of the form

$$\int \mathrm{d}q \, \frac{q^{2n}}{\tilde{v}^m(q)} \quad \text{with} \quad \tilde{v} = 1 + q^2 \left(1 + q^2 \zeta_1\right)^2 \quad \text{and} \quad n, m \in \mathbb{N}. \tag{4.119}$$

This can be done analytically. When using Mathematica, it is advantageous to factorize $\tilde{v}(q)$, which is a third order polynomial in q^2, and leave the roots $a_i(\zeta_1)$ unspecified, first. However, different signs of $\Im(\sqrt{a_i})$, where the square root is chosen to have a phase $\varphi \in (-\pi/2, \pi/2]$, require separate integration. Considering plots of $a_i(\zeta_1)$ around $\zeta_1 = 0$ we have concluded that only one of the a_i can acquire both signs, while the other two always give $\Im(\sqrt{a_i}) > 0$. This allowed us to save some case analysis.

Remark: Our first naive plots of $\Im(\sqrt{a_i})$ exhibited regions of rapid oscillation. We have found that these were correlated with a_i being essentially real and negative. Obviously, numerical errors added a small random imaginary part to the actually real values of a_i making the relevant solution of $\sqrt{a_i}$ jump between $\varphi \approx -\pi/2$ and $\varphi \approx \pi/2$.

Plotting $F_1(\zeta_1) - F_2(\zeta_1)$ we find that the imaginary part stays constantly near zero while the real part displays two sign changes in-between of branches rising to large positive values. Regarding one of the nulls, FindRoot returns $\zeta_{1R} = -0.0816$. At the other one, the slope with which $\Re(F_1 - F_2)$ traverses the ζ_1-axis is too small to allow for an exact localization. However, the range of possible intersection spreads around zero, and we know that there must be a zero-crossing at $\zeta_1 = 0$. The results can be cross-checked by taking a look at $1/F_1 - 1/F_2$. Particularly, we find that determining ζ_{1R} from $\Re(1/F_1 - 1/F_2)$ increases the accuracy because the slope of traversal comes out to be larger than for $\Re(F_1 - F_2)$. Finally, we have $(\zeta_1^*, g_*^2) = (-0.0816, 2.9425)$.

The linearized flow equations amount to

$$\begin{aligned} \lambda \, \mathrm{d}c_n &= \left(-2 + \frac{2}{3} n\right) \mathrm{d}c_n \quad \forall n \neq 3, \\ \lambda \, \mathrm{d}c_3 &= a \, \mathrm{d}c_2 + b \, \mathrm{d}c_3 + e \, \mathrm{d}\zeta_1, \\ \lambda \, \mathrm{d}\zeta_1 &= \tilde{a} \, \mathrm{d}c_2 + \tilde{b} \, \mathrm{d}c_3 + \tilde{e} \, \mathrm{d}\zeta_1. \end{aligned} \tag{4.120}$$

To get a, b, e, and tilded, further integrals of the above form, see Eq. (4.119), have to be computed. This time, in addition to $\mathrm{d}c_3$ and $\mathrm{d}\zeta_1$ also $\mathrm{d}c_2$ is excluded from the familiar diagonal part of the stability matrix. Solving the eigenvalue problem for

$$\begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ a & b & e \\ \tilde{a} & \tilde{b} & \tilde{e} \end{pmatrix} \tag{4.121}$$

completes the critical exponents and eigendirections with

	λ_2	λ_+	λ_-
λ	$-2/3$	0.6687	-75.7460
$(\delta w, \mathrm{d}\zeta_1) \propto$	$(\chi^2 \pm 0.4766\chi^3, -0.0118)$	$(\chi^3, \pm 0.0070)$	$(\chi^3, \mp 268.0604)$

As before, the upper signs correspond to a positive and the lower ones to a negative g_*. In contrast to the LPA′ computation employing the same regulator, the spectrum of the stability matrix becomes non-degenerate. Interestingly, we encounter, as in case II of the previous section, a change in the number of relevant directions. With

$$\omega = \lambda_+ = 0.6687, \quad e_\omega \approx 27\ \% \tag{4.122}$$

our present ω is a little bit closer to [11] than the corresponding LPA′ result. However, the deviation is three to four times larger than with the Litim-type regulators.

Next, we want to include ζ_2 in our truncation. After deriving the p^4-projection of the ζ_k-flow we end up with three fixed point equations, altogether. For $\zeta_2^* = 0$, two of them are reduced to the ones accompanying the preceding truncation. Consequently, they fix ζ_1^* and g_*^2 to the above values. Plugging them, along with $\zeta_2^* = 0$, into the new fixed point equation leads to a contradiction, revealing that ζ_2^* has to be non-vanishing. Similarly to above, we solve all fixed point equations for g_*^2, multiply them by $\zeta_1^* \zeta_{2*}^{17}$, and call the right hand sides $F_i(\zeta_1^*, \zeta_2^*)$. This definition implies $F_i(\zeta_1, 0) = 0$. Thus, the fixed point values (ζ_1^*, ζ_2^*) become the points of intersection of all three F_i or, equivalently, simultaneous roots of two $F_i - F_j$ with $\zeta_2 \neq 0$ and the corresponding g_*^2 finite.

The integrals contributing to the F_i are similar to the ones involved in the previous truncation:

$$\int \mathrm{d}q\, \frac{q^{2n}}{\tilde{v}^m(q)} \quad \text{with} \quad \tilde{v} = 1 + q^2 \left(1 + q^2\zeta_1 + q^4\zeta_2\right)^2 \quad \text{and} \quad n, m \in \mathbb{N} \tag{4.123}$$

As $\tilde{v}$ is a polynomial of fifth degree now, its roots must be calculated numerically. Apart from that, we can proceed as discussed above. Immediately taking into account the integrals arising during the derivation of the critical exponents, we find that we need considerably less integrals of the form (4.123) if we convert the integrands to the common denominator $\tilde{v}^8$. Then we still have to regard all $1 \leq n \leq 30$ at $m = 8$. Employing Mathematica, we succeeded in solving these integrals analytically. However, the gained F_i-expressions turned out to be so huge that we failed in identifying (ζ_1^*, ζ_2^*), above all due to a lack of memory space. Maybe, one could reduce this problem by preferring a greater amount of simpler integrals to fewer complicated ones.

However, we have decided to spare memory consumption by backing away from the analytical treatment of the integrals. First, we recomputed ζ_1^*, g_*^2 and $\omega = \lambda_+$ for the truncation terminating at ζ_1, employing numerical integration routines. The results deviated by less than 10^{-7} from the original values. After this promising performance, we proceeded to ζ_2. To identify (ζ_1^*, ζ_2^*) we searched for the common roots of $\Re(F_1 - F_3)$ with $\Re(F_2 - F_3)$ and $\Re(1/F_1 - 1/F_3)$ with $\Re(1/F_2 - 1/F_3)$. The imaginary parts have been neglected: Obviously, any root of a complex number is a root of its real part, too. Moreover, all $\Im(F_i)/\Re(F_i)$ appeared to stay invariably near zero, thus awaking no hope to simplify the root search by considering the imaginary parts. FindRoot has been initialized with $\zeta_2 = \pm 0.001$ and the last fixed point result for ζ_1.

Remark: It was not possible to start from $\zeta_2 = 0$ because the way of implementing the F_i turned $F_i(\zeta_1, 0)$ into removable singularities. The reason behind this is that $\zeta_2 = 0$ reduces the polynomial order of $\tilde{v}$.

Unfortunately, none of this computations yielded a proper null. Also, examining plots at the initial ζ_1 did not help us to guess a better starting point for the FindRoot routine.

4.3. Beyond LPA′ Including a Kähler Potential

Instead of allowing for additional derivatives in the kinetic term of the Γ_k-ansatz, we now incorporate the Kähler potential introduced in section 2.5.1:

$$\Gamma_k = -\frac{1}{4} \int \mathrm{d}^3x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \; K(\Phi, \Phi^\dagger) - \left\{ \frac{1}{2i} \int \mathrm{d}^3x \, \mathrm{d}^2\theta \; W_k(\Phi) + \mathrm{h.\,c.} \right\} \tag{4.124}$$

With $Z_k(\phi, \phi^\dagger) := \partial_\phi \partial_{\phi^\dagger} K(\phi, \phi^\dagger)$ this amounts to

$$\begin{aligned} \Gamma_k = \int \mathrm{d}^3x \; \Bigg(& Z_k \left(\partial_j \phi \partial^j \phi^\dagger + \frac{i}{2} \bar{\psi} \sigma^j \partial_j \psi - \frac{i}{2} (\partial_j \bar{\psi}) \sigma^j \psi - f f^\dagger \right) \\ & + \frac{1}{2} \left\{ \partial_\phi Z_k \left(\psi^T \sigma^2 \psi f^\dagger + i \bar{\psi} \sigma^j \psi \partial_j \phi \right) + \mathrm{h.\,c.} \right\} - \frac{1}{2} \partial_\phi \partial_{\phi^\dagger} Z_k \psi^T \sigma^2 \psi \bar{\psi} \sigma^2 \psi^* \\ & - \left\{ W_k'(\phi) f - \frac{1}{2} W_k''(\phi) \psi^T \sigma^2 \psi + \mathrm{h.\,c.} \right\} \Bigg) \end{aligned} \tag{4.125}$$

The formulation in Fourier space is deferred to appendix C. As we stick to the definition (4.69) of Ψ, the regulator matrices remain as obtained in section 4.2, see Eq. (4.70) and (4.71).

To extract the flows of W_k and Z_k from the Wetterich equation, we start, as in section 4.1, from a projection on constant fields, setting $\psi=\bar{\psi}=0$. For our ansatz, this results in

$$\Gamma_k = -(2\pi)^3\delta(0)\left(Z_k f f^\dagger + \{W_k' f + \mathrm{h.\,c.}\}\right). \tag{4.126}$$

The corresponding $\Gamma_k^{(2)}$ is also provided in appendix C. Inverting $\Gamma_k^{(2)}(p,q)+R_k\delta(p-q)$ causes no difficulties, because $\Gamma_k^{(2)}(p,q)$ is a purely bosonic, block diagonal matrix proportional to $\delta(p-q)$. Taking the f-derivative of the projected Wetterich equation at $f=f^\dagger=\phi^\dagger=0$ yields

$$\partial_t W_k(\phi)=0; \tag{4.127}$$

we, again, receive a confirmation of the non-renormalization theorem. From the f-$f^\dagger$-derivative at vanishing f and $f^\dagger$ we get

$$\begin{aligned}\partial_t Z_k(\phi,\phi^\dagger) = &-\frac{1}{(2\pi)^3}\int \mathrm{d}^3q\ \frac{1}{v^3}\\ &\Big(\partial_t r_1\left(h^2(M+M^\dagger)|W_k'''|^2 - h\left\{(2M^{\dagger 2}+u)W_k'''\partial_{\phi^\dagger}Z_k + \mathrm{h.\,c.}\right\}\right.\\ &\left.+(M+M^\dagger)((u-2q^2h^2)|\partial_\phi Z_k|^2 + hv\partial_\phi\partial_{\phi^\dagger}Z_k)\right)\\ &+\partial_t r_2\left(-hu|W_k'''|^2 + (u-2q^2h^2)\left\{M^\dagger W_k'''\partial_{\phi^\dagger}Z_k + \mathrm{h.\,c.}\right\}\right.\\ &\left.\left.+2q^2h(2|M|^2+u)|\partial_\phi Z_k|^2 - uv\partial_\phi\partial_{\phi^\dagger}Z_k\right)\right)\end{aligned} \tag{4.128}$$

with

$$h=Z_k+r_2,\ \ M=W_k''+r_1,\ \ u=|M|^2-q^2h^2,\ \text{and}\ v=|M|^2+q^2h^2. \tag{4.129}$$

Setting $Z_k(\phi,\phi^\dagger)=Z_0$ recovers the LPA$'$ result.

To ensure that the obtained flow of Z_k is correct, we have decided to recompute it from a second projection. Assuming only ϕ ($\phi^\dagger$) and f ($f^\dagger$) to be constant and applying $f=f^\dagger=0$, we find

$$\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(p)}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\psi(q)} = Z_k\sigma^j q_j\delta(p-q). \tag{4.130}$$

With this approach, $\Gamma_k^{(2)}(p,q)$ becomes a full supermatrix with a nontrivial momentum-dependence, again. To master the inversion we proceed almost as in section 4.2: We rewrite the Wetterich equation as

$$\partial_t\Gamma_k = \frac{1}{2}\,\mathrm{STr}\left(\tilde{\partial}_t\ln(\Gamma_k^{(2)}+R_k)\right) \tag{4.131}$$

and expand the right hand side about the $\psi/\bar{\psi}$-independent contribution $\Gamma_0(p,q)$ to

$$\Gamma_k^{(2)}(p,q)+R_k\delta(p-q) \equiv \Gamma_0(p,q)+\Delta\Gamma(p,q). \tag{4.132}$$

As above, the gained simplification is due to the fact that $\Gamma_0(p,q)$ turns out to be proportional to $\delta(p-q)$. As we are going to compute

$$\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(p)}\frac{1}{2}\,\mathrm{STr}\left(\tilde{\partial}_t\ln(\Gamma_k^{(2)}+R_k)\right)\frac{\overleftarrow{\delta}}{\delta\psi(q)}\Bigg|_{\psi=\bar{\psi}=0}, \tag{4.133}$$

we have to consider only expansion terms involving one or two factors of $\Delta\Gamma$. Moreover, we can immediately omit all contributions to $\Delta\Gamma$ which are proportional to $\psi_i\psi_j$, $\bar{\psi}_i\bar{\psi}_j$, or a product of more than two fermionic components. The accordingly simplified $\Gamma_k^{(2)}$ is also included in appendix C.

Remark: We have decided to apply the ψ-$\bar{\psi}$-derivative inside the trace. Then it suffices to compute $\overrightarrow{\delta}_{\bar{\psi}}\cdots\overleftarrow{\delta}_{\psi}|_0$ of the diagonal blocks and δ_{ψ_i}, $\delta_{\bar{\psi}_i}$ of the off-diagonal blocks in $\Delta\Gamma$. As we use the component notation of $\delta_\psi/\delta_{\bar{\psi}}$ for the first time, here, let us mention that

$$\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(p)}\cdots\frac{\overleftarrow{\delta}}{\delta\psi(q)}=\begin{pmatrix}\frac{\overrightarrow{\delta}}{\delta\bar{\psi}_1(p)}\cdots\frac{\overleftarrow{\delta}}{\delta\psi_1(q)} & \frac{\overrightarrow{\delta}}{\delta\bar{\psi}_1(p)}\cdots\frac{\overleftarrow{\delta}}{\delta\psi_2(q)}\\ \frac{\overrightarrow{\delta}}{\delta\bar{\psi}_2(p)}\cdots\frac{\overleftarrow{\delta}}{\delta\psi_1(q)} & \frac{\overrightarrow{\delta}}{\delta\bar{\psi}_2(p)}\cdots\frac{\overleftarrow{\delta}}{\delta\psi_2(q)}\end{pmatrix}. \tag{4.134}$$

Thus, the trace in Eq. (4.133) becomes a sum of two by two matrices constituting the main diagonal.

Finally, we have to set the momentum to zero. We find that the result coincides with Eq. (4.128).

We immediately decide on $r_1=0$ because this admits the ansatz $Z_k^*(\phi,\phi^\dagger)=Z_k^*(\rho)$ where $\rho\equiv\phi\phi^\dagger$. In analogy to section 4.2, we introduce the notation

$$Z_k(\phi,\phi^\dagger)=Z_0\zeta_k(\phi,\phi^\dagger)\ \text{ with }\ Z_0=Z_k(0,0). \tag{4.135}$$

Thus, Z_0 denotes, as before, the wave function renormalization. When proceeding to renormalized, dimensionless quantities, this time, we forbear to extract Z_0 from r_2. Instead, we recall that the physics cannot depend on Z_0; thus, we arbitrarily set it to one where it is not encased in η. We have found that this considerably simplifies our computations. That way we arrive at

$$\begin{aligned}&\left(\partial_t-\frac{1+\eta}{2}\chi\partial_\chi-\frac{1+\eta}{2}\chi^\dagger\partial_{\chi^\dagger}-\eta\right)\zeta_k\\&=\frac{1}{2\pi^2}\int\mathrm{d}q\,q^2\frac{1}{v^3}(\partial_t-q\partial_q)r_2\left(hu|w_k'''|^2-(u-2q^2h^2)\left\{w_k''^\dagger w_k'''\partial_{\chi^\dagger}\zeta_k+\text{h. c.}\right\}\right.\\&\quad\left.-2q^2h(2|w_k''|^2+u)|\partial_\chi\zeta_k|^2+uv\partial_\chi\partial_{\chi^\dagger}\zeta_k\right)\end{aligned} \tag{4.136}$$

with

$$h=\zeta_k+r_2,\quad u=|w_k''|^2-q^2h^2,\quad\text{and}\quad v=|w_k''|^2+q^2h^2. \tag{4.137}$$

Projecting Eq. (4.136) on vanishing χ, $\chi^\dagger$ we get an expression for η which, in contrast to our previous results, is not proportional to g^2. Thus, it is not obvious which monomial superpotential provides a solution of the fixed point equations. However, guided by our previous findings, we premise $W_* = c_3^* \chi^3/3$, $c_3^* \neq 0$ implying $\eta = 1/3$ for our further analysis.

Let us specify r_2 to be the Litim-type regulator

$$r_2 = a\left(\frac{1}{q} - 1\right)\Theta(1-q^2). \tag{4.138}$$

The factor $a > 0$ has been introduced to allow for an optimization of the ω-result by means of a minimum sensitivity condition as discussed in section 3.4. With this r_2, $\eta_* = 1/3$, and the ansatz $\zeta_*(\chi, \chi^\dagger) = \zeta_*(\rho)$, $\rho = \chi\chi^\dagger$ the fixed point equation becomes

$$\begin{aligned} 0 = \frac{4}{3}\rho\zeta_*' + \frac{1}{3}\zeta_* + \frac{a}{2\pi^2}\int_0^1 \mathrm{d}q\,\frac{q}{v_*^3}\Big(&4g_*^2 h_* u_* - 8g_*^2\rho(u_* - 2q^2h_*^2)\zeta_*' \\ &-2q^2h_*(8g_*^2\rho + u_*)\rho\zeta_*'^2 + u_*v_*(\zeta_*' + \rho\zeta_*'')\Big) \end{aligned} \tag{4.139}$$
$$\text{where}\;\; h_* = \zeta_* + a\left(\frac{1}{q} - 1\right),\;\; u_* = 4g_*^2 - q^2h_*^2,\;\; \text{and}\;\; v_* = 4g_*^2 + q^2h_*^2.$$

For $\rho \neq 0$ and $\zeta_*(\rho) \neq a$, the integral can be evaluated by computing

$$\int_0^1 \mathrm{d}q\, q^n\Big((q - A - iB)(q - A + iB)\Big)^{-3} \quad \text{for}\;\; A, B \in \mathbb{R},\;\; B \neq 0,\;\; \text{and}\;\; n \in \{0, 1, \cdots, 5\} \tag{4.140}$$

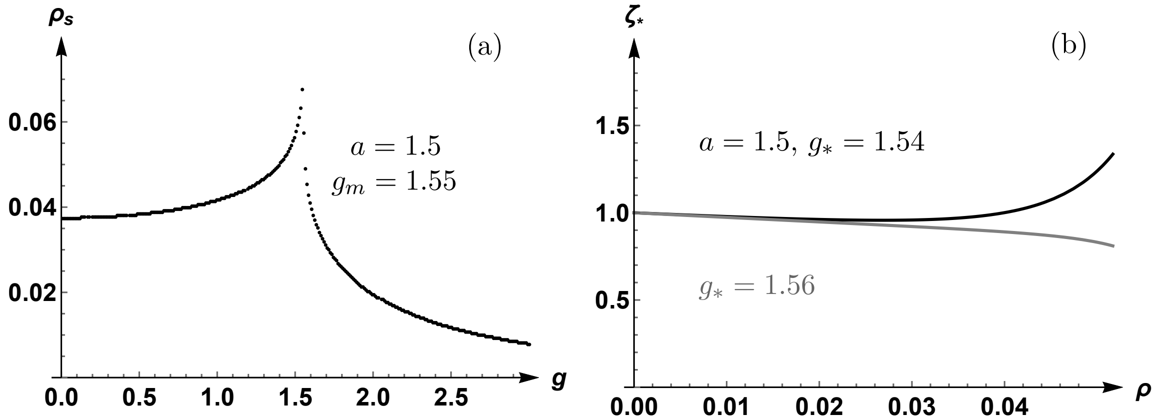
and replacing $A \pm iB$ by the roots of v_* afterwards.

Eq. (4.139) is a second order nonlinear differential equation for $\zeta_*(\rho)$. One initial condition is fixed by the definition (4.135) of ζ_k. According to it, $\zeta_*(0) = 1$. At $\rho = 0$, the fixed point equation looses its ζ_*''-content. This allows to compute $\zeta_*'(0)$.

Remark: Pay attention to the fact that for $\rho = 0$ we have to evaluate the integral in Eq. (4.139) separately.

Now, we can employ numerical methods to solve the fixed point equation for a fixed value of g_* and a.

Remark: Because the order of the differential equation decreases at $\rho = 0$, we cannot construct a solution starting from $\rho = 0$ directly. However, we are looking for a result continuously connected to $\rho = 0$. Therefore, we approximate the initial conditions by demanding $\zeta_*(0) = \zeta_*(\epsilon)$, $\zeta_*'(0) = \zeta_*'(\epsilon)$ for some $\epsilon \ll 1$.



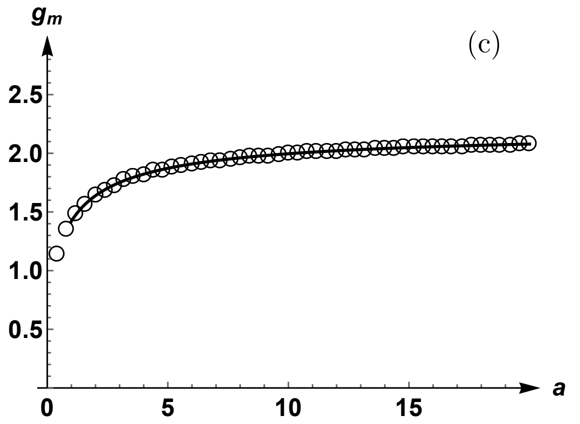


Figure 4.1.:
Shooting from the Origin.
(a) Spike plot allowing to identify g_m at a given a. (b) $\zeta_*(\rho)$ as obtained at a $|g_*| < g_m$ (black) and a $|g_*| > g_m$ (gray). The ρ-axis ends at 90 % of the smaller ρ_s belonging to the gray curve. (c) The dependence of g_m on a. The solid line is due to interpolation; the parameters are given in the text.

How can we distinguish the true fixed point value of g? We make use of a method known as shooting from the origin, which is presented e. g. in [51]. Beyond truncation, a physically acceptable $\zeta_*(\rho)$ has to be regular for all $\rho \geq 0$. However, all numerical solutions of Eq. (4.139) we have computed hit singularities. The graphs of their respective position ρ_s as a function of g_*, $\rho_s(g_*) = \rho_s(|g_*|)$, exhibit sharp maximums at some $|g_*| = g_m(a)$, see Figure 4.1 (a).

Remark: Our Mathematica representation of the fixed point equation is based on the integrals presented in Eq. (4.140). This introduces an artificial singularity at $\zeta_*(\rho) = a$. Therefore, when creating such spike plots as in Figure 4.1 (a), we have considered only solutions ζ_* terminating at $\zeta_*(\rho_s) \neq a$.

Figure 4.1 (b) illustrates that the $\zeta_*(\rho)$ are found to be qualitatively different for $|g_*| \lessgtr g_m$. For $|g_*| \approx g_m$ they converge at small ρ but still differ widely close to the singularity. At fixed a, g_m is believed to provide the best fixed point approximation obtainable from the truncation. Figure 4.1 (c) conveys the dependence of g_m on a. The interpolating function has been estimated to

$$g_m(a) \overset{\text{IP}}{=} \frac{2.1869a^2 + 11.6067a + 3.4571}{a^2 + 6.4777a + 4.6596}. \tag{4.141}$$

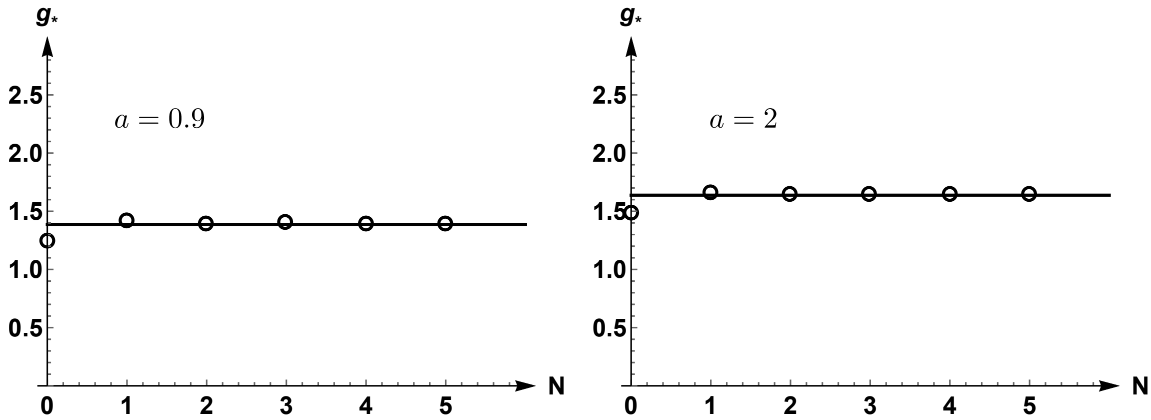


Figure 4.2.: Polynomial Truncation vs. Shooting. The solid line is located at $g_* = g_m(a)$ as obtained from the interpolation. With increasing order N of the polynomial truncation, $|g_*(a, N)|$ converges to g_m.

Instead of resorting to the shooting method, one, of course, can also approximate the fixed point solution from a polynomial truncation of ζ_*, as it has been done in section 4.2. To this end, we set

$$\zeta_*(\rho) = 1 + \sum_{n=1}^{N} \zeta_n^* \rho^n \tag{4.142}$$

where N is the order of truncation. Then the fixed point equation is approximately fulfilled if all its projections on ρ^n, $n \leq N$ hold.

Remark: Substituting all trigonometric functions in our original Mathematica representation of the fixed point equation by complex logarithms employing

$$\arctan(x) = \frac{i}{2} \Big(\ln(1 - ix) - \ln(1 + ix) \Big) \tag{4.143}$$

removes the artificial singularity at $\rho = 0$. Thus, projecting on different powers of ρ becomes an easy task.

Solving the system of $N + 1$ equations provides numerous roots. However, we are interested only in real couplings. For all cases considered, this has left us with an — up to the sign of g_* — unique solution. The polynomial computation confirms our above findings. Figure 4.2 shows that, for various a, $|g_*(N)|$ converges to g_m. In appendix C we provide the fixed point couplings obtained with $0 \leq N \leq 5$ for several a.

To approach the critical exponents, we stick to a polynomial truncation of ζ_k but refrain from combining χ, $\chi^\dagger$ to ρ:

$$\zeta_k(\chi, \chi^\dagger) = 1 + \sum_{n=1}^{N/2} \zeta_n (\chi\chi^\dagger)^n + \sum_{n=0}^{N} \sum_{m=1+n}^{N-n} (\zeta_{nm} \chi^n \chi^{\dagger m} + \zeta_{mn} \chi^m \chi^{\dagger n}), \quad \zeta_{mn} = \zeta_{nm}^\dagger. \tag{4.144}$$

Thus, to order N we include all terms proportional to $\chi^n \chi^{\dagger m}$ with $n + m \leq N$. In principle, starting from here we can straightforwardly derive the stability matrix and its spectrum for any N. However, practically, the computation turns out to be extremely time-consuming; we have not pushed it beyond $N = 5$. The obtained results suggest the following

Conjecture: The couplings g and ζ_i generate, independently of N, a diagonal block of the stability matrix.

To proof this, we analyze which non-vanishing elements can appear in the rows and columns corresponding to g and ζ_i.

Proof: Let us start with the columns. Because of the non-renormalization theorem, the beta-functions originating from the superpotential can contribute only to their g-row. Furthermore, all ζ_{nm}-elements vanish. To see this, let us compute such an entry in the following way. First, we take the flow equation for ζ_k and set all ζ_{nm} and c_n, $n \neq 3$ to their fixed point values, namely to zero. We are allowed to do so because we are not going to take any derivatives with respect to these couplings. Then we are left with an equation depending only on ρ, but not on χ, $\chi^\dagger$ separately. However, the beta-functions corresponding to ζ_{nm} are projections of this equation on $\chi^n \chi^{\dagger m}$ with $n \neq m$. Hence, with the fixed point values already plugged in, they vanish. Consequently, taking derivatives with respect to g or ζ_i cannot give anything else than zero.

Now, let us turn to the ζ_i-rows and have a look at their ζ_{nm}-, ζ_{mn}-entries for some particular (n, m). Without loss of generality, we assume $n - m =: \Delta > 0$. If we set all couplings but ζ_{nm} and ζ_{mn} in the flow of ζ_k to their fixed point values, we get an expression depending only on

$$\rho, \ \ \chi^\Delta \zeta_{nm}, \ \ \chi^{\dagger\Delta} \zeta_{mn}, \ \ \text{and} \ \ \zeta_{nm}\zeta_{mn}. \tag{4.145}$$

Taylor expanding the flow equation in these terms we see that the beta-function of a ζ_i, which is a ρ^i-projection of $\partial_t \zeta_k$, can contain ζ_{nm}, ζ_{mn} only enclosed to the product $\zeta_{nm}\zeta_{mn}$. As both ζ_{nm} and ζ_{mn} vanish at the fixed point, the considered elements of the stability matrix become zero.

The same argument applies to the respective entries in the g-row. The flow of g depends only via the anomalous dimension on the different couplings. However, η is determined, just as the beta-functions of ζ_i, by a ρ^i-, namely a ρ^0-projection of $\partial_t \zeta_k$. Thus, all ζ_{nm}- and ζ_{mn}-elements vanish, again.

Finally, we have to consider the c_n-contributions, $n \neq 3$ to the rows. Setting everything apart from a c_n, $n \neq 3$ to the respective fixed point value, we find that $\partial_t \zeta_k$ is a function of

$$\rho, \ \ \chi^{n-3} c_n, \ \ \text{and} \ \ \chi^{\dagger(n-3)} c_n. \tag{4.146}$$

Thus, a ρ-projection can contain only c_n^2. However, as $c_n^* = 0$, this implies that these elements of the stability matrix vanish, too.

Hence, we have found that any g- or ζ_i-row as well as any of the respective columns contains only g- and ζ_i-entries.

□

This allows us to compute ω from the stability submatrix corresponding to the couplings g and ζ_i. Particularly, we return to a polynomial expansion in ρ. Therefore, we say that a truncation is of order N if it contains all ζ_i with $i \leq N$. This is a factor of 2 different from the notation just used. Identifying the smallest positive eigenvalue of the submatrix with ω, we get the following values:

N	$a = 1.5$	$a = 1.7$	$a = 1.9499$	$a = 2.3$
0	1.0000	1.0000	1.0000	1.0000
1	0.8422	0.8446	0.8454	0.8444
2	0.8317	0.8317	0.8307	0.8284
3	0.8330	0.8338	0.8336	0.8321
4	0.8338	0.8345	0.8342	0.8326

We have not checked if all corresponding eigendirections indeed come with a non-vanishing dg. However, the closeness of these results to our previous ones makes it highly improbable that we have picked some wrong eigenvalue. Despite the considerable simplification we have achieved thanks to the above conjecture, the computation still required too much memory space to go beyond $N = 4$. Therefore, we have failed to reach a satisfying convergence.

Eventually, we want to make use of the freedom to choose a. We expect that $\omega(a)$ exhibits some extreme. With the idea of a minimum sensitivity optimization in mind, we assume the corresponding ω to be particularly close to the exact result. At $N = 2$, the computational effort is not yet significant — we easily discern a maximum at $a_m = 1.9499$. However, the tabulated ω show that a_m drifts with N. Thus, our best result at $N = 4$ becomes

$$\omega = 0.8345 \tag{4.147}$$

computed with $a = 1.7$. This lies between the values obtained, employing the Litim-type regulator (II), from the truncations in section 4.1 and 4.2. The deviation from [11] amounts to $e_\omega \approx 8\ \%$.

5. Conclusions

The present thesis addresses the three-dimensional $\mathcal{N} = 2$ Wess-Zumino model. We have employed a functional renormalization group approach — the Wetterich equation — to analyze its critical behavior. Particular attention has been payed to the fixed point with a superpotential $W_*(\phi) \propto \phi^3$ and an anomalous dimension of $\eta_* = 1/3$.

To render the Wetterich equation algorithmically solvable we have relied on three different truncations. All of them recover the non-renormalization theorem $\partial_t W_k = 0$. This implies that we always find

$$-\lambda_n = 2 - \frac{2}{3}n, \quad n \in \mathbb{N} \setminus \{3\} \tag{5.1}$$

amongst the critical exponents. For two of the truncations also all other eigenvalues λ of the respective stability matrix have been computed. Concerning the third one, we have restricted ourselves to the critical exponent we were particularly interested in: to ω, the smallest positive λ generating leading order corrections $\propto \phi^3$ to the superpotential.

Measured against $\omega_b = 0.9098(20)$ as conjectured by means of conformal bootstrapping in [11] our best results are

- from the LPA$'$ truncation: $\omega = 0.8\bar{3}$,
- beyond LPA$'$ — with additional momentum-dependence: $\omega = 0.8443$,
- and beyond LPA$'$ — with a Kähler potential included: $\omega = 0.8345$.

All of these values have been obtained with Litim-type regulators. Their relative deviation e_ω from ω_b is about $7 - 8$ %. The authors of [11] have compared with an ϵ-expansion about four dimensions yielding $\omega = 1 + \mathcal{O}(\epsilon^2)$ [5, 7]. Our results are somewhat closer to ω_b. Interestingly, they approach it from the other side.

When going beyond LPA$'$, after formulating the respective ansatz for the effective action we have additionally employed a polynomial truncation of ζ_k to extract the critical exponents. Recall that ζ_k is the momentum- or field-dependent factor accompanying the wave function renormalization Z_0. Unfortunately, we have not been able to proceed to appreciably high orders. Considering a Kähler potential, we observe at least the commencement of convergence. With the momentum-dependent $\zeta_k(p^2)$ no statement can be made in this respect. Thus, it is a pending task to break down the barriers on the way to higher polynomial orders. Apart from enhancing our results this could provide an answer to the interesting question if either the momentum-dependence or the Kähler potential has superior relevance for the accuracy of ω.

There is another promising strategy to improve our results which we have not explored so far. In the context of the Kähler potential we have discussed that ω can be derived even if we restrict $\zeta_k(\chi, \chi^\dagger)$ to $\zeta_k(\chi\chi^\dagger)$. This considerably simplifies the (linearized) flow equation for ζ_k. Thus, one could try to derive the critical exponents by numerical methods abandoning the polynomial truncation altogether.

To analyze the non-trivial fixed point within the truncation comprising a Kähler potential we have employed the shooting-method starting from the origin. This provided us with approximate solutions for $\zeta_*(\chi\chi^\dagger)$ defined between $\chi\chi^\dagger = 0$ and a singularity at $\chi\chi^\dagger \ll 1$. Instead, we could also shoot from infinity. This would allow us to obtain a global fixed point solution ζ_* and thus get acquainted with its overall shape.

Throughout this thesis we have focused on three dimensions. However, there are several interesting tasks arising when we broaden our perspective to include a variable D. First of all, we have observed that our flow equation for $\zeta_k(p^2)$ almost agrees with the one for the dimensionally reduced model [25]. The results differ only in the dimensionality of the spacetime being thus connected by analytical continuation. We suppose that it is not very demanding to proof that this coincidence is mandatory.

Furthermore, one can investigate the "multicritical" fixed points with more than one relevant direction. They are assumed to occur when the dimension is continuously lowered from four to two. First, the D-dependent evolution of spike plots obtained by means of shooting from the origin can be studied. Secondly, one can ϵ-expand the fixed point equation for $\zeta_*(\chi\chi^\dagger)$ about the different upper critical dimensions. We expect this to reveal the corresponding anomalous dimensions η_*. Finally, an ϵ-expansion of the flow equation for $\zeta_k(\chi\chi^\dagger)$ can be used to analyze the D-dependence of ω. For scalar theories the shooting approach to multicritical fixed points is discussed e. g. in [52] and the ϵ-expansion in [53].

We see that, though some interesting results have been obtained, there are plenty of open questions waiting for being addressed.

6. Acknowledgment

First of all, I would like to thank my supervisors Professor Andreas Wipf and Doctor Luca Zambelli for providing me with the versatile, exciting topic of this thesis. I owe my deepest gratitude to Prof. Wipf for his outstandingly motivating, proficient and comprehensible explanations. Furthermore, Dr. Zambelli's detailed involvement with the particular computations has been a great help for me.

I am in debt to Tobias Hellwig, Julia Borchardt, Dorothee Tell, Friederike Schulze, Silvia Kunz, and Dr. Zambelli for fast but careful proofreading. Moreover, I would like to offer my special thanks to everyone who has generously taken the time to discuss the problems arising during this thesis or to just listen to my expositions thereof. This thesis would not have been possible without such sympathy.

Finally, I want to thank the Heinrich Böll Foundation for financial support throughout my studies.

7. Appendix

A. LPA′ Truncation

In case I (see section 4.1), $\Gamma_k^{(2)}$ is a block diagonal matrix with the bosonic block

$$\Gamma_B^{(2)} = \begin{pmatrix} 2Z_0q^2 - 2(\frac{\partial^3 u}{\partial\phi_1^3}f_1 + \frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2}f_2) & -2(\frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2}f_1 - \frac{\partial^3 u}{\partial\phi_1^3}f_2) & -2\frac{\partial^2 u}{\partial\phi_1^2} & -2\frac{\partial^2 u}{\partial\phi_1\partial\phi_2} \\ -2(\frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2}f_1 - \frac{\partial^3 u}{\partial\phi_1^3}f_2) & 2Z_0q^2 + 2(\frac{\partial^3 u}{\partial\phi_1^3}f_1 + \frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2}f_2) & -2\frac{\partial^2 u}{\partial\phi_1\partial\phi_2} & 2\frac{\partial^2 u}{\partial\phi_1^2} \\ -2\frac{\partial^2 u}{\partial\phi_1^2} & -2\frac{\partial^2 u}{\partial\phi_1\partial\phi_2} & -2Z_0 & 0 \\ -2\frac{\partial^2 u}{\partial\phi_1\partial\phi_2} & 2\frac{\partial^2 u}{\partial\phi_1^2} & 0 & -2Z_0 \end{pmatrix} \delta(p-q) \quad (7.1)$$

and the fermionic one

$$\Gamma_F^{(2)} = \begin{pmatrix} -Z_0\sigma^j q_j & (\frac{\partial^2 u}{\partial\phi_1^2} + i\frac{\partial^2 u}{\partial\phi_1\partial\phi_2})\sigma^2 \\ (\frac{\partial^2 u}{\partial\phi_1^2} - i\frac{\partial^2 u}{\partial\phi_1\partial\phi_2})\sigma^2 & -Z_0{\sigma^j}^T q_j \end{pmatrix} \delta(p-q). \quad (7.2)$$

Case II gives a full matrix

$$\Gamma_k^{(2)} \equiv \begin{pmatrix} \Gamma_B^{(2)} & \Gamma_{BF}^{(2)} \\ \Gamma_{FB}^{(2)} & \Gamma_F^{(2)} \end{pmatrix}. \quad (7.3)$$

$\Gamma_B^{(2)}$ differs from above only in the first main diagonal block, which becomes

$$\begin{pmatrix} 2Z_0q^2 + \left\{\frac{1}{2}(\frac{\partial^4 u}{\partial\phi_1^4} - i\frac{\partial^4 u}{\partial\phi_1^3\partial\phi_2})\psi^T\sigma^2\psi + \mathrm{h.c.}\right\} & \left\{\frac{1}{2}(i\frac{\partial^4 u}{\partial\phi_1^4} + \frac{\partial^4 u}{\partial\phi_1^3\partial\phi_2})\psi^T\sigma^2\psi + \mathrm{h.c.}\right\} \\ \left\{\frac{1}{2}(i\frac{\partial^4 u}{\partial\phi_1^4} + \frac{\partial^4 u}{\partial\phi_1^3\partial\phi_2})\psi^T\sigma^2\psi + \mathrm{h.c.}\right\} & 2Z_0q^2 - \left\{\frac{1}{2}(\frac{\partial^4 u}{\partial\phi_1^4} - i\frac{\partial^4 u}{\partial\phi_1^3\partial\phi_2})\psi^T\sigma^2\psi + \mathrm{h.c.}\right\} \end{pmatrix} \delta(p-q), \quad (7.4)$$

$\Gamma_F^{(2)}$ remains unaffected, and

$$\Gamma_{BF}^{(2)} = \begin{pmatrix} (\frac{\partial^3 u}{\partial\phi_1^3} - i\frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2})\psi^T\sigma^2 & (\frac{\partial^3 u}{\partial\phi_1^3} + i\frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2})\bar{\psi}\sigma^2 \\ (i\frac{\partial^3 u}{\partial\phi_1^3} + \frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2})\psi^T\sigma^2 & (-i\frac{\partial^3 u}{\partial\phi_1^3} + \frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2})\bar{\psi}\sigma^2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \delta(p-q), \quad (7.5)$$

$$\Gamma^{(2)}_{FB} = \begin{pmatrix} (\frac{\partial^3 u}{\partial\phi_1^3} + i\frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2})\sigma^2\psi^* & (-i\frac{\partial^3 u}{\partial\phi_1^3} + \frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2})\sigma^2\psi^* & 0 & 0 \\ (\frac{\partial^3 u}{\partial\phi_1^3} - i\frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2})\sigma^2\psi & (i\frac{\partial^3 u}{\partial\phi_1^3} + \frac{\partial^3 u}{\partial\phi_1^2\partial\phi_2})\sigma^2\psi & 0 & 0 \end{pmatrix}\delta(p-q). \tag{7.6}$$

B. Beyond LPA′: $\mathbf{Z_k(q^2)}$

For this truncation, see section 4.2, $\Gamma^{(2)}_k(p,q)$ is block-diagonal with

$$\Gamma^{(2)}_B = \begin{pmatrix} q^2 Z_k(q^2)\delta(p-q) & -\frac{1}{\sqrt{2\pi}^3}W_k'''^\dagger(p-q)f^\dagger & 0 & -\frac{1}{\sqrt{2\pi}^3}W_k''^\dagger(p-q) \\ -\frac{1}{\sqrt{2\pi}^3}W_k'''(p-q)f & q^2 Z_k(q^2)\delta(p-q) & -\frac{1}{\sqrt{2\pi}^3}W_k''(p-q) & 0 \\ 0 & -\frac{1}{\sqrt{2\pi}^3}W_k''^\dagger(p-q) & -Z_k(q^2)\delta(p-q) & 0 \\ -\frac{1}{\sqrt{2\pi}^3}W_k''(p-q) & 0 & 0 & -Z_k(q^2)\delta(p-q) \end{pmatrix} \tag{7.7}$$

and

$$\Gamma^{(2)}_F = \begin{pmatrix} Z_k(q^2)\sigma^j q_j\delta(p-q) & \frac{1}{\sqrt{2\pi}^3}W_k''^\dagger(p-q)\sigma^2 \\ \frac{1}{\sqrt{2\pi}^3}W_k''(p-q)\sigma^2 & Z_k(q^2)\sigma^{jT}q_j\delta(p-q) \end{pmatrix}. \tag{7.8}$$

C. Beyond LPA′: $\mathbf{Z_k(\phi,\phi^\dagger)}$

In Fourier space, the ansatz for Γ_k employed in section 4.3 reads

$$\begin{aligned}\Gamma_k = \int \mathrm{d}^3q \Bigg(& Z_k(-q)\frac{1}{\sqrt{2\pi}^3}\int \mathrm{d}^3q' \Big(q'(q'-q)\phi(q')\phi^\dagger(q'-q) \\ & \qquad + \frac{1}{2}\bar{\psi}(q')\sigma^j(q+2q')_j\psi(q+q') - f(q')f^\dagger(q'-q)\Big) \\ & + \frac{1}{2}\frac{1}{(2\pi)^3}\Big\{(\partial_\phi Z_k)(-q)\int \mathrm{d}^3q'\,\mathrm{d}^3q''\ \Big(\psi^T(q')\sigma^2\psi(q''-q')f^\dagger(q''-q) \\ & \qquad + \bar{\psi}(q')\sigma^j(q+q'-q'')_j\psi(q'')\phi(q+q'-q'')\Big) + \mathrm{h.\,c.}\Big\} \\ & - \frac{1}{2}\frac{1}{\sqrt{2\pi}^9}(\partial_\phi\partial_{\phi^\dagger}Z_k)(-q)\int \mathrm{d}^3q'\,\mathrm{d}^3q''\,\mathrm{d}^3q'''\ \psi^T(q')\sigma^2\psi(q'')\bar{\psi}(q''')\sigma^2\psi^*(q'+q''-q'''-q) \\ & - \Big\{W_k'(-q)f(q) - \frac{1}{2}\frac{1}{\sqrt{2\pi}^3}W_k''(-q)\int \mathrm{d}^3q'\psi^T(q')\sigma^2\psi(q-q') + \mathrm{h.\,c.}\Big\}\Bigg).\end{aligned} \tag{7.9}$$

$\Gamma_k^{(2)}$ at constant fields and $\psi=\bar{\psi}=0$ is constituted from the two diagonal blocks

$$\Gamma_B^{(2)} = \delta(p-q)$$
$$\begin{pmatrix} -\partial_\phi\partial_{\phi^\dagger}Z_k ff^\dagger + q^2 Z_k & -\partial^2_{\phi^\dagger}Z_k ff^\dagger - W_k'''^\dagger f^\dagger & -\partial_{\phi^\dagger}Z_k f^\dagger & -\partial_{\phi^\dagger}Z_k f - W_k''^\dagger \\ -\partial^2_\phi Z_k ff^\dagger - W_k''' f & -\partial_\phi\partial_{\phi^\dagger}Z_k ff^\dagger + q^2 Z_k & -\partial_\phi Z_k f^\dagger - W_k'' & -\partial_\phi Z_k f \\ -\partial_\phi Z_k f & -\partial_{\phi^\dagger}Z_k f - W_k''^\dagger & -Z_k & 0 \\ -\partial_\phi Z_k f^\dagger - W_k'' & -\partial_{\phi^\dagger}Z_k f^\dagger & 0 & -Z_k \end{pmatrix} \tag{7.10}$$

and

$$\Gamma_F^{(2)} = \begin{pmatrix} Z_k\sigma^j q_j & \partial_{\phi^\dagger}Z_k f\sigma^2 + W_k''^\dagger\sigma^2 \\ \partial_\phi Z_k f^\dagger\sigma^2 + W_k''\sigma^2 & Z_k\sigma^{jT}q_j \end{pmatrix}\delta(p-q). \tag{7.11}$$

At constant ϕ ($\phi^\dagger$) and $f=f^\dagger=0$, $\Gamma_k^{(2)}$ becomes, after all terms proportional to $\psi_i\psi_j$, $\bar{\psi}_i\bar{\psi}_j$, or a product of more than two fermionic components are omitted,

$$\Gamma_k^{(2)} \equiv \begin{pmatrix} \Gamma_B^{(2)} & \Gamma_{BF}^{(2)} \\ \Gamma_{FB}^{(2)} & \Gamma_F^{(2)} \end{pmatrix} \tag{7.12}$$

with

$$\Gamma_B^{(2)} = \begin{pmatrix} a_1 & a_2 & 0 & -W_k''^\dagger\delta(p-q) \\ a_3 & a_4 & -W_k''\delta(p-q) & 0 \\ 0 & -W_k''^\dagger\delta(p-q) & -Z_k\delta(p-q) & \\ -W_k''\delta(p-q) & 0 & 0 & -Z_k\delta(p-q) \end{pmatrix}, \tag{7.13}$$

$$\begin{aligned} a_1 &= q^2 Z_k\delta(p-q) + \frac{1}{(2\pi)^3}\partial_\phi\partial_{\phi^\dagger}Z_k\int \mathrm{d}^3k\ \bar{\psi}(k)\sigma^j(k+p)_j\psi(k+p-q), \\ a_2 &= \partial^2_{\phi^\dagger}Z_k\int \mathrm{d}^3k\ \bar{\psi}(k)\sigma^j(k+p-q)_j\psi(k+p-q), \\ a_3 &= \partial^2_\phi Z_k\int \mathrm{d}^3k\ \bar{\psi}(k)\sigma^j k_j\psi(k+p-q), \\ a_4 &= q^2 Z_k\delta(p-q) + \frac{1}{(2\pi)^3}\partial_\phi\partial_{\phi^\dagger}Z_k\int \mathrm{d}^3k\ \bar{\psi}(k)\sigma^j(k-q)_j\psi(k+p-q), \end{aligned} \tag{7.14}$$

$$\Gamma_F^{(2)} = \left(\begin{matrix} Z_k\sigma^j q_j\delta(p-q) - \frac{2}{(2\pi)^3}\partial_\phi\partial_{\phi^\dagger}Z_k\int \mathrm{d}^3k\ \sigma^2\psi^*(k+q-p)\psi^T(k)\sigma^2 \\ W_k''\sigma^2\delta(p-q) \end{matrix} \quad \cdots \right.$$
$$\left. \cdots \quad \begin{matrix} W_k''^\dagger\sigma^2\delta(p-q) \\ Z_k\sigma^{jT}q_j\delta(p-q) - \frac{2}{(2\pi)^3}\partial_\phi\partial_{\phi^\dagger}Z_k\int \mathrm{d}^3k\ \sigma^2\psi(k)\bar{\psi}(k+q-p)\sigma^2 \end{matrix} \right), \tag{7.15}$$

$$\Gamma^{(2)}_{BF} = \begin{pmatrix} \frac{1}{\sqrt{2\pi}^3}\partial_{\phi^\dagger} Z_k \bar{\psi}(q-p)\sigma^j q_j & \\ \frac{1}{\sqrt{2\pi}^3}\partial_\phi Z_k \bar{\psi}(q-p)\sigma^j (q-p)_j + \frac{1}{\sqrt{2\pi}^3} W_k''' \psi^T(p-q)\sigma^2 & \dots \\ \frac{1}{\sqrt{2\pi}^3}\partial_\phi Z_k \psi^T(p-q)\sigma^2 & \\ 0 & \end{pmatrix}$$

$$\begin{pmatrix} & \frac{1}{\sqrt{2\pi}^3}\partial_{\phi^\dagger} Z_k \psi^T(p-q)\sigma^{jT}(q-p)_j + \frac{1}{\sqrt{2\pi}^3} W_k'''^\dagger \bar{\psi}(q-p)\sigma^2 \\ \dots & \frac{1}{\sqrt{2\pi}^3}\partial_\phi Z_k \psi^T(p-q)\sigma^{jT} q_j \\ & 0 \\ & \frac{1}{\sqrt{2\pi}^3}\partial_{\phi^\dagger} Z_k \bar{\psi}(q-p)\sigma^2 \end{pmatrix}, \tag{7.16}$$

and

$$\Gamma^{(2)}_{FB} = \begin{pmatrix} \frac{1}{\sqrt{2\pi}^3}\partial_\phi Z_k \sigma^j p_j \psi(p-q) & \\ \frac{1}{\sqrt{2\pi}^3}\partial_\phi Z_k \sigma^{jT}(p-q)_j \psi^*(q-p) + \frac{1}{\sqrt{2\pi}^3} W_k''' \sigma^2 \psi(p-q) & \dots \end{pmatrix}$$

$$\dots \begin{matrix} \frac{1}{\sqrt{2\pi}^3}\partial_{\phi^\dagger} Z_k \sigma^j (p-q)_j \psi(p-q) + \frac{1}{\sqrt{2\pi}^3} W_k'''^\dagger \sigma^2 \psi^*(q-p) \\ \frac{1}{\sqrt{2\pi}^3}\partial_{\phi^\dagger} Z_k \sigma^{jT} p_j \psi^*(q-p) \end{matrix} \dots$$

$$\begin{matrix} \dots \end{matrix} \begin{matrix} \frac{1}{\sqrt{2\pi}^3}\partial_{\phi^\dagger} Z_k \sigma^2 \psi^*(q-p) & 0 \\ 0 & \frac{1}{\sqrt{2\pi}^3}\partial_\phi Z_k \sigma^2 \psi(p-q) \end{matrix} \Bigg). \tag{7.17}$$

The polynomial truncation of $\zeta_*(\rho)$ to order N gives

- for $a = 0.9$:

N	g_*^2	ζ_1^*	ζ_2^*	ζ_3^*	ζ_4^*	ζ_5^*
0	1.5584	0	0	0	0	0
1	2.0041	−3.9007	0	0	0	0
2	1.9426	−3.3628	13.6244	0	0	0
3	1.9516	−3.4417	11.6370	−43.6257	0	0
4	1.9486	−3.4156	12.2942	−43.6257	586.3878	0
5	1.9497	−3.4245	12.0688	−51.0256	385.3642	−5158.8833

- for $a = 1.5$:

N	g_*^2	ζ_1^*	ζ_2^*	ζ_3^*	ζ_4^*	ζ_5^*
0	1.9739	0	0	0	0	0
1	2.4635	−2.8773	0	0	0	0
2	2.4287	−2.6726	4.9596	0	0	0
3	2.4272	−2.6636	5.1754	5.9379	0	0
4	2.4273	−2.6644	5.1568	5.4250	−12.5051	0
5	2.4274	−2.6650	5.1433	5.0535	−21.5633	−203.2030

- and for $a = 2$:

N	g_*^2	ζ_1^*	ζ_2^*	ζ_3^*	ζ_4^*	ζ_5^*
0	2.1932	0	0	0	0	0
1	2.7467	−2.7057	0	0	0	0
2	2.7073	−2.5131	4.4385	0	0	0
3	2.7046	−2.4998	4.7438	7.8374	0	0
4	2.7052	−2.5024	4.6848	6.3249	−34.6299	0
5	2.7052	−2.5027	4.6772	6.1277	−39.1448	−94.8318

Bibliography

1. Wipf, A. *Introduction to Supersymmetry* Lecture Notes. 2016. <http://www.tpi.uni-jena.de/qfphysics/homepage/wipf/lecturenotes.html>.
2. Sohnius, M. F. Introducing supersymmetry. *Physics reports* **128,** 39–204 (1985).
3. Wess, J. & Bagger, J. *Supersymmetry and supergravity* (Princeton university press, 1992).
4. Weinberg, S. *The quantum theory of fields* (Cambridge university press, 2000).
5. Thomas, S. *Emergent supersymmetry* KITP talk. 2005. <http://online.kitp.ucsb.edu/online/qpt_c05/thomas/pdf/Thomas_KITP.pdf>.
6. Grover, T., Sheng, D. & Vishwanath, A. Emergent space-time supersymmetry at the boundary of a topological phase. *Science* **344,** 280–283 (2014).
7. Lee, S.-S. Emergence of supersymmetry at a critical point of a lattice model. *Physical Review B* **76,** 075103 (2007).
8. Lee, S.-S. TASI lectures on emergence of supersymmetry, gauge theory and string in condensed matter systems. *arXiv preprint arXiv:1009.5127* (2010).
9. Wess, J. & Zumino, B. Supergauge transformations in four dimensions. *Nuclear Physics B* **70,** 39–50 (1974).
10. Giedt, J. Deconstruction and other approaches to supersymmetric lattice field theories. *International Journal of Modern Physics A* **21,** 3039–3093 (2006).
11. Bobev, N., El-Showk, S., Mazáč, D. & Paulos, M. F. Bootstrapping the three dimensional supersymmetric Ising model. *Phys. Rev. Lett.* **115,** 051601 (5 2015).
12. Bobev, N., El-Showk, S., Mazac, D. & Paulos, M. F. Bootstrapping SCFTs with four supercharges. *arXiv preprint arXiv:1503.02081* (2015).
13. Wilson, K. G. Renormalization group and critical phenomena. I. Renormalization group and the Kadanoff scaling picture. *Physical review B* **4,** 3174 (1971).
14. Rosten, O. J. On the renormalization of theories of a scalar chiral superfield. *Journal of High Energy Physics* **2010,** 1–46. ISSN: 1029-8479 (2010).
15. Hellwig, T., Wipf, A. & Zanusso, O. Scaling and superscaling solutions from the functional renormalization group. *Physical Review D* **92,** 085027 (2015).
16. Heilmann, M., Hellwig, T., Knorr, B., Ansorg, M. & Wipf, A. Convergence of derivative expansion in supersymmetric functional RG flows. *arXiv preprint arXiv:1409.5650* (2014).
17. Mastaler, M, Synatschke-Czerwonka, F & Wipf, A. Supersymmetric renormalization group flows. *Physics of Particles and Nuclei* **43,** 593–599 (2012).

18. Cardy, J. *Scaling and Renormalization in Statistical Physics* (Cambridge University Press, New York, 1996).

19. Wilson, K. G. & Kogut, J. The renormalization group and the ϵ expansion. *Physics Reports* **12,** 75 –199. ISSN: 0370-1573 (1974).

20. Berges, J., Tetradis, N. & Wetterich, C. Non-perturbative renormalization flow in quantum field theory and statistical physics. *Physics Reports* **363.** Renormalization group theory in the new millennium. {IV}, 223 –386. ISSN: 0370-1573 (2002).

21. Rosten, O. J. Fundamentals of the exact renormalization group. *Physics Reports* **511,** 177–272 (2012).

22. Synatschke, F., Gies, H. & Wipf, A. Phase diagram and fixed-point structure of two-dimensional $\mathcal{N} = 1$ Wess-Zumino models. *Phys. Rev. D* **80,** 085007 (8 2009).

23. Synatschke, F., Gies, H. & Wipf, A. The phase diagram for Wess-Zumino models. *arXiv preprint arXiv:0909.4189* (2009).

24. Synatschke, F., Braun, J. & Wipf, A. N= 1 Wess-Zumino model in d= 3 at zero and finite temperature. *Physical Review D* **81,** 125001 (2010).

25. Synatschke-Czerwonka, F., Fischbacher, T. & Bergner, G. Two-dimensional $\mathcal{N} = (2, 2)$ Wess-Zumino model in the functional renormalization group approach. *Phys. Rev. D* **82,** 085003 (8 2010).

26. *Quantum Fields and Strings: A Course for Mathematicians* (eds Deligne, P., Kazhdan, D. & Etingof, P.) (American Mathematical Society, Rhode Island, 1999).

27. Weinberg, S. *The Quantum Theory of Fields* Russian Translation (FIZMATLIT, Moscow, 2003).

28. De Traubenberg, M. R. Clifford algebras in physics. *Advances in Applied Clifford Algebras* **19,** 869–908. ISSN: 1661-4909 (2009).

29. McKeon, D. G. C. & Sherry, T. N. Supersymmetry and superfields in three euclidean dimensions. *International Journal of Theoretical Physics* **41,** 2369–2393. ISSN: 1572-9575 (2002).

30. Giedt, J. R-symmetry in the Q-exact 2d lattice Wess–Zumino model. *Nuclear Physics B* **726,** 210 –232. ISSN: 0550-3213 (2005).

31. Bergner, G., Kaestner, T., Uhlmann, S. & Wipf, A. Low-dimensional supersymmetric lattice models. *Annals of Physics* **323,** 946 –988. ISSN: 0003-4916 (2008).

32. Landau, L. On the theory of phase transitions. *Ukrainian Journal of Physics* **53.** Originally published in Zh. Eksp. Teor. Fiz. 7, pp. 19–32 (1937) in Russian, 25–35 (2008).

33. Sachdev, S. *Quantum Phase Transitions* 2nd ed. (Cambridge University Press, New York, 2011).

34. Wetterich, C. Exact evolution equation for the effective potential. *Physics Letters B* **301,** 90 –94. ISSN: 0370-2693 (1993).

35. Polchinski, J. Renormalization and effective lagrangians. *Nuclear Physics B* **231,** 269 –295. ISSN: 0550-3213 (1984).

36. Metzner, W., Salmhofer, M., Honerkamp, C., Meden, V. & Schönhammer, K. Functional renormalization group approach to correlated fermion systems. *Rev. Mod. Phys.* **84,** 299–352 (1 2012).

37. Heilmann, M., Hellwig, T., Knorr, B., Ansorg, M. & Wipf, A. Convergence of derivative expansion in supersymmetric functional RG flows. *Journal of High Energy Physics* **2015,** 1–27. ISSN: 1029-8479 (2015).

38. Liao, S.-B., Polonyi, J. & Strickland, M. Optimization of renormalization group flow. *Nuclear Physics B* **567,** 493 –514. ISSN: 0550-3213 (2000).

39. Litim, D. F. Optimisation of the exact renormalisation group. *Physics Letters B* **486,** 92 –99. ISSN: 0370-2693 (2000).

40. Litim, D. F. Mind the gap. *International Journal of Modern Physics A* **16,** 2081–2087 (2001).

41. Litim, D. F. Optimized renormalization group flows. *Phys. Rev. D* **64,** 105007 (10 2001).

42. Seiberg, N. Naturalness versus supersymmetric non-renormalization theorems. *Physics Letters B* **318,** 469 –475. ISSN: 0370-2693 (1993).

43. Strassler, M. J. in *Strings, Branes and Extra Dimensions* 561–639 (WORLD SCIENTIFIC). doi:10.1142/9789812702821_0011.

44. Aharony, O., Hanany, A., Intriligator, K., Seiberg, N. & Strassler, M. Aspects of N = 2 supersymmetric gauge theories in three dimensions. *Nuclear Physics B* **499,** 67 –99. ISSN: 0550-3213 (1997).

45. Fei, L., Giombi, S., Klebanov, I. R. & Tarnopolsky, G. Yukawa CFTs and emergent supersymmetry. arXiv: 1607.05316 [hep-th] (2016).

46. Litim, D. F. & Vergara, L. Subleading critical exponents from the renormalisation group. *Physics Letters B* **581,** 263 –269. ISSN: 0370-2693 (2004).

47. Synatschke, F., Bergner, G., Gies, H. & Wipf, A. Flow equation for supersymmetric quantum mechanics. *Journal of High Energy Physics* **2009,** 028 (2009).

48. Morris, T. R. The renormalization group and two dimensional multicritical effective scalar field theory. *Physics Letters B* **345,** 139 –148. ISSN: 0370-2693 (1995).

49. Vafa, C. & Warner, N. Catastrophes and the classification of conformal theories. *Physics Letters B* **218,** 51–58 (1989).

50. Gies, H. & Wetterich, C. Renormalization flow of bound states. *Phys. Rev. D* **65,** 065001 (6 2002).

51. Morris, T. R. On truncations of the exact renormalization group. *Physics Letters B* **334,** 355–362 (1994).

52. Codello, A. Scaling solutions in a continuous dimension. *Journal of Physics A: Mathematical and Theoretical* **45,** 465006 (2012).

53. O'dwyer, J & Osborn, H. Epsilon expansion for multicritical fixed points and exact renormalisation group equations. *Annals of Physics* **323,** 1859–1898 (2008).

Statement of Authorship

I, Polina Feldmann, confirm that the work presented in this thesis has been performed and interpreted solely by myself except where explicitly identified to the contrary. I have clearly marked and acknowledged all quotations or references that have been taken from the works of others. I confirm that this thesis is submitted in partial fulfillment for the degree Master of Science in Physics at the Friedrich Schiller University Jena and has not been submitted elsewhere in any other form for the award of any other degree or qualification.