Effective actions in statistical and quantum field theory

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1 Invitation: percolations in $d = 2$

1.1 A forest on fire

Models of **percolation** are probably the best examples of systems exhibiting a critical behavior because their geometrical nature can be visualized. The example given below is an oversimplification, but it has the relevant details to clearly outline the importance of critical properties of a system.

As a concrete example, let's try to visualize a physical system which turns our to be a concrete example of percolative behavior. Imagine that we have a forest, which we assume to be shaped like a square, separating a small town on the north from a highway on the south. For simplicity, we assume the trees to be positioned in a square grid, but this will not really change the final result that much. Let us finally assume that we are in the middle of a hot summer and that it did not rain much in the past months, so some trees have withered. Withered trees are much more likely to catch and trasmit fire.

The town's fire department is concerned with potential fire hazards which include sparks coming from the nearby highway (but also lightning strikes of some future storm). The department has promoted an investigation to determine how many – on average – are the withered trees in the forest using some **sampling**. The results returned the **estimate** that about N of the total number N_{tot} of trees have withered, therefore a tree has probability $p = N/N_{\text{tot}}$ to have dried. The number p is representative of our **ignorance** of the details of the system: we could in principle have determined which exact trees have withered, but this is certainly very difficult for a big forest. Going further, p hides the details of the dynamics that have made the forest as is now: the actual number of days without water, the age of the trees, the effect of wind in breaking dried branches, or even the microscopic details of the trees themselves including how they grow, how fast etc.

Based on the value of p, how should the department act?

The naive expectation would be that for increasing p we have an increasing risk of any fire propagating to town (which is true), and that risk and p might be correlated linearly (which is not true). If this is what the department has concluded, then it would have the incentive of spending resources to keep the value p low – for example by clearing withered trees – uniformly. This might not be the most effective strategy.

In truth there is a **critical** value of p, which we call p_c, that lies somewhere between $0 < p_c < 1$ and that for our purpose is approximately $p_c \simeq 0.5$. If $p < p_c$ there is no true risk of fire hazard, if instead $p > p_c$ even a single spark could doom the town. Why? The simplest way to understand this is to look at how big are **clusters** of withered trees, because a single separate withered tree can burn alone, but a cluster burns together and propagates the fire through the forest. In other words, we are interested on how the size of the clusters increase with p: we find out is that as long as $p < p_c$ the clusters are very small compared to the size of the system, while at $p \simeq p_c$ and beyond the size of the biggest cluster jumps quickly to about the size of the system itself. In practice, if a spark hits at $p < p_c$ we might burn down a small cluster trees, but if a spark hits at $p \simeq p_c$ we burn down almost the entire forest.

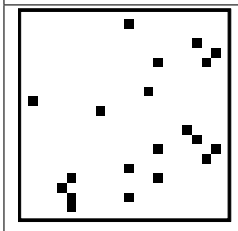
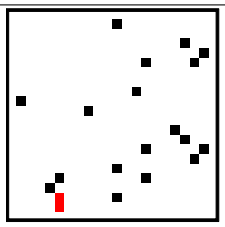
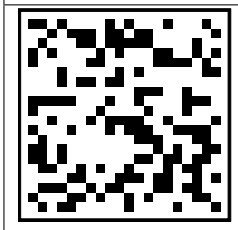
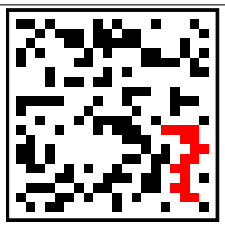
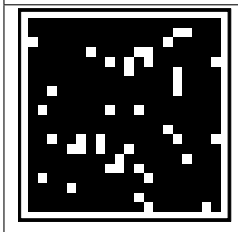
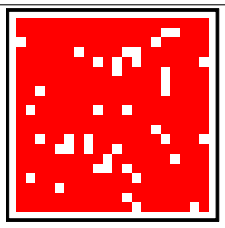
The best strategy of the department is thus to ensure that the number p is smaller than p_c and *absolutely* do not let it cross p_c. The knowledge of the system teaches us something more too: as long as $p < p_c$ fires are rather under control, so it is possible, and oftentimes convenient, to light some trees on fire under this controlled condition. A fire at $p < p_c$ does not propagate far away, and clears the forest of some withered trees, thus lowering p overall. North America's fire departments do sometimes light areas on fire in a controlled manner to reduce the overall risk of global fire in the long term.

Another place in which the department should invest resources is the determination of p_c. This value depends on the details of the system under consideration: in the case of our simplified example the most important detail is the shape of the grid (square vs any other arrangement). Our estimate $p_c \simeq 0.5$ is based on considerations that will be discussed more below.

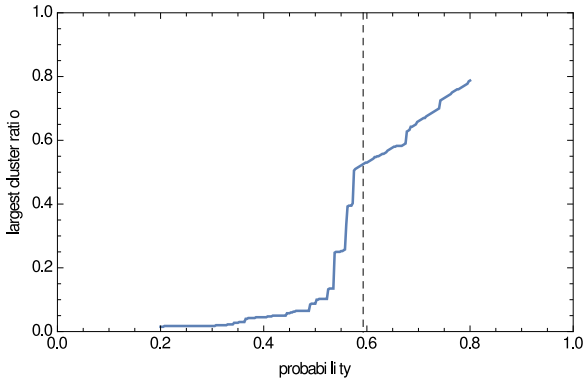
1.2 Population's behavior

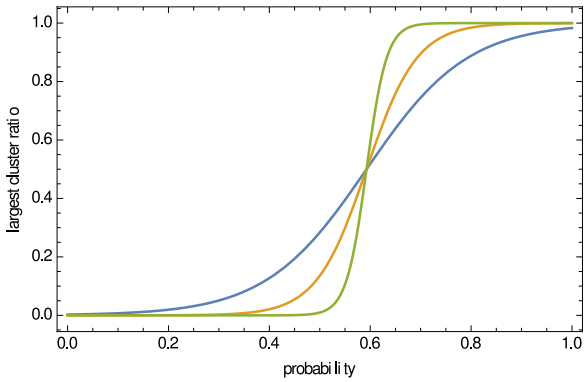
The one below is an example of **site percolation**, corresponding to the forest of the previous example. The size of the lattice is 20×20, black squares are populated sites (withered trees), while white square are not populated (healthy trees). The sites colored in red are the largest connected clusters (clusters of withered trees which represent the biggest fire hazards). Since we have a **statistical** lack of knowledge on the system, we should generate several sample populations of the lattice at each value of p, and then average our measures over them. However, it is illustrative (and computationally convenient) to populate a lattice for increasing p from $p = 0$ to $p = p_c$, because it corresponds to progressively add populated sites to the lattice.

For this type of lattice, as anticipated, $p_c = 0.5927 \simeq 0.5$. From top to bottom we give in each row a snapshot of the sample population, the hihlight of the largest cluster, and some data. The data include the value p in relation with p_c, the size of the largest cluster and the size of the populated sites. We observe that passing between the second and third row the size of the cluster and the size of the population become comparable.

Site percolation	**Largest cluster**	**Data**
		$p = 0.05 \ll p_c$ cluster $= 2$ sites $= 20$
		$p = 0.375 < p_c$ cluster $= 17$ sites $= 150$
		$p = 0.5925 \simeq p_c = 0.5927$ cluster $= 210$ sites $= 237$
		$p = 0.9 \gg p_c$ cluster $= 360$ sites $= 360$

If we plot the size of the largest cluster (over the system size) versus the probability p, we get a curve that has a sharp increase close to p_c, even for a lattice as small as our 20×20, and even if we did not **average** over more than one configuration (figure on the left). If we statistically average over an ensamble of populations, the curve gets *smoother*, while if we increase the sistem size it gets *steeper* (figure on the right).





Notice that if $L^2 = 400$ is already a big sized system, imagine what an Avogadro number of sites gives!!!

Now some buzzwords which we will hear over and over: The value p_c separates between two **phases**: a non-percolated one and a percolated one. Some observable quantities "diverge" close the p_c, that is at the phase transition. For example, the mean size of the clusters grows with the system size close to p_c, and therefore diverges if the system increases. These quantities behave as

$$\text{Observable} \sim \left(\frac{p - p_c}{p_c} \right)^a$$

for some **critical exponent** a.

1.3 Glimpses into the future

An instructive question is: what happens if the lattice is one-dimensional? It is straightforward to realize that a one dimensional chain can percolate only if *all* sites are populated, meaning that $p_c = 1$. Therefore the one dimensional case has only one phase, the non percolated one, unless p is strictly one. In the jargon of statistical mechanics the model is said to be **trivial**.

However, as long as the dimensionality of the lattice remains two (or bigger, but smaller than a certain value which we will discuss in the future), we can have a **nontrivial** phase transition. The value p_c depends on the details of the lattice: some example are Honeycomb $p_c = 0.6962$, Square $p_c = 0.592746$, Triangular $p_c = 0.5$, and Bond $p_c = 0.5$. This motivated our conclusion that $p_c \simeq 0.5$ rather generally for planar regular lattices (this must be taken with care though).

An amazing fact is that critical exponents *do not* depend on the details of the $2d$ lattice, and are therefore representative of some *universal physics* which goes beyond the details of the lattice itself. We anticipate that it depends parametrically only on the dimensionality of the system! The modern understanding is to say that all these lattices belong to the same **universality class**, which includes all models that exhibit the same **critical behavior** in terms of **universal critical exponents**.

An interesting aspect is that the critical exponents of a universality class can be computed with **continuum field theory** methods. For example, the percolation universality class can be studied using a special type of ϕ^3 theory (which is a theory with a cubic interaction, but it's a bit more complicate than this, which is basically why we will use another model as toy model for the course...). A mysterious fact is that at p_c the system exhibit an increased symmetry. You can imagine that rotating and translating sections of the forest do not change the physical properties, but it turns out that a bigger group of transformations – known as **conformal transformations** – do not change the physics.

2 Introduction: quantum vs statistical field theory

This table is meant to provide a small and easy-to-use dictionary between quantities of quantum field theory and statistical field theory.

	Quantum field theory	**Statistical field theory**
Origin Parent theories	Quantum mechanics Single-particle relativistic mechanics	Thermodynamics Statistical mechanics
Uncertainty	Quantum	Ignorance: Temperature T, Probability p
Basic constant	Planck's $\hbar$ measures units of action	Boltzmann's k_B k_B converts T in energy: $\beta^{-1} = k_B T$
Microphysics	Bare action $S[\phi]$	Hamiltonian $\mathcal{H}[\sigma]$
Prototypical example	ϕ^4 theory $S = \int \left\{ \frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{4!}\phi^4 \right\}$	Ising model $\mathcal{H} = -\frac{J}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j; \qquad \sigma_i = \pm 1$
Configurations	Quantum phase $\mathrm{e}^{\mathrm{i}S[\phi]/\hbar}$	Boltzmann weight $\mathrm{e}^{-\beta\mathcal{H}[\sigma]}$
Path integral	Quantum superposition $\mathrm{e}^{\mathrm{i}W} \simeq \int D\phi \ \mathrm{e}^{\mathrm{i}S[\phi]}$	Distribution $\mathrm{e}^{-\beta F} = \sum_{\{\sigma\}} \mathrm{e}^{-\beta\mathcal{H}[\sigma]}$
Functionals	W-generator, effective action $W[J]$, $\Gamma[\bar{\phi}]$	Free energy, thermod. potentials $F[\bar{m}]$
n-point functions	Scattering processes $P(x_1 \dots x_n) = \lvert\langle \phi(x_1) \dots \phi(x_n) \rangle\rvert^2$	Correlators $\langle \sigma_{i_1} \dots \sigma_{i_n} \rangle$
Euclidean field theory	Wick rotation: $t = i\tau$ $\int D\phi \ \mathrm{e}^{-S_E[\phi]}$	Field representation: $\int D\phi \ \mathrm{e}^{-S_E[\phi]}$

Quantum field theory (QFT) and statistical field theory (SFT) could be understood as evolutions of quantum mechanics (QM) and statistical mechanics (STM) respectively. Both QFT and SFT are born to simultaneously deal with many degrees of freedom and/or uncertainty in our knowledge of the behavior of such degrees of freedom. The origin of the uncertainty is however very different: in the first case the uncertainty is intrinsic and due to the quantum nature of the interactions, while in the second case the uncertainty is more pragmatically related to our limited knowledge of the microscopic state of a macroscopic system (for example because of a thermal bath).

3 From spins to fields: The Ising model

The partition function of the Ising model is

$$Z = \sum_{\{\sigma\}} \mathrm{e}^{-\beta \mathcal{H}[\sigma]} \tag{1}$$

for $\beta = (k_B T)^{-1}$. The microscopic energy associated to each spin configuration is

$$\mathcal{H} = -\frac{J}{2} \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i \tag{2}$$

in which the summation extends on nearest-neighbor pairs $\langle i, j \rangle$ of a **square lattice** in dimension d. Each single spin σ_i can take the following values $\sigma_i = \pm 1$, so if there are N total lattice sites the total number of configurations is 2^N. The **free energy** of the system is defined

$$Z = \mathrm{e}^{-\beta N F}\,; \qquad\qquad F = -\frac{1}{\beta N} \ln F \tag{3}$$

One is generally interested in the **thermodynamic limit** $N \to \infty$. Final remark: at $h = 0$ we have that the Hamiltonian is invariant under a *parity* transformation $\mathbb{Z}_2 : \sigma_i \to -\sigma_i$

The couplings are:

- J is the *nearest-neighbor coupling* that weights the strength of the spin interactions. For J large and positive the Hamiltonian $\mathcal{H}$ becomes smaller. Configurations which minimize the energy are those in which the spins tend to be aligned.
- h is the *uniform magnetic field*. For strong h, the spins tend to align with its sign $h/|h|$.

A nonzero magnetic field drives the spins in the direction of its sign: we therefore expect that $h = 0$ separates between two phases in which the spins are aligned differently with $\mathbb{Z}_2$ **symmetry breaking**. Take instead $h = 0$, based on the above considerations we expect two limits

- $J \gg k_B T$: it costs a lot of energy to flip a spin if it is aligned with a neighbor, therefore we expect all spins to be aligned in the same direction, but at $h = 0$ such direction is arbitrary. This becomes a $\mathbb{Z}_2$ **spontaneous symmetry breaking** phase.
- $J \ll k_B T$: no configuration is favored in the limit and they are all equally likely: spins are random and $\mathbb{Z}_2$ is **preserved**.

There are thus four **phases** in which $\mathbb{Z}_2$ is either broken or preserved by our expectations on the lowest energy configurations. We can characterize the phases by measuring the **magnetization**

$$m_i = \langle \sigma_i \rangle = \frac{1}{Z} \sum_{\{\sigma\}} \sigma_i \, \mathrm{e}^{-\beta \mathcal{H}[\sigma]} \tag{4}$$

or the **magnetization per spin**

$$m = \frac{1}{N} \sum_i m_i = \frac{1}{N} \frac{1}{Z} \sum_{\{\sigma\}} \sum_i \sigma_i \, \mathrm{e}^{-\beta \mathcal{H}[\sigma]} = \frac{1}{N} \frac{1}{\beta} \frac{\partial \log Z}{\partial h} = -\frac{\partial F}{\partial h} \tag{5}$$

They are equally good **order parameters** if the system is **uniform**.

We use m as follows:

- If $m \neq 0$ we are in the **symmetry broken phase** also known as **ferromagnetic phase** or **ordered phase**.
- If $m = 0$ we are in the **symmetric phase** also known as **paramagnetic phase** or **disordered phase**.

Based on the above expectations: there must be a value T_c (known as **critical temperature** or Curie temperature) which separates the two phases at $h = 0$, and at T_c occurs a **phase transition**. Notice that for what we know at this stage T_c could be zero, but if $T_c > 0$ it would be more interesting!

3.1 Quantities associated to the free energy

Besides the **magnetization per spin**

$$m(T,h) = -\left.\frac{\partial F}{\partial h}\right|_T \tag{6}$$

We define the **entropy per spin**

$$s(T,h) \;=\; -\left.\frac{\partial F}{\partial T}\right|_h \tag{7}$$

The **magnetic susceptibility**

$$\chi(T,h) \;=\; -\left.\frac{\partial^2 F}{\partial h^2}\right|_T \tag{8}$$

The **specific heat**

$$c(T,h) \;=\; -\frac{1}{\beta}\left.\frac{\partial^2 F}{\partial T^2}\right|_h \tag{9}$$

Incidentally note that all the above quantities depend parametrically on N even though we do not display it explicitly.

3.2 Ising model in $d = 1$

We introduce here a lot of useful nomenclature in the simplified setting of the Ising model in one dimension, which can be exactly solved. In one dimension we choose N lattice sites and the boundary condition such that $\sigma_1 = \sigma_{N+1}$ (we will ultimately be interested in the thermodynamic limit $N \to \infty$ which should be independent of the boundary condition). The Hamiltonian simplifies

$$\mathcal{H} \;=\; -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i \;=\; -\sum_{i=1}^{N}\left\{J\sigma_i\sigma_{i+1} + \frac{h}{2}\left(\sigma_i + \sigma_{i+1}\right)\right\} \tag{10}$$

and the partition function

$$Z \;=\; \sum_{\{\sigma\}} \mathrm{e}^{\sum_{i=1}^{N}\left\{\beta J\sigma_i\sigma_{i+1} + \frac{\beta h}{2}(\sigma_i+\sigma_{i+1})\right\}} \tag{11}$$

$$\;=\; \sum_{\{\sigma\}}\prod_{i=1}^{N} \mathrm{e}^{\beta J\sigma_i\sigma_{i+1} + \frac{\beta h}{2}(\sigma_i+\sigma_{i+1})} \;=\; \sum_{\{\sigma\}}\prod_{i=1}^{N} T(\sigma_i,\sigma_{i+1}) \tag{12}$$

We defined the **transfer matrix**

$$T(\sigma_i,\sigma_{i+1}) = \mathrm{e}^{\beta J\sigma_i\sigma_{i+1} + \frac{\beta h}{2}(\sigma_i+\sigma_{i+1})} \tag{13}$$

In the basis $\sigma_i \otimes \sigma_{i+1}$ the matrix becomes

$$\mathbf{T} = \begin{bmatrix} T(+1,+1) & T(+1,-1) \\ T(-1,+1) & T(-1,-1) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\beta(J+h)} & \mathrm{e}^{-\beta J} \\ \mathrm{e}^{-\beta J} & \mathrm{e}^{\beta(J-h)} \end{bmatrix} \tag{14}$$

The partition function is the trace of the N-th power of the matrix, which can be expressed in terms of the matrix's eigenvalues

$$Z \;=\; \mathrm{Tr}\,\mathbf{T}^N \;=\; \lambda_1^N + \lambda_2^N \tag{15}$$

The eigenvalues are

$$\lambda_{1,2} = \mathrm{e}^{\beta J}\cosh(\beta h) \pm \mathrm{e}^{-\beta J}\sqrt{1+\mathrm{e}^{4\beta J}\sinh(\beta h)^2}$$

Since we are interested in the limit $N \to \infty$ we use the fact that $0 < \lambda_2 < \lambda_1$ at $T > 0$

$$Z = \lambda_1^N \left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right) \sim \lambda_1^N + \ldots \qquad \text{for} \quad N \to \infty$$

We can now compute all the quantities of interest.

We obtain the free energy in the limit

$$F(T,h) \;=\; \lim_{N\to\infty}\left(-\frac{1}{\beta N}\ln Z\right) = -\frac{1}{\beta}\ln\lambda_1 \tag{16}$$

$$\;=\; -\frac{1}{\beta}\ln\left\{\mathrm{e}^{\beta J}\cosh(\beta h) + \mathrm{e}^{-\beta J}\sqrt{1+\mathrm{e}^{4\beta J}\sinh(\beta h)^2}\right\} \tag{17}$$

We use the free energy to compute the magnetization per spin

$$m(T,h) \;=\; -\left.\frac{\partial F}{\partial h}\right|_T = \frac{\mathrm{e}^{2J\beta}\sinh(h\beta)}{\sqrt{1+\mathrm{e}^{4J\beta}\sinh(h\beta)^2}} \tag{18}$$

For our purpose it is sufficient to note that

$$m(T,h) \;\propto\; \sinh(\beta h) \tag{19}$$

We have that at zero magnetic field $m(T, h = 0) = 0$, therefore in $d = 1$ there is no ferromagnetism in $d = 1$ at $T > 0$ and $h = 0$. In other words: $T_c = 0$ for the Ising models in $d = 1$! The specific heat $c(T, h)$ is a complicated expression, but at $h = 0$ it simplifies enormously

$$c(T,0) \;=\; -\frac{4\beta^3 J^2 \mathrm{e}^{2J\beta}}{(1+\mathrm{e}^{2J\beta})^2} \tag{20}$$

The **spin-spin correlator** can be calculated in a way analog to the partition function itself. Assuming $j > i$, we have

$$\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle \propto \frac{\lambda_1^{N-(j-i)}\lambda_2^{j-i} + \lambda_2^{N-(j-i)}\lambda_1^{j-i}}{\lambda_1^N + \lambda_2^N} \;\sim\; \left(\frac{\lambda_2}{\lambda_1}\right)^{j-i} \qquad \text{for} \quad N \to \infty \tag{21}$$

It clearly depends on the distance between the sites i and j, which is $r = \mathrm{dist}(i,j) = j - i$. At $h = 0$ and in the thermodynamic limit, we have

$$\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle \propto \tanh(\beta J)^r = \mathrm{e}^{-r/\xi} \tag{22}$$

with $\xi = 1/|\ln\tanh(\beta J)|$ (the absolute value is because it is a function that is always negative). The quantity ξ is known as **correlation length** and diverges at $T = T_c = 0$.

3.3 Ising model in $d = 2$

An exact solution for the Ising model in $d = 2$ at $h = 0$ was first given by Onsager in 1944 (it will not be part of this lecture). He proved that (for the isotropic square lattice)

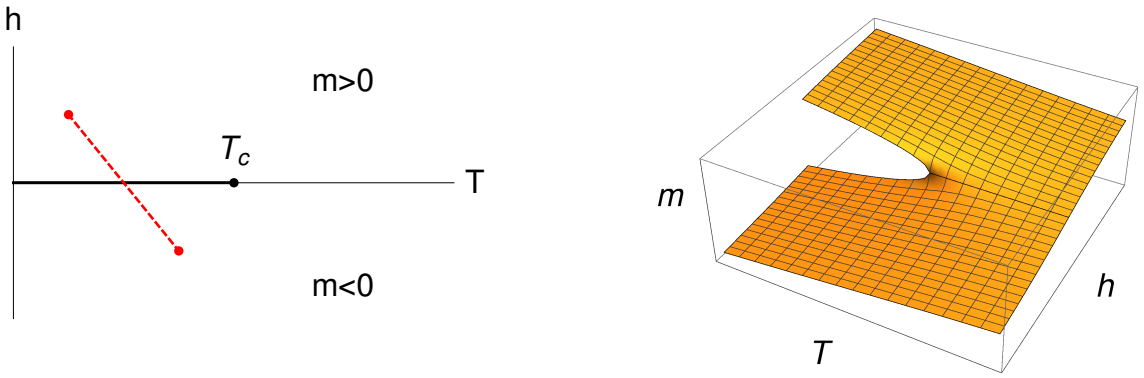
$$k_B T_c = \frac{2J}{\ln(1+\sqrt{2})} \simeq 2.2691 \cdot J \tag{23}$$

Therefore in $d = 2$ one can have the two distinct phases at nonzero temperture and $h = 0$.

3.4 Ising model in $d > 1$: Landau-Ginzburg

In more than one dimension, likewise $d = 2$, the Ising model exhibits a behavior that is more interesting than the $d = 1$ case. This section is more of a narration of some ideas: you can imagine that you have numerically computed the partition function by some approximation and gained access to some observable properties such as the magnetization.

In $d > 1$ we have that in general $T_c > 0$, so we can have four distinct phases in the phase diagram. The phase diagram is shown on the left, while the plot of the magnetization is on the right:



There are several different ways to move around this phase diagram as a function of h and T.

- By lowering h at $T < T_c$ we cross the transition $h = 0$ along the thick black curve (red dashed line). The system jumps discontinuously from $m > 0$ to $m < 0$, hence the transition is known as **discontinuous**.
- By lowering h at $T > T_c$ we cross $h = 0$ along the thinner part. The system crosses over from $m > 0$ to $m < 0$ passing through $m = 0$ at $h = 0$, hence this transition is **continuous**.
- If $h = 0$ and $T > T_c$ we are in the symmetric phase $m = 0$, but by lowering T to the critical value we can enter the spontaneously broken phase. The transition is continuous because the order parameter does not jump to a finite value, however the correlation length diverges like it happened in $d = 1$.

As discussed in the recap of thermodynamic, we can think at the free energy as the functional whose minimization returns the statistical expectation values. Let F be a function of a uniform quantity $\overline{m}$ with couplings that depend parametrically on h and T, and let $m = m(T, h)$ be the parametrically dependent minimum. Now we want to backward engineer the minimal **effective** function $F(\overline{m})$ which has minimum $m(T, h)$ as above and for which $F(T, h) = F(m(T, h))$. In the language of field theory $F(\overline{m})$ is an **off-shell** functional and $F(m(T, h))$ is its **on-shell** evaluation.

From the plot of the magnetization it is clear that the location of the minimum has to respond to the magnetic field. In agreement with the definition of magnetization we expect the term

$$\delta F \sim -\overline{m}\, h$$

This term is however not enough to explain the entirety of $m(T,h)$. The function F has to have a nontrivial minimum whenever $h \neq 0$, so there must be at least a quadratic term

$$\delta F \sim r(T)\,\overline{m}^2$$

The quadratic term is enough to explain the part of $m(T,h)$ for $T > T_c$ if $r(T > T_c) > 0$ because the minima of a parabola have a continuous dependence with respect to their parametric dependence. For $T < T_c$ however the jump is discontinuous: the simplest way in which this can be achieved is if $r(T < T_c) < 0$ and simultaneously we include

$$\delta F \sim u(T)\,\overline{m}^4$$

with $u(T) > 0$ for each $T > 0$. Putting everything together we have

$$F[\overline{m}] = -h\,\overline{m} + r(T)\,\overline{m}^2 + u(T)\,\overline{m}^4$$

Further contributions such as $\overline{m}^6$ may in principle be allowed and they give **corrections** to the above minimal description.

Having established that $r(T > T_c) > 0$ and $r(T < T_c) < 0$ we might ask what is the behavior of $r(T)$ close to T_c. Since $T_c > 0$ it is convenient to define the **reduced temperature**

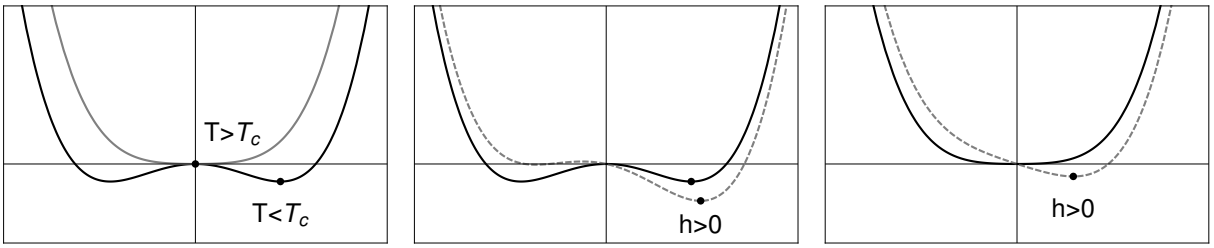
$$t = \frac{T - T_c}{T_c} \tag{24}$$

Given that r has to change sign with $T - T_c$ we expect that r and t are related by some odd power. The simplest choice is that

$$r(T) \sim T - T_c \sim t$$

Other choices are in principle possible, but this is the most natural. One – perhaps convoluted – way to see this is by noticing that in units of k_B, the formula agrees with the naive expectation that if to the field $\overline{m}$ is given a canonical dimension then $r(T)$ has the dimension of a mass.

The first plot below shows the function $F(T, h = 0)$ above and below T_c; the second plot shows what happens to $F(T < T_c, 0)$ when h is turned on to a positive value; the third plot shows what happens to $F(T > T_c, 0)$ when h is turned on to a positive value.



3.5 Definition of the thermodynamical exponents and first estimation

For $T \to T_c > 0$ and $h = 0$ the quantities $m(T,h)$, $c(T,h)$ and $\chi(T,h)$ behave according to specific **scaling forms**. In the disordered phase at $t > 0$

$$c \sim t^{-\alpha}\,; \qquad \chi \sim t^{-\gamma} \tag{25}$$

and in the ordered phase at $t < 0$

$$c \sim (-t)^{-\alpha'}\,; \qquad m \sim (-t)^{\beta}\,; \qquad \chi \sim (-t)^{-\gamma'} \tag{26}$$

Additionally using the equation of state at $T = T_c$ and for $h \to 0$

$$m \sim h^{\frac{1}{\delta}} \tag{27}$$

The exponents α, β, γ and δ are known as **thermodynamical exponents**. We can give a preliminary estimate on these quantities using the analysis of the previous section. We get the so-called **mean field estimates**

$$\alpha = \alpha' = 0\,; \qquad \beta = \frac{1}{2}\,; \qquad \gamma = \gamma' = 1\,; \qquad \delta = 3 \tag{28}$$

Notice that the heat-capacity has a discontinuity which is approached on both sides with exponent zero. Notice also that exponents that exist on both sides of the transition take the same values. The question is, how do they compare with experiments?

	mean field	$d=2$	$d=3$	$d=4$
α	0	log	0.1101	0.
β	$\frac{1}{2}$	$\frac{1}{8}$	0.3264	0.5
γ	1	$\frac{7}{4}$	1.2371	1.
δ	3	15	4.7898	3.

This is an *OK* result. Notable features are that the mean field prediction surprisingly agrees with the findings in $d = 4$, gives decent estimates for $d = 3$, and is far off the $d = 2$ results. The main problem is that the mean field estimate does not depend on the dimension d, while the above numbers clearly do.

In $d \geq 4$ the theory is often referred to as **trivial**. The critical exponents can be trivially estimated on the basis of the mean field analysis (we will return on the meaning of triviality in a quantum field theoretical sense later). In the case $d = 4$ quantum corrections are responsible for logarithmic contributions.

3.6 Ginzburg-Landau and field-theoretical critical exponents

In the Landau-Ginzburg discussion we have worked with a uniform field $\overline{m}$ and we have implicitly assumed that the role of statistical fluctuations about the minimum value $m = m(T, h)$ is negligible. To take the effect of fluctuations into account let us promote $\overline{m}$ to a **field**

$$\phi(x) = \overline{m} + \delta\phi(x)$$

We expect the corresponding free energy to become a functional which takes into account the fact that there is energy associated to fluctuations $\delta\phi(x)$ about the uniform value $\overline{m}$ which becomes $m(T, h)$ on-shell. The simplest effect would be

$$F[\phi] = \int \mathrm{d}^d x \left\{ -h\,\phi + r\,\phi^2 + u\,\phi^4 + \zeta\,(\partial\phi)^2 \right\} \tag{29}$$

Now we want to estimate how important are the fluctuations $\delta\phi(x)$ as compared to the contributions from the uniform field. Let us concentrate on the case $h = 0$: the uniform configuration that minimizes the free energy is $\overline{m} = 0$ for $T > T_c$ and $\overline{m} = \sqrt{-r/(2u)}$ for $T < T_c$. Either way, we can insert a configuration with small fluctuations about that minimum in F to find

$$F[\overline{m} + \delta\phi] = \zeta \int \mathrm{d}^d x \left\{ (\partial(\delta\phi))^2 + \frac{1}{\xi^2}(\delta\phi)^2 \right\} + \mathcal{O}(\delta\phi)^3 \tag{30}$$

We have introduced a positive quantity with units of length known as **correlation length** which depends on the temperature. In fact it is different between the two sides of the transition

$$\xi^2 = \frac{1}{r} \qquad \text{for} \quad T > T_c \tag{31}$$

$$\xi^2 = -\frac{1}{2r} \qquad \text{for} \quad T < T_c \tag{32}$$

Remember that $r \propto T - T_c$, so it is negative below T_c! The correlation length scales with its own critical exponent (known as **correlation length's critical exponent** because we lack imagination)

$$\xi \sim t^{-\nu} \qquad \text{for} \quad T > T_c \tag{33}$$

$$\xi \sim (-t)^{-\nu} \qquad \text{for} \quad T < T_c \tag{34}$$

The uniform field is not concerned with distance, but the effect of fluctuations is long(er)-range and the range is related to the length ξ. To see this imagine the generalization of the spin-spin correlator seen in the case $d = 1$ in the continuum. We have that neglecting further interactions

$$G(x,y) = \langle \delta\phi(x)\delta\phi(y)\rangle \propto \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{e}^{-ik\cdot(x-y)}}{k^2 + \frac{1}{\xi^2}} \tag{35}$$

$$= \frac{1}{(2\pi)^d} \frac{1}{(\xi r)^{d/2-1}} K_{d/2-1}(|x-y|/\xi) \tag{36}$$

To take into account interactions we replace the Bessel function with a new function Φ (which is different on the two sides of the transition) and include a new exponent

$$G(x,y) = \frac{1}{|x-y|^{d-2+\eta}} \Phi\left(|x-y|/\xi\right) \tag{37}$$

The new exponent η is known as **anomalous dimension**.

The two new exponents ν and η are markedly different from the thermodynamical ones because they are clearly of field-theoretical nature. Their mean field estimates are $\nu = \frac{1}{2}$ and $\eta = 0$ and the discussion of the experimental validity of these estimates follows closely the one for the other exponents.

	mean field	$d = 2$	$d = 3$	$d = 4$
ν	$\frac{1}{2}$	1	0.63	0.5
η	0	$\frac{1}{4}$	0.036	0

Using the simpler form based on the Bessel function we can observe that $G(x,y)$ has two distinct limits:

$$G(x,y) \sim \frac{1}{|x-y|^{d-2}} \qquad \text{for} \quad |x-y| \ll \xi \tag{38}$$

$$G(x,y) \sim \frac{1}{|x-y|^{d/2-1}} \mathrm{e}^{-|x-y|/\xi} \qquad \text{for} \quad |x-y| \gg \xi \tag{39}$$

which together are known ad Ornstein-Zernicke correlation.

3.7 Ginzburg criterion

The uniform part of the field contributes to the energy with $\langle\phi\rangle = m$ in our semiclassical treatment, while fluctuations contribute with the longer range correlator $\langle\phi(x)\phi(y)\rangle$. Fluctuations are less important if $\langle\phi^2\rangle \ll \langle\phi\rangle^2$. An estimate of this inequality was invented by Ginzburg

$$R = \frac{\int_{|x|<\xi} \mathrm{d}^d x \, \langle\phi(x)\phi(y)\rangle}{\int_{|x|<\xi} \mathrm{d}^d x \, m^2} \tag{40}$$

To estimate the numerator it we use the limit $|x| \ll \xi$, while the denominator just evaluates to $m^2\xi^d$. We find that close to T_c and using the mean field estimates

$$R \propto \frac{\xi^{2-d}}{m^2} \sim t^{d/2-1} t^{-1} = t^{(d-4)/2} \tag{41}$$

Therefore, according to the mean field estimates for the scaling *above* $d = 4$ fluctuations become increasingly *less* important as $T \to T_c$, instead *below* $d = 4$ fluctuations become important and might even dominate over the uniform contributions. At $d = 4$ the size of the two contributions is expected to be the same.

The dimension $d_c = 4$ is known as the **upper critical dimension** of the field theory, and more generally of the universality class of the Ising model.

3.8 Scaling

We have seen in the recap lecure of thermodynamics the properties that can be extracted from assuming that the thermodynamic potentials are **homogeneous scaling functions** as follows

$$F_s(\lambda T, \lambda h) = \lambda F_s(T, h)\,. \tag{42}$$

We introduced the subscript s on the free energy to underline that we assume that it has a scaling form in proximity of $T = 0 = h$, that is relevant if $T_c = 0$. Using the above scaling hypothesis we can derive that at $T = 0 = h$ the quantities of interest behave as

$$m = -F_s^{(0,1)}(1, 0)\,; \qquad \chi = -F_s^{(0,2)}(1, 0)/T\,; \qquad c = -F_s^{(2,0)}(1, 0) \tag{43}$$

As we discussed, for $d > 2$ we have that $T_c > 0$ and it is convenient to introduce the reduced temperature $t = (T - T_c)/T_c$. In proximity of T_c the scaling has to lead to nontrivial values for the exponents. Consequently we expect a generalization with scaling in proximity of $t = 0 = h$ and non-integer powers. We assume that the free energy is a **generalized homogeneous scaling function** in the pair (t, h) rather than (T, h)

$$F_s(\lambda^{a_t} t, \lambda^{a_h} h) = \lambda F_s(t, h) \tag{44}$$

Based on the special form we deduce two important conclusions:

- Since the free energy depends only on two exponents **field scaling power** a_h and **temperature scaling power** a_t, we deduce that it is sufficient to determine only two critical exponents to uniquely determine all others (we will see briefly which ones are the best!)
- Since the free energy is a dimensionful quantity we argue that the scaling function must be somehow related to the scale dependence (that is the renormalization group!)

We can also derive explicit expressions for the thermodynamical quantities close to $t = 0$

$$m = -t^{\frac{1-a_h}{a_t}} F_s^{(0,1)}(1, 0)\,; \qquad \chi = -t^{\frac{1-2a_h}{a_t}} F_s^{(0,2)}(1, 0)\,; \qquad c = -t^{\frac{1-2a_t}{a_t}} F_s^{(2,0)}(1, 0) \tag{45}$$

which imply the following form for the critical exponents

$$\alpha = -\frac{1 - 2a_t}{a_t}\,; \qquad \beta = \frac{1 - a_h}{a_t}\,; \qquad \gamma = -\frac{1 - 2a_h}{a_t} \tag{46}$$

We use the equation of state $M_s(t, h) = F_s^{(0,1)}(t, h)$ to relate the magnetization to h through the exponent δ at T_c. We get that close to the critical point

$$M_s(0, h) = h^{\frac{1-a_h}{a_h}} M_s(0, 1) \tag{47}$$

which implies

$$\delta = \frac{a_h}{1 - a_h} \tag{48}$$

On this general basis we can derive relations known as **scaling relations** among the thermodynamical exponents. The first one does not involve the equation of state. We elimitate a_t and a_h by first determining

them through the definitions of β and γ, and then substitute the result in the equation of α. We get the **Rushbrooke equality**

$$\alpha + 2\beta + \gamma = 2 \tag{49}$$

To derive further relations we have to use the equation of state so we have two additional ways of elimiating the pair of unknowns. Solving for the unknowns using the equations of β and δ and inserting them in the relation of α we get the **Griffiths equality**

$$\alpha + \beta(1 + \delta) = 2 \tag{50}$$

Solving for the unknowns using the equations of γ and δ and inserting them in the relation of β we get the **Widom equality**

$$\gamma = \beta(\delta - 1) \tag{51}$$

Close to, but away from, T_c the equation of state can be written

$$M_s = M_s(t, h) = h^{\frac{1-a_h}{a_h}} M_s(t \cdot h^{-a_t/a_h}, 1) \equiv h^{1/\delta} M_s^h(t_h) \tag{52}$$

where we introduced the **scaled temperature** $t_{\rm res} = t \cdot h^{-a_t/a_h}$ and the **scaled magnetization** $M_{\rm res}(t_{\rm res}) = M_s(t_{\rm res}, 1)$. Incidentally we define the exponent $\Delta = a_h/a_t$ so that $t_{\rm res} = t \cdot h^{-1/\Delta}$ which is called **gap exponent**. By construction $M_{\rm res}$ is a function of only $t_{\rm res}$, and it turns out that close to the critical point the **scaled equation of state** $M_{\rm res} = M_{\rm res}(t_{\rm res})$ which relates the scaled quantities is a **universal** function. While critical exponents are relatievly easy quantities to determine with field theoretical methods (as we will see), the rescaled equation of state is much more nontrivial!

3.9 Scaling forms

Using the results of the previous section we can rewrite several quantities close to T_c in their **scaling form** in a way similar to what we have done for the scaled equation of state. We begin with the free energy

$$\begin{aligned} F_s(t, h) &= t^{\frac{1}{a_t}} F_s(1, h \cdot t^{-\frac{a_h}{a_t}}) && (53) \\ &\equiv t^{2-\alpha} g_F\left(\frac{h}{t^\Delta}\right) && (54) \end{aligned}$$

The above relation could be taken as a starting point for the derivation of the scaling relations of the previous section. The same kind of manipulation can be performed on the magnetization (which is analogous to what we have done to obtain the scaled equation of state). If a quantity is (generalized) homogeneous all derived quantities are (generalized) homogeneous too. This includes all the transforms (and therefore all other theormodynamical potentials) as well as its derivatives.

However we are interested in the correlation length, which is not necessarily generalized homogeneous because it is not straightforwardly derived from F_s, but rather from the off shell functional discussed previously. Let us assume that it also has a generalized homogeneous form based on this circumstantial relation.

$$\xi_s(t, h) = t^{-\nu} g_\xi\left(\frac{h}{t^\Delta}\right) \tag{55}$$

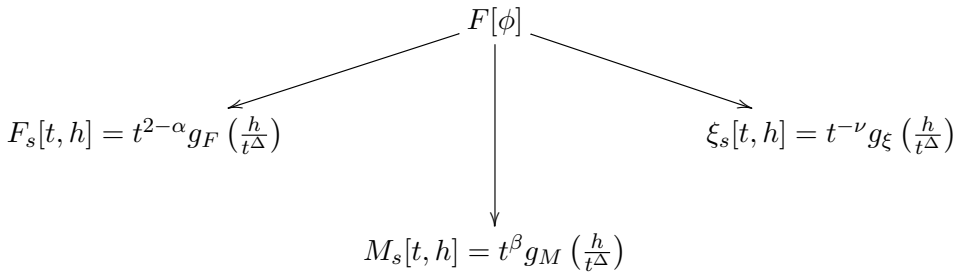
The overall exponent is fixed by our original definition of ν itself, while the argument of the function is fixed by the analogy with the free energy.

3.10 Hyperscaling

The **correlation length diverges** with the exponent ν in proximity of the phase transition. In the assumption that no other length scale diverges at the same time, we can think at it as the dominating scale

of the system close to T_c. In other words: we assume that ξ is responsible for all singular contributions close to T_c (essentially we are moving to a field-theoretical interpretation of the transition).

The scaling form of the correlation length at the critical point is not implied by the scaling form of the free energy. However, we can justify our choice on the assumption that both quantities can be deduced from a parent quantity $F[\phi]$ as follows

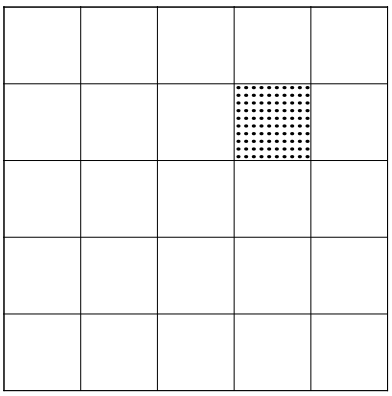


We have already encountered it: it is the off-shell functional that we have constructed naively to explain the phase diagram.

Now imagine that we are approaching T_c so that ξ is still large but finite. We have in general three scales: a the microscopic length (for example the distance between any pair of neighoring spins), L the size of the system, and ξ the correlation length. (Notice that in the above analyses we have often worked in units of a, but for the purpose of this analysis it is useful to introduce it. By construction the system size will be a large integer times a, being for example $L = a \cdot N$ in the $d = 1$ Ising model). Close to T_c we have the ordering

$$a \ll \xi \lesssim L$$

Imagine now that we subdivide the full system in cells of size ξ



By construction the microscopic degrees of freedom are correlated within the cells, so it is natural to argue that the system admits a **mesoscopic** description in which we average degrees of freedom over the cells' size (this is our first hint at renormalization group!) A good averaging procedure is one that does not alter the physics, and in particular it does not alter the partition function. Since the logarithm of the partition function is **dimensionless and extensive** we expect it to scale as the volume in the new units (there is a number L/ξ of cells in the new description)

$$\ln Z = g \cdot \left(\frac{L}{\xi}\right)^d + \dots \tag{56}$$

The coefficient g is a regular function of all dimensionless parameters and the dots hide details that are negligible in the separation $a \ll \xi; L$.

We can compute the divergent behavior of the free energy as

$$t^{2-\alpha} g_f\left(\frac{h}{t^\Delta}\right) = f_s(t,h) \sim \frac{\ln Z}{L^d} \sim \xi_s^{-d} \sim t^{d\nu} \tag{57}$$

We deduce the **Josephson identity**

$$2 - \alpha = d\nu \tag{58}$$

Finally, let us recall that the susceptibility is defined as the second derivative with respect to the magnetic field and that at zero magnetic field it diverges with the exponent γ. The magnetic field can be thought as the **source** of excitations of our field. It is not difficult to convince ourselves that

$$\chi \sim \int \mathrm{d}^d x \langle \phi(x)\phi(y)\rangle = \int \mathrm{d}^d x \frac{1}{|x-y|^{d-2+\eta}} \Phi(|x-y|/\xi) \tag{59}$$

We have already established that the function Φ decays exponentially, so at criticality we estimate the behavior of the integral over a ball of radius ξ

$$\chi \sim \int^{\xi} \mathrm{d}^d x \frac{1}{|x-y|^{d-2+\eta}} \tag{60}$$

This agrees with our cell-based description. We estimate the integral

$$\chi \sim \xi^{2-\eta} \sim t^{-\nu(2-\eta)} \tag{61}$$

Which implies the **Fisher's identity**

$$\gamma = (2-\eta)\nu \tag{62}$$

The relations derived in this section which use the correlator at two points are known as the hyperscaling relations. Combining them with the thermodynamical relations we see that the knowledge of the field-theoretical exponents η and ν is sufficient to determine all the theormodynamical exponents:

$$\begin{array}{ll} \alpha = 2 - \nu d & \gamma = \nu(2-\eta) \\ \beta = \frac{1}{2}(d-2+\eta)\nu & \delta = \frac{d+2-\eta}{d-2+\eta} \end{array} \tag{63}$$

Since we have mentioned the renormalization group already let's spend a couple more thoughts on in and on how it comes about. We have established that the two-point correlator seems to have a crucial role in our interpretation of the phase transition, or at least in describing our point of view on it. At the critical point the exponential drop-off disappears and we have that its form becomes particularly simple

$$\langle \phi(x)\phi(0)\rangle \to G_{\text{crit.}}(x) = \frac{1}{|x|^{d-2+\eta}} \tag{64}$$

It is essentially a powerlaw function! Powerlaws have the property of being homogeneous under a rescaling

$$G_{\text{crit.}}(\lambda\, x) = \lambda^p\, G_{\text{crit.}}(x) \tag{65}$$

with $p = -d + 2 - \eta$. This is essentially analog to the scaling analysis of theormodynamical quantities, but in the field-theoretical framework the scaling is a physical transformation known as **dilatation**. It tells us that if we change the scale at which we look at the system by a factor p, we will see the same physics, at least as long as the two point function is concerned. However, following the original assumption that all critical properties are encoded in ξ, we might argue that similar properties will hold for all correlators. That is, we assume that the system at criticality is **scale invariant** in the sense that it is **self-similar** in the way described above.

Modulo notable exceptions, which include many systems with conformal invariance and some constructive fractals, it is not always straightforward to study the physics of self-similar systems. The renormalization group employs the idea of breaking explicitly scale invariance introducing a reference scale: the system can "flow" according to this scale but the physics (observables) is kept fixed. Using the renormalization group flow scale invariant points, which are thus critical theories, appear as fixed points of the flow!

3.11 Field theory of the Ising model

We have established that the left hand side of the path-integral of the Ising model

$$\mathrm{e}^{-F[\phi]} \simeq \sum_{\{\sigma\}} \mathrm{e}^{-\beta \mathcal{H}[\sigma]} \tag{66}$$

admits a field-theoretical description for which the field ϕ can be interpreted as being the sum of a mean value of the spins and fluctuations over such mean value. We have importantly noted that ϕ^4 is a necessary interaction to capture all the physics of the phase diagram. We have also discussed that close to criticality it is possible to imagine a useful description of the path integral over some mesoscopic variable, say φ, which comes from averaging the system over intermediate scales.

Since the separation between microscopic and macroscopic lengths at criticality becomes infinite, it is feasible to imagine an averaging procedure over an intermediate scale which has degrees of freedom compact enough to interpret them as a continuous field, so to equate

$$\mathrm{e}^{-F[\phi]} \simeq \int D\varphi \, \mathrm{e}^{-S[\varphi]} \tag{67}$$

The question is then: what is the interaction governing the **bare action** $S[\varphi]$? Answering this question is indeed possible using mesoscopic methods, but there is a rather elegant way to do it without having to average our degrees of freedom thus leaving unaltered the physics by conctruction.

Let us adopt the notation in which the Ising interaction is written as a matrix

$$J_{ij} = \begin{cases} J & \text{if } \langle i,j \rangle \text{ is true} \\ 0 & \text{otherwise} \end{cases} \tag{68}$$

and promote the magnetic field to be a local function h_i so that the interaction becomes $h \sum_i \sigma_i \to \sum_i h_i \sigma_i$. Let us finally rescale J_{ij} and h_i by β to simplify the notation. The partition function becomes

$$Z = \sum_{\{\sigma\}} \exp\left\{ \sum_{i,j} J_{ij} \sigma_i \sigma_j + \sum_i h_i \sigma_i \right\} \tag{69}$$

We introduce a Gaussian integral in the auxiliary variables φ_i

$$\int \mathrm{d}\varphi_i \exp\left\{ -\frac{1}{4} \sum_{i,j} \varphi_i \left[J^{-1}\right]_{ij} \varphi_j + \sum_i \varphi_i \sigma_j \right\} = \mathcal{N} \exp\left\{ \sum_{i,j} J_{ij} \sigma_i \sigma_j \right\} \tag{70}$$

and use it to rewrite the partition function

$$Z = \mathcal{N} \sum_{\{\sigma\}} \prod_i \int \mathrm{d}\varphi_i \exp\left\{ -\frac{1}{4} \sum_{i,j} \varphi_i \left[J^{-1}\right]_{ij} \varphi_j + \sum_i \left(\varphi_i + h_i\right) \sigma_j \right\} \tag{71}$$

Given that the measure is translational invariant we can perform the change $\varphi_i \to \varphi_i - h_i$

$$Z = \mathcal{N} \sum_{\{\sigma\}} \prod_i \int \mathrm{d}\varphi_i \exp\left\{ -\frac{1}{4} \sum_{i,j} \left(\varphi_i - h_i\right) \left[J^{-1}\right]_{ij} \left(\varphi_j - h_j\right) + \sum_i \varphi_i \sigma_j \right\} \tag{72}$$

We are now able to perform the summation over the spins σ_i one by one

$$\sum_{\sigma_i = \pm 1} \exp\left\{ \sum_i \varphi_i \sigma_j \right\} = 2 \cosh(\varphi_i) \tag{73}$$

and thus we can rewrite

$$Z = \mathcal{N}' \int \prod_i \mathrm{d}\varphi_i \exp\left\{ -\frac{1}{4} \sum_{i,j} (\varphi_i - h_i) \left[J^{-1}\right]_{ij} (\varphi_j - h_j) + \sum_i \log\cosh(\varphi_i) \right\} \tag{74}$$

As a final transformation we rotate and rescale the auxiliary variable $\varphi_i \to \frac{1}{2}\sum_j \left[J^{-1}\right]_{ij} \phi_j$ and obtain

$$Z = \mathcal{N}'' \int \prod_i \mathrm{d}\varphi_i \exp\left\{ \sum_{i,j} J_{ij}\varphi_i\varphi_j + \sum_i \log\cosh\left[2(J\cdot\varphi)_i\right] + \sum_i h_i\varphi_i \right\} \tag{75}$$

in which we have hidden in the normalization a factor $\exp\left\{-\frac{1}{4} h \cdot J^{-1} \cdot h\right\}$ that does not depend on φ_i.

We clearly see that the magnetic field acts as a source of the quantities ϕ_i, much like it was actiong on the spins σ_i, but this time around the sourced field has continuous values $\varphi_i \in \mathbb{R}$ rather than discrete! Notice also that the measure can be taken as the definition thorough discretization of the measure of a continuous field too

$$\int \prod_i \mathrm{d}\varphi_i \to \int D\varphi$$

Let us introduce coordinates $\mathbf{r}_i$ in our space in which $\mathbf{r}_i$ is the location of the lattice site i. By definition the continuous field is

$$\varphi_i = \varphi(\mathbf{r}_i)$$

Analogously the quadratic interaction becomes an interaction between two coordinates

$$J_{ij} = K(\mathbf{r}_i - \mathbf{r}_j)$$

The discrete symmetries of the statistical model survive the process of transforming to φ: the interaction is a function of $|\mathbf{r}_i - \mathbf{r}_j|$ because it is symmetric (which in the continuum becomes translational and rotational invariant), and the magnetic field still breaks $\mathbb{Z}_2$ invariance (which this time acts on φ_i as $\mathbb{Z}_2 : \varphi_i \to -\varphi_i$).

Let us now perform a lattice Fourier transform

$$\varphi(\mathbf{r}_i) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \varphi(\mathbf{k}) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}_i} \tag{76}$$

$$K(\mathbf{r}_i - \mathbf{r}_j) = \frac{1}{N} \sum_{\mathbf{k}} K(\mathbf{k}) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \tag{77}$$

(a similar formula can be defined for the magnetic field). We use the transformation to rewrite the action terms as

$$\sum_{i,j} J_{ij}\varphi_i\varphi_j = \sum_{\mathbf{k}} K(\mathbf{k}) \left|\varphi(\mathbf{k})\right|^2 \tag{78}$$

$$\sum_i \log\cosh\left[2(J\cdot\varphi)_i\right] = 2\sum_i ((J\cdot\varphi)_i)^2 + \cdots = \sum_{\mathbf{k}} 2|K(\mathbf{k})|^2 \left|\varphi(\mathbf{k})\right|^2 + \ldots \tag{79}$$

Now let's move to the interesting stuff: the exponent of the interior of the partition function can be interpreted as a continuum action $S[\varphi]$. The computation of the quadratic part of this action requires the first order of the expansion of $\ln\cosh$. It becomes

$$S_0[\varphi] = \sum_{\mathbf{k}} \left(K(\mathbf{k}) - 2|K(\mathbf{k})|^2\right) \left|\varphi(\mathbf{k})\right|^2 \tag{80}$$

Assuming the simplest expansion $K(\mathbf{k}) \simeq K_0(1 - \rho\mathbf{k}^2)$ we get that the kinetic term becomes

$$S_0[\varphi] = K_0 \sum_{\mathbf{k}} \left[(4K_0 - 1)\rho^2\mathbf{k}^2 + (1 - 2K_0)\right] \left|\varphi(\mathbf{k})\right|^2 \tag{81}$$

The natural identification mean field is that $1-2K_0$ is proportional to the reduced temperature. We impose the *mean field* condition

$$1-2K_0 = \frac{T-T_c}{T_c} \quad \Rightarrow \quad 4K_0 - 1 = 1 + \mathcal{O}(T-T_c) \tag{82}$$

Close to criticality we have

$$S_0[\varphi] \simeq \frac{1}{2}\sum_{\mathbf{k}} \left[\rho^2 \mathbf{k}^2 + \frac{T-T_c}{T_c}\right] |\varphi(\mathbf{k})|^2 \tag{83}$$

and with a further rescaling $\varphi \to \rho^{-1/2}\varphi$ we get a canonical form

$$S_0[\varphi] \simeq \frac{1}{2}\sum_{\mathbf{k}} \left[\mathbf{k}^2 + m^2\right] |\varphi(\mathbf{k})|^2 \tag{84}$$

with mass $m^2 = \frac{t}{\rho}$. At the critical temperature the mass of the field's excitations goes to zero which is another way to say that the correlation length (which is its inverse) diverges.

Having made the connection with field theory more direct, we can now use a lot of our understanding form the field theory lecture. We can, for example, associate a canonical dimension $(d-2)/2$ to the field φ and classify the importance of further interactions in terms of their (ir)relevance with the canonical field-theoretical analysis. If we neglect symmetry breaking interactions, the first nontrivial self-interaction that complements our action is φ^4. There is a catch however; look for example at the local interactions:

$$\sum_i \log\cosh\left[2(J\cdot\varphi)_i\right] = 2\sum_i ((J\cdot\varphi)_i)^2 - \frac{4}{3}\sum_i ((J\cdot\varphi)_i)^4 + \frac{64}{45}\sum_i ((J\cdot\varphi)_i)^6 + \dots \tag{85}$$

$$\sim \cdots - \frac{4}{3}\int K_0^4\varphi^4 + \frac{64}{45}\int K_0^6\varphi^6 + \dots \tag{86}$$

3.12 Why ϕ^4? Detective work

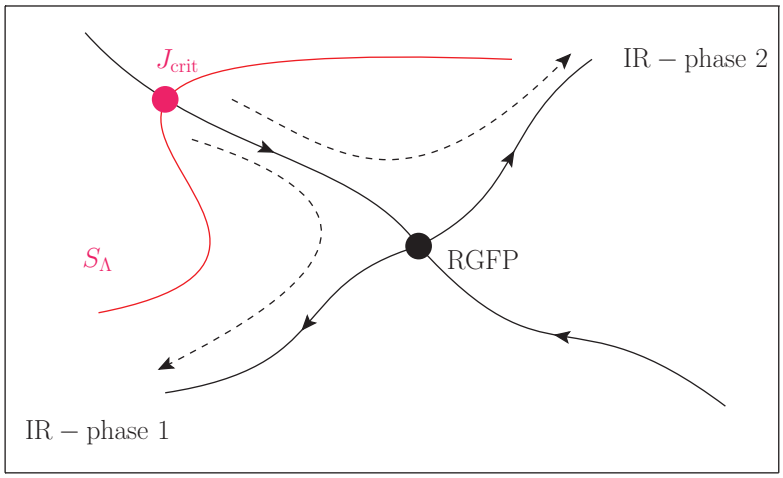
So far we have discovered that:

- At criticality the Ising system is scale invariant.
- The off shell effective theory $F[\phi]$ needs at least a ϕ^4 interaction to explain the full phase diagram of the Ising model.
- We have enough parameters to tune the mass of the field theory to zero, correspondingly the correlation length to infinity, but no other (besides the magnetic field).
- The theory becomes trivial at $d = 4$.

Instead of pinpointing directly to scale invariant models (which can be done in some instances), we try to discover scale invariant theories in hindsight: we first construct equations outlining the scale dependence of field theories which preserves the physics, and then look for theories which do not change with the scale. The renormalization group is the method that does exactly this!

Having already seen hints of the process of regulatization/renormalization from the QFT lectures we can already guess, based on the form of the effective interaction and on its triviality at $d = 4$ that the field theory underlining the universality class of the Ising model is the well-known ϕ^4 model. The way we use it is however very different from what you've learnt from QFT. The field theory analysis of the previous section can be understood as the derivation of the microscopic/ultraviolet action $S_\Lambda[\varphi]$ which we input in the process of renormalization and which depends both on a ultraviolet scale $\Lambda \sim 1/a$ (a is the lattice spacing) and parametrically on the size of the interactions J_{ij} and h_i.

In practice this is how it works: The red line represents how $S_\Lambda[\varphi]$ changes parametrically as a function of J_{ij} and eventual other microscopic parameters. We have embedded the parametrized actions in the bigger space (the whole rectangle) of all theories in which we have also drawn the RG flow of the ϕ^4 theory computed with some method (to be established). The RG fixed point has IR relevant and IR irrelevant directions (arrows point to the IR). What is important for us is where the **critical surface** of relevant intersects the parametrized red line: that intersection is the critical point of the Ising model!



It goes without saying that, as evinced from the figure, the fixed point and the critical point are not the same thing!!! More generally the **phase diagram** and the **RG diagram** are not the same thing. In fact, the critical point, though related to scale invariance, cannot really be scale invariant because it has a physical scale dependence on the lattice spacing a. However, by looking at the system for increasingly large scales (the thermodynamical limit), we can systematically wash out the dependence on a and by tuning to criticality use the RG to fall into the RG fixed point with arbitrary precison, which in practice means that we observe scale invariance of the physical system.

Finally, having in mind the Ising system, we displayed two possible outcomes for the RG in the IR, and hence for IR physics. If we start slightly on the left of the critical point we end up in phase 1, which could be the ordered phase, while if we start slightly on the right of the critical point we end up in phase 2, which could be the disordered phase. That is to say that some points of the **RG diagram** are endpoints of the flow and have a special meaning in the sense of the **phase diagram** being associated to a specific phase. More often than not, these points are the Gaussian fixed point(s), generally associated to the high-temperature (disorder) and low-temperature phases (order). The Gaussian fixed points would be the actual Gaussian point of the field theory (corresponding to the free theory), but also the point at infinity, which is also often called Gaussian by lattice theorists.

3.13 Generalizations

The Ising critical point can be understood as a *bicritical* point since it comes from the degeneracy of two minima related by $\mathbb{Z}_2$ symmetry. The action would be

$$S[\varphi] = \int \mathrm{d}^d x \Big\{ \frac{1}{2}(\partial\phi)^2 + g_1\varphi + g_2\varphi^2 + g_4\varphi^4 \Big\} \tag{87}$$

Let us discuss briefly some standard generalization of the Ising according to the above summary.

The *Blume-Capel model* is defined as a simple generalization of the Ising model including a spin variable $\sigma_i = \pm 1$ and a *vacancy* variable $t_i = 0, 1$ indicating whether the site is empty or occupied. An Hamiltonian with nearest-neighbor interactions is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j t_i t_j + \Delta \sum_i t_i - h \sum_i s_i t_i - h_3 \sum_{\langle i,j \rangle} (s_i t_i t_j + s_j t_j t_i) - K \sum_{\langle i,j \rangle} t_i t_j \tag{88}$$

There are three new couplings: h_3 a "staggered" magnetic field and Δ and K controlling the population of the sites. You can imagine that there is a critical point of J as a function of Δ when all other couplings are zero, which terminates in a *tricritical* critical point (J_c, Δ_c). This critical point is associated to the **tricritical Ising model**.

One way to understand the generalization would be to consider the product $t_i \times \sigma_i = \{0,1\} \times \pm 1 = \{-1,0,0,1\} \simeq \{-1,0,1\} = S_i$. Therefore we have moved to bicritical to multicritical by increasing the spin of the spin variable. However to observe a phase with higher ciritcality we also need to have one more coupling, which in this case would be Δ. An action for the new variable would be

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j + \Delta \sum_i S_i^2 - h \sum_i S_i \tag{89}$$

On the field theory side we need an action with up to three minima that might become degenerate and more couplings. Let us guess the minimum field theory requirement:

$$S[\varphi] = \int \mathrm{d}^d x \Big\{ \frac{1}{2} (\partial\phi)^2 + g_1 \varphi + g_2 \varphi^2 + g_3 \varphi^3 + g_4 \varphi^4 + g_6 \varphi^6 \Big\} \tag{90}$$

Now let us state one simple fact: the above action has upper critical dimension $d_c = 3$.

Further generalizations in this direction are rather straightforward: increase the spin of the spin variable S_i and simultaneously increase the number of interactions such as Δ. If this procedure is followed one ends up with a spin-$(n-1)/2$ model that in the continuum has n possible degenerate minima and field theory action

$$S[\varphi] = \int \mathrm{d}^d x \Big\{ \frac{1}{2} (\partial\phi)^2 + g_1 \varphi + g_2 \varphi^2 + \cdots + g_{2n-2} \varphi^{2n-2} + g_{2n} \varphi^{2n} \Big\} \tag{91}$$

These are together known as the **multicritical models** and the nth one has upper critical dimension $d_c = 2n/(n-1)$. It is easy to see that they are not mean field only in dimension $d = 2$!

Another generalization goes in the direction of changing the symmetry rather than the structure of the interactions. Let us promote the spin variable to a vector with $n = N + 1$ components $\vec{S}_i = \{S_{1,i}, \ldots, S_{n,i}\}$ which is constrained $\vec{S}_i \cdot \vec{S}_i = 1$. This is the n-component model and has symmetry $O(N)$ (isometries of the sphere). The Hamiltonian would be

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{S}_i \vec{S}_j - h \sum_i \vec{S}_i \tag{92}$$

You can imagine that the field theoretical description promotes the field φ to a multiplet $\Phi = \{\varphi_1, \ldots, \varphi_N\}$. However there is no reason to believe that the interaction changes

$$S[\varphi] = \int \mathrm{d}^d x \Big\{ \frac{1}{2} (\partial\Phi)^2 + \sum_l g_{1,l} \Phi_l + g_2 \Phi^2 + g_4 \Phi^4 \Big\} \tag{93}$$

except that now the magnetic field can turn on any component. The analysis of the free energy leads to the standard "sombrero hat" discussion which generalizes our previous sections. The upper critical dimension is $d_c = 4$ which would make us expect that the model is nontrivial in $d = 2$ and $d = 3$, but this is not entirely true. Let us state an implication of the **Mermin-Wagner theorem** (without proof): In $d = 2$ the only nontrivial $O(N)$ model is the case $N = 1$ corresponding to the Ising model. In dimension $d = 3$ they are however interesting and nontrivial!

The final generalization involves again the enhancement of the number of components, but rethinks the interaction. Consider a lattice variable which can take q distinct values/states: $\zeta_i = \{1, \ldots, q\}$ and construct the interaction

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta(\zeta_i, \zeta_j) \tag{94}$$

The case $q = 2$ can be related to the Ising model making the correspondence between states and spins $\{1,2\} \leftrightarrow \{-1,1\}$ (and opportunely redefining J). The general case is called the q-states **Potts model** and

is invariant under the permutations S_q of the states (notice that $S_2 \simeq \mathbb{Z}_2$ as expected). With the exception of the case $q = 2$ this model is generally associated to a field theory with a type of φ^3 interaction and $d_c = 6$. The model is very interesting because it can be analitically continued in q. There are several continuations of interest, but the most important one is the continuation to $q = 0$ which leads to the model of percolation described in the invitation! (Imagine the one-state model as a model populating the sites.)

4 Renormalization: general notions and real-space

Having understood that critical theories are also scale-invariant theories we now seek a method to identify the signatures of a scale invariant theory and, possibly, to compute some of its critical properties.

Rather than looking directly for scale invariant configurations, it is in general simpler to study the dependence on the scale of physical systems by looking at it at increasingly large scales (less resolution) while simultaneously keeping the physical observable fixed. This action of looking at the system for increasingly large scales can be used to construct a semigroup that acts on the system's defining path integral (or actions, or Hamiltonian) which is known as the renormalization group.

In general a step of the renormalization group is composed of two parts:

- A coarse-graining of the system, which reduces the number of degrees of freedom.
- A rescaling of the system, which maps the new coarse-grained system size over the old one.

The first part is necessary to ensure that we are loking at the system with less resolution, and correspondingly at a larger scale. The second part is necessary to compare the changes between the old and new system and understand what changes it underwent.

There is a wild variety of possible realizations of the above two procedures which go under the generic name of **renormalization group schemes**. Schemes differ in method and purpose, ranging from the desire of computing either just critical exponents or full observables.

4.1 Real-space renormalization in $d = 1$

Consider the one dimensional Ising model, define a rescaled coupling $K \equiv \beta J$, and take $h = 0$

$$Z = \sum_{\{\sigma\}} \prod_{n\in\mathbb{N}_0} \mathrm{e}^{K(\sigma_{2n-1}\sigma_{2n}+\sigma_{2n}\sigma_{2n+1})} \tag{95}$$

$$= \sum_{\{\sigma\}} \mathrm{e}^{K(\sigma_1\sigma_2+\sigma_2\sigma_3)}\mathrm{e}^{K(\sigma_3\sigma_4+\sigma_4\sigma_5)}\dots \tag{96}$$

A **decimation** process is one for which we sum over part of the degrees of freedom to obtain an effective decription over the others. Begin by performing the sum over the even spins using

$$\sum_{\sigma_{2n}=\pm 1} \mathrm{e}^{K(\sigma_{2n-1}\sigma_{2n}+\sigma_{2n}\sigma_{2n+1})} = \mathrm{e}^{K(\sigma_{2n-1}+\sigma_{2n+1})} + \mathrm{e}^{-K(\sigma_{2n-1}+\sigma_{2n+1})}$$

We are left with a sum $\{\sigma'\}$ restricted only on the sites which had odd labels

$$Z = \sum_{\{\sigma'\}} \prod_{n\in\mathrm{odd}} \left\{\mathrm{e}^{K(\sigma_n+\sigma_{n+2})} + \mathrm{e}^{-K(\sigma_n+\sigma_{n+2})}\right\} \tag{97}$$

Now we want to interpret this summation as a partition function over the odd sites which is of the form of the original one, but has a different coupling K'. We are looking for a normalization $f(K)$ and a new coupling K' as follows

$$e^{K(\sigma_{2n-1}+\sigma_{2n+1})} + e^{-K(\sigma_{2n-1}+\sigma_{2n+1})} = f(K)e^{K'\sigma_{2n-1}\sigma_{2n+1}}$$

Inserting the values ± 1 in the spins we find only two independent equations

$$e^{2K} + e^{-2K} = f(K)e^{K'} \qquad 2 = f(K)e^{-K'} \tag{98}$$

which are simply solved as

$$K' = \frac{1}{2}\ln\cosh(2K) \qquad f(K) = 2\cosh(2K)^{\frac{1}{2}} \tag{99}$$

We now can relabel the odd spins and write the partition function making the dependence on N and K explicit

$$Z(N,K) \quad = \quad f(K)^{N/2} Z(N/2, K') \tag{100}$$

This relation establishes the link between the partiton function and its decimated form. For large N we know that the free energy is extensive with the system, therefore up to a factor of β we have at large N that

$$F(K) = \frac{1}{N}\log Z(N,K) \tag{101}$$

Taking the log on both sides of the relation for the decimated partiton function we get

$$F(K) = \frac{1}{2}\ln f(K) + \frac{1}{2}F(K') \tag{102}$$

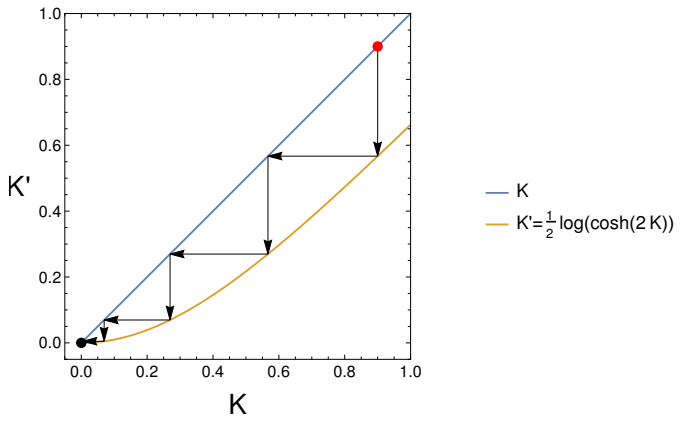
which can be solved for $F(K')$ as

$$F(K') = 2F(K) - \ln(2\sqrt{\cosh(2K)}) \tag{103}$$

The renormalization group flow of the coupling K is given by the equation

$$K' = \frac{1}{2}\ln\cosh(2K) \tag{104}$$

Applying interatively this relation sends $K \to 0$ very quickly. This is telling up that decimating the sistem is equivalent to lowering the coupling K and that the lowering does not stop until $K = 0$, which corresponds to the high temperature phase of the system in which disorder is guaranteed. This is in agreement with the fact that the system in $d = 1$ has no phase transition and we are always in the disordered phase! We could also say that the end point $K = 0$ is a class representative for the disordered phase.



Applying the transformation in reverse, *which you should not do in general with RG transformations because most are irreversible*, you get that K is sent to infinity which is the low temperature phase. We can use this

reversed procedure to compute the partition function approximately at any value of K. Take K' very small, say $K' = 0.01$, for which we can estimate the free energy by just counting the spin states $F(K') \simeq \ln 2$. Now apply

$$K = \frac{1}{2} \operatorname{arccosh} \mathrm{e}^{2K'} \tag{105}$$

$$F(K) = \frac{1}{2} \ln 2 + \frac{1}{2} K' + \frac{1}{2} F(K') \tag{106}$$

ten times; we get a result almost identical to the thermodynamical limit and gets better with more steps

K	$F(K)$	exact
0.01000	0.69315	0.69320
0.10033	0.69815	
0.32745	0.74581	
0.63625	0.88320	
0.97271	1.10629	1.10630
1.31671	1.38608	
1.66264	1.69797	
2.00905	2.02688	
2.35558	2.36454	
2.70215	2.70663	2.70663
3.04872	3.05096	

Improvement can be made by setting the initial condition to a smaller value so that the initial ansatz of $\ln 2$ is more correct.

The reason why the real-space renormalization works so well in $d = 1$ is because there is no approximation involved in deriving any of the relations for the free energy or the running coupling. The iteration of the decimation procedure does end up summing all the spins and we have not neglected any contribution in so doing.

4.1.1 Real-space renormalization in $d = 2$

In $d = 2$ we are not as lucky as in $d = 1$, unfortunately. Imagine that we are summing over spins in a checkboard pattern. Fix one spin to be summed over and its four neighbors. The summation involves the chosen spin σ and the four cardinal neighbors σ_C with $C = N, S, W, E$.

$$\sum_C \sum_{\sigma=\pm 1} \mathrm{e}^{K\sigma\sigma_C} = \mathrm{e}^{K(\sigma_N+\sigma_S+\sigma_W+\sigma_E)} + \mathrm{e}^{-K(\sigma_N+\sigma_S+\sigma_W+\sigma_E)} \tag{107}$$

Spins which were originally separated by $\sqrt{2}$ are nearest neighbors after the sum (example: south-west and south-east), while spins that were separated by 2 are next-to-nearest neighbors (example: north-south and east-west). It is not possible to perform the same trick of the previous section in rewriting the interaction. At the least we can make the ansatz

$$\mathrm{e}^{K(\sigma_N+\sigma_S+\sigma_W+\sigma_E)} + \mathrm{e}^{-K(\sigma_N+\sigma_S+\sigma_W+\sigma_E)} \tag{108}$$

$$= f(K) \mathrm{e}^{\frac{1}{2}K_1(\sigma_N\sigma_W+\sigma_W\sigma_S+\sigma_S\sigma_E+\sigma_E\sigma_N)+K_2(\sigma_N\sigma_S+\sigma_E\sigma_W)+K_3\sigma_N\sigma_S\sigma_W\sigma_E} \tag{109}$$

The equations obtained evaluating the spins to ± 1 are much more complicate. Neglecting f and K_3 we get

$$K_1 = \frac{1}{4} \log \cosh(4K) \qquad K_2 = \frac{1}{8} \log \cosh(4K) \tag{110}$$

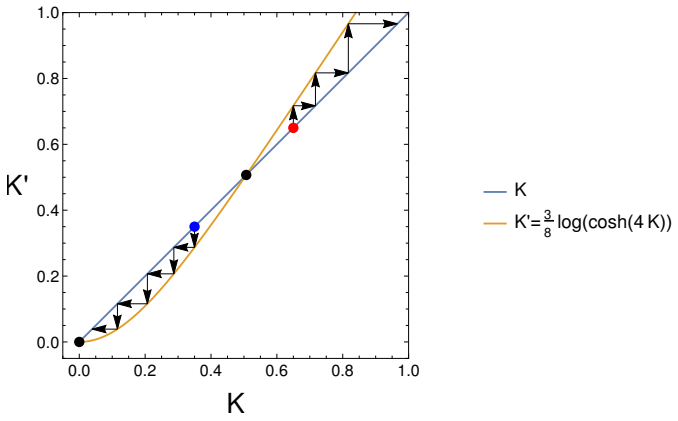
One could imagine projecting the result only on the $K_1 \to K$ coupling that represents the original interaction. If this is done, however, we reproduce the same situation as in the case $d = 1$. A meaningful approximation

is by observing that the interaction K_2 enhances K_1, so we can imagine setting $K_{1,2} \to K$. This means that after a decimation the coupling becomes in our approximate scheme

$$K' = K'(K) = K_1 + K_2 = \frac{3}{8} \log \cosh(4K) \tag{111}$$

This new recursion has a fixed point (UV) at $K_c \simeq 0.556981$. The critical value should be compared to the exact result from the Onsager solution which is $K_c = 0.44069$.

Points on the left of K_c are sent to $K = 0$ in the IR, while points on the right are sent to $K = \infty$. This has the following interpretation: for $T > T_c$ we have $K < K_c$ (recall $K \sim \beta$ which reverses the sign $(T - T_c) \sim (K_c - K)$) and the RG send us to $K = 0$ which is the high temperature fixed point which is representative of the disordered phase; for $T < T_c$ we have $K > K_c$ and the RG send us to $K = \infty$ which is the low temperature fixed point which is representative of the ordered phase! The idea of class representatives still works!



Having made the approximation of neglecting K_3 and projecting K_1 and K_2 into one another we are completely in RG territory and we certainly cannout use a recursion relation to compute the partition function (we haven't derived it anyway). We can however attempt an estimation of a critical exponent, specifically α because we are looking at the free energy. First we find the expression of the decimated free energy from the one of $f(K)$

$$F(K') = 2F(K) - \log(2 \operatorname{arccosh}(2K)^{1/2} \cosh(4K)^{1/8})$$

Then assume that it has the non-analytic term

$$F(K) = A \left| K - K_c \right|^{2-\alpha} + \text{ regular}$$

We perform the Taylor expansion of

$$K'(K) = K_c + (K - K_c) \left. \frac{\mathrm{d}K'}{\mathrm{d}K} \right|_{K_c} + \dots$$

and use it inside

$$F(K') = A \left| K' - K_c \right|^{2-\alpha} \simeq A \left| K_c + (K - K_c) \left. \frac{\mathrm{d}K'}{\mathrm{d}K} \right|_{K_c} - K_c \right|^{2-\alpha} = A \left| K - K_c \right|^{2-\alpha} \left| \left. \frac{\mathrm{d}K'}{\mathrm{d}K} \right|_{K_c} \right|^{2-\alpha}$$

Using the decimation of the free energy we have to compare (neglecting regular terms)

$$F(K') \sim A \left| K - K_c \right|^{2-\alpha} \left| \left. \frac{\mathrm{d}K'}{\mathrm{d}K} \right|_{K_c} \right|^{2-\alpha} = 2A \left| K - K_c \right|^{2-\alpha}$$

which implies

$$\left|\left.\frac{\mathrm{d}K'}{\mathrm{d}K}\right|_{K_c}\right|^{2-\alpha} = 2$$

and therefore

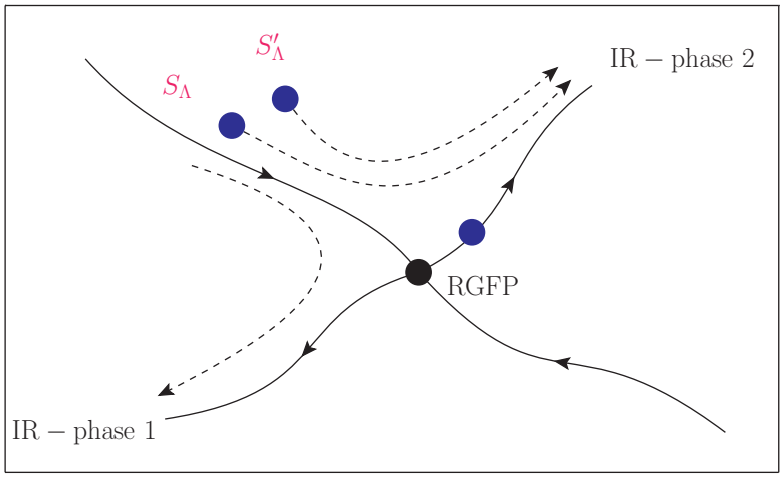
$$\alpha = 2 - \frac{\log 2}{\log\left|\left.\frac{\mathrm{d}K'}{\mathrm{d}K}\right|_{K_c}\right|} \simeq 0.131$$

which is to be compared with the result $\alpha = 0(\log)$ of the Onsager solution.

4.2 Renormalization in statistical vs quantum field theory

The renormalization group is applied, together with the paradigm of universality, to both the physics of low energy systems like a ferromagnet and the physics of high energy particles. While the methods are very much the same, their interpretations are very different. The renormalization group is, for obvious reasons, a very useful method when it comes to understand the large-scale behavior of a low energy system. In fact, we can use it to argue that the large scale physics of a system is essentially insensitive of the microscopic details, which is equivalent to say that the remaining physics is thus a universal feature.

Universality therefore helps low energy physics by decreasing the importance of the microscopic details when looking at macroscopic or long range observables. However, the good properties in the low energy regime become potential dangers if one is interested to high energies. Recall the figure from before: Suppose that we find that our system is in either phase, but that we are interested in the details of the microscopic theory.



Both the microscopic actions S_λ and S'_λ as well as infinitely many other equally good microscopic actions approach the same physics in the IR. This means that if we try to follow the RG "backward" – which we should never do so don't tell anybody! – it becomes increasingly difficult for us to distinguish among those actions unless we perform experiments up to the energy scale Λ.

This is a backward way to approach the problem of constructing infrared theories of unknown ultraviolet degrees of freedom. The premise of the **effective approach** to the construction of infrared theories is to resort on a combination of renormalization group methods and expansion in powers of $\frac{1}{\Lambda}$ to distinguish which of all possible bare theories including S_λ and S'_λ reproduces the correct infrared physics.

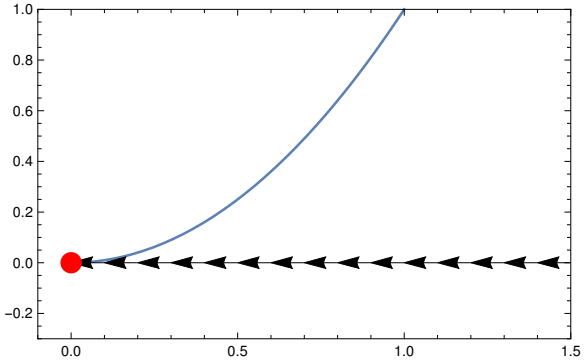
The big problem comes if one wants to find an **ultraviolet complete** theory, meaning one theory that works for all energies and has $\Lambda \to \infty$. Suppose that we are on one of the trajectories emanating from S_λ and S'_λ. If our experiments reach energies larger than Λ we will not be able to make predictions anymore because the renormalization group pushes us further away: we need to perform more and more experiments to determine the location of our microscopic model at thigher energies. There is however one way to find a theory which is under control in the limit $\lambda \to \infty$: it is sufficient to stay *exactly* on the trajectory emanating from the fixed point as shown in the figure. If the theory is required to stay on such trajectory we already know its

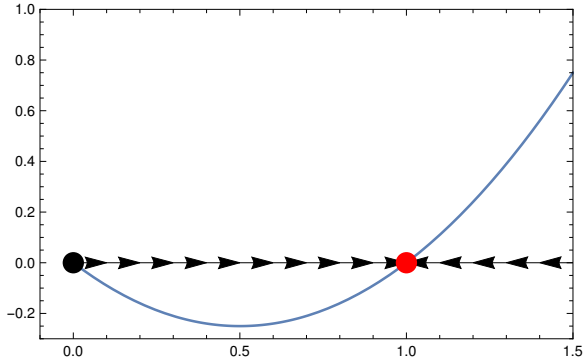
limit $\lambda \to \infty$! **Asymptotically free** and **asymptotically safe** theories are all examples of theories which are required to lie on a specific critical surface in the space of all couplings that ensures that the UV is under control.

Let us try to analyze ϕ^4 in the light of this discussion. We anticipate the beta function computed in dimensional regularization (we will make sense of dimensional regularization later on)

$$\beta_\lambda = (d-4)\lambda + A\lambda^4$$

The constant A has a specific value, but we take $A = 1$ for simplicity. The function is plotted for $d = 4$ on the left and for $d = 3$ (representative for $d < 4$) on the right, the arrows point in the direction of the infrared.





In $d = 4$ the IR fixed point is Gaussian, while in $d < 4$ the IR fixed point is nontrivial. In both cases the microscopic details are washed away by the RG flow, the difference is that in $d = 4$ we end up in the Gaussian theory and correspondingly the physics is ruled by mean field exponents. In $d < 4$ the scaling is nontrivial and this is seen thorugh the presence of a nontrivial IR fixed point which gives corrections to the scaling exponents.

The first plot is also useful for the analysis of the ϕ^4 model as a UV theory. Following the arrows backward we see that the coupling grows higher and higher, regardless of where we start from. In practice, the integrated flow diverges for a finite (albeit very high) scale at a singularity known as the **Landau pole**. This means that even a weakly interacting theory in the IR becomes a strongly interacting one in the UV. This problem can be resolved by moving the initial condition closer and closer to the Gaussian point, and eventually the Landau pole disappears iff the UV of the theory is chosen to be the Gaussian point itself. In other words: ϕ^4 is UV complete theory only if it is **trivial**! Models like QCD avoid this necessity of being trivial to make sense in the UV by having beta functions that are negative, which imply that their coupling becomes smaller in the UV because of universality, but at the same time they switch the position of the strongly interacting physics to the IR, which is what leads to the very complicate low energy nuclear physics.

5 Renormalization: from scaling to the renormalization group

From now on the field ϕ should be thought as a mesoscopic variable, which describes the intermediate physics of some microscopic degrees of freedom such as those of the Ising model.

Thanks to the hyperscaling hypothesis we have connected the scaling behavior of the reduced temperature to the one of the diverging correlation length. Now that we are strong of field theory arguments we ask what happens to the system if we scale

$$\xi \to \xi/\lambda$$

close to criticality. From $\xi \sim t^{-\nu}$ with t the reduced temperature, we argue that

$$t \to \lambda^{\Delta_t} t \qquad \text{with} \quad \Delta_t = 1/\nu$$

From the two-point function $\langle\phi(r)\phi(0)\rangle \sim t^{-d+2-\eta}$ we deduce

$$\phi(x) \to \lambda^{\Delta_\phi}\phi(x) \qquad \text{with} \quad \Delta_\phi = \frac{d-2+\eta}{2}$$

Notice that if the Fourier transform of $\phi(x)$ is denoted $\tilde{\phi}(p)$ we expect $\Delta_\phi = \Delta_{\tilde{\phi}} + d$, and therefore

$$\Delta_{\tilde{\phi}} = -\frac{d+2-\eta}{2}$$

so we have to be careful when working in momentum space. From the coupling of the magnetic field $\int B\phi$ we deduce

$$\Delta_\phi + \Delta_B = d \qquad \text{which} \quad \text{implies} \quad \Delta_B = \frac{d+2-\eta}{2}$$

Using that on shell the magnetization scales as $\phi \sim t^\beta$ and the equation of state $\phi \sim B^{1/\delta}$ it is easy to prove the relations of the exponents β and δ with the field theory exponents from $\beta = \nu\Delta_\phi$ and $\delta = \Delta_B/\Delta_\phi$.

Any other quantity will behave with some appropriate scaling behavior. Consider an arbitrary contribution to the free energy

$$F[\phi] = \int d^d x\, g_{\mathcal{O}}\, \mathcal{O}(x)$$

and the rescaling $x \to x' = x/\lambda$ on the interaction

$$\mathcal{O}(x) \to \mathcal{O}'(x') = \lambda^{\Delta_{\mathcal{O}}}\mathcal{O}(x)$$

which implies that the coupling scales

$$g_{\mathcal{O}} \to \lambda^{d-\Delta_{\mathcal{O}}} g_{\mathcal{O}}$$

The interaction is said to be

- **relevant** if $\Delta_{\mathcal{O}} > 0$,
- **irrelevant** if $\Delta_{\mathcal{O}} < 0$,
- and **marginal** if $\Delta_{\mathcal{O}} = 0$.

Naively we understand that relevant directions are increasingly important at large scales, irrelevant ones can be neglected, and marginal has to be treated with care.

5.1 Momentum-shell and Gaussian scaling

Consider a path integral of the form

$$\mathrm{e}^{-F} = \int D\varphi\, \mathrm{e}^{-S[\varphi]} \tag{112}$$

Let us introduce an ultraviolet scale Λ related to the microscopic degrees of freedom of the theory. We also introduce a new scale μ related to the ultraviolet one by a rescaling $\mu = \lambda\Lambda$ and split the Fourier transformed field in a **momentum-shell**

$$\varphi_k = \varphi_k^+ + \varphi_k^-$$

in which φ_k^+ are the high wavelength modes $\mu < k < \Lambda$, and φ_k^- are the short wavelength modes $0 < k < \mu$. Under rather general assumptions we can imagine the action to decompose as follows

$$S[\varphi] = S_0[\varphi^+] + S_0[\varphi^-] + S_{\text{int}}[\varphi^+, \varphi^-]$$

in which we included the free action

$$S_0[\varphi] = \frac{1}{2}\int \mathrm{d}^d x \Big\{(\partial\varphi)^2 + m_0^2\Big\} = \frac{1}{2}\int^\Lambda (k^2 + m_0^2)\varphi_k\varphi_{-k}$$

and interactions amond the different modes. By construction

$$S_0[\varphi^-] = \frac{1}{2}\int_0^{\lambda\Lambda}(k^2+m_0^2)\varphi_k^-\varphi_{-k}^-\,; \qquad S_0[\varphi^+] = \frac{1}{2}\int_{\lambda\Lambda}^{\Lambda}(k^2+m_0^2)\varphi_k^+\varphi_{-k}^+$$

The procedure that is analog to the decimation in real space is to perform the path-integral in steps: we integrate explicitly the high frequencies

$$\begin{aligned} \mathrm{e}^{-F} &= \int D\varphi^- D\varphi^+ \,\mathrm{e}^{-S_0[\varphi^+]-S_0[\varphi^-]-S_{\rm int}[\varphi^+,\varphi^-]} && (113)\\ &= \int D\varphi^- \,\mathrm{e}^{-S_0[\varphi^-]}\int D\varphi^+ \,\mathrm{e}^{-S_0[\varphi^+]-S_{\rm int}[\varphi^+,\varphi^-]} && (114)\\ &= \mathcal{N}\int D\varphi^- \,\mathrm{e}^{-S_0[\varphi^-]}\langle \mathrm{e}^{-S_{\rm int}[\varphi^+,\varphi^-]}\rangle_+ && (115)\end{aligned}$$

In the last step we have considered the average over a path-integral for the φ^+ modes which are weighted by the action $S_0[\varphi^+]$. We can interpret the path integral as over the low modes only

$$\mathrm{e}^{-F} = \int D\varphi^- \,\mathrm{e}^{-S'[\varphi^-]} \qquad (116)$$

Equating we find the relation

$$S'[\varphi^-] = S_0[\varphi^-] - \log\langle \mathrm{e}^{-S_{\rm int}[\varphi^+,\varphi^-]}\rangle_+ \qquad (117)$$

which is an effective "mesoscopic" action for the φ^- modes obtained by integrating the high modes.

The computation of such an action is generally rather hard. The reason is that we have not yet implemented any of the methods of quantum field theory to manipulate the partition function, which still contains, for example, disconnected diagrams. We are not interested now in the full result, but we chase one simple observation that can be understood by neglecting the interactions. So now let's take $S_{\rm int}=0$ and proceed noticing that in this case

$$\log\langle \mathrm{e}^{-S_{\rm int}[\varphi^+,\varphi^-]}\rangle_+ = \log\langle 1\rangle_+ = \text{const.}$$

The effective description of the low modes becomes thus trivially

$$S'[\varphi^-] = S_0[\varphi^-] = \frac{1}{2}\int_0^{\lambda\Lambda}(k^2+m_0^2)\varphi_k^-\varphi_{-k}^- \qquad (118)$$

Now we ask the question: Is the system scale invariant? To correctly pose it we have to be able to compare the new action with the old one $S_0[\varphi]$. The two actions are in form equal, but they extend to two different ranges of momentum integration. To amend this we choose the new integration variable $k' = k/\lambda \to k$

$$S'[\varphi^-] = \frac{1}{2}\int_0^{\Lambda}\left(\lambda^{d+2}k'^2+\lambda^d m_0^2\right)\varphi_{k'}^-\varphi_{-k'}^- \qquad (119)$$

and rescale the field such that the normalization of the kinetic term is left invariant $\varphi_k'^- = \lambda^{-(d+2)/2}\varphi_k^- \to \varphi_k^-$ (which agrees with the scaling $\Delta_{\tilde\phi}$ at $\eta=0$). We obtain

$$S'[\varphi^-] = \frac{1}{2}\int_0^{\Lambda}\left(k^2+\lambda^{-2}m_0^2\right)\varphi_k^-\varphi_{-k}^- \qquad (120)$$

in which we also renamed the integration variable in the second line. The mass term receives an explicit rescaling: we define the new mass

$$m^2(\lambda) = \lambda^{-2}m_0^2$$

of the rescaled theory

$$S'[\varphi^-] = \frac{1}{2}\int_0^{\Lambda}\left(k^2+m^2(\lambda)\right)\varphi_k^-\varphi_{-k}^- \qquad (121)$$

The new field can now be compared with the old and we can frame the previous question as: what are the conditions such that $S'[\varphi] = S[\varphi]$?

One way to answer this question is by considering the **renormalization group time** $t = \log(\mu/\Lambda) = \log\lambda$ and the zeroes of the beta function

$$\beta_{m^2} \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda} m^2(\lambda) = -2\, m^2(\lambda)$$

There are two interesting possibilities.

The first one is that $m^2 = 0$: this is the **Gaussian fixed point** which we argue to be related to the **low temperature limit** of the field theory (recall the analysis of the blocking of the Ising model: this corresponds to the $K \to \infty$ limit). At $m^2 = 0$ and in absence of interactions the path integral is dominated by free fluctuations which are weighted only by their energy through a Boltzmann's distribution. The second one happens at $m^2 = \infty$: this is the "other" Gaussian fixed point which we call **high temperature fixed point** and corresponds to the **high temperature limit** of the field theory. These two fixed points exist in general for any interaction S_{int}. The above analysis highlights also the necessity of rescaling the system, and specifically field and couplings, to appropriately compare it with itself at another scale!

Can we use the Gaussian fixed point to determine the critical exponents of the theory? Yes, consider that the square of the correlation length is the inverse of the mass and that under a rescaling. We find the scaling of the correlation length from

$$\nu = -\left(\frac{\mathrm{d}m^2}{\mathrm{d}\lambda}\right)^{-1}$$

which is the simplest Gaussian example can be estimated as $\nu = \frac{1}{2}$.

5.2 Interactions and the ϵ-expansion

We expect that for nontrivial S_{int} there can be a new point at which scale invariance is achieved. For example, one could introduce $S_{\mathrm{int}} = g_0 \int \varphi^4$ and compute the new contribution

$$-\log\langle \mathrm{e}^{-S_{\mathrm{int}}[\varphi^+,\varphi^-]}\rangle_+$$

We expect that the integration of the high fluctuations will contribute to the **renormalization** of the couplings. If one is interested to φ^2 and φ^4

$$-\log\langle \mathrm{e}^{-S_{\mathrm{int}}[\varphi^+,\varphi^-]}\rangle_+ \simeq \int a(\mu,\Lambda;g_0,m_0^2)\varphi^2 + \int b(\mu,\Lambda;g_0,m_0^2)\varphi^4 + \dots$$

The computation of a and b is not particularly easy because it contains diagrams which are disconnected... which is why we are going to do it in the next section with a similar but simpler method. However we can argue the structure of the result: taking into account the necessary rescaling of field and momentum variables, as well as the contributions of a and b, and neglecting both dots and higher orders we get

$$\beta_{m^2} = -2m^2 + A(m^2)g\,, \qquad \beta_g = (d-4)g + B(m^2)g^2 \tag{122}$$

This system has a new fixed point which is positive for $d < 4$:

$$g^* = (4-d)/B(m^{*2}) \qquad m^{*2} = (4-d)\frac{A(m^{*2})}{2B(m^{*2})}$$

The nontrivial fixed point becomes the Gaussian one if we follow $d \to 4$. This is where the theory approaches the mean field value and becomes Gaussian. *By construction at the Gaussian point we have that $\nu = \frac{1}{2}$ and $\eta = 0$, which we can combine with the hyperscaling relations to establish that the thermodynamical exponents take the mean field values.*

We can study the **stability matrix** which linearizes the flow close to the fixed point

$$\mathcal{M} \equiv \left[\begin{array}{cc} \frac{\partial \beta_{m^2}}{\partial m^2} & \frac{\partial \beta_{m^2}}{\partial g} \\ \frac{\partial \beta_g}{\partial m^2} & \frac{\partial \beta_g}{\partial g} \end{array} \right] \Bigg|_{m^{*2},g^*} \tag{123}$$

The negative of its eigenvalues are generically called **critical exponents** and denoted θ_i. A convenient way to show the critical exponents is by taking $d = 4 - \epsilon$. We have

$$\theta_1 = 2 - \frac{A'(m^{*2})}{B(m^{*2})}\epsilon\,, \qquad \theta_2 = -\epsilon \tag{124}$$

The scaling of the mass is replaced by the inverse of θ_1 which includes a **mixing** of the original operators of the free theory φ^2 and φ^4. In other words, there is an operator which combines φ^2 and φ^4 and which has definite scaling properties at the critical point. We find

$$\nu = \frac{1}{2} + \frac{A'(m^{*2})}{4B(m^{*2})}\epsilon \tag{125}$$

5.3 Propertime equation and our first local potential

Having established that the computation of a and b is difficult because the partition function is neither connected nor irreducible generator, let us simplify our life by considering the effective action that is the generator of the 1PI diagrams. At the leading order

$$\Gamma[\varphi] = S_\Lambda[\varphi] + \frac{1}{2}\,\mathrm{tr}\log S^{(2)} \tag{126}$$

Let us represent the effects of the one loop fluctuations as a **propertime integral** over the parameter s

$$\Gamma[\varphi] = S_\Lambda[\varphi] - \frac{1}{2}\,\mathrm{tr}\int_0^\infty \frac{\mathrm{d}s}{s}\exp(-s\,S^{(2)}) \tag{127}$$

We have that in momentum space $S^{(2)} = q^2 + V''(\varphi)$ in which the potential denotes collectively $V(\varphi) = \frac{1}{2}m^2\varphi^2 + g\varphi^4$. This is sometimes known as the **local potential approximation**.

The variable s is thus a dual variable to the energy q^2 of the propagating fluctuations. The generalization of the momentum-shell procedure in the dual variable becomes

$$\Gamma_{\mu,\Lambda}[\varphi] = S_\Lambda[\varphi] - \frac{1}{2}\,\mathrm{tr}\int_{1/\Lambda^2}^{1/\mu^2} \frac{\mathrm{d}s}{s}\exp(-s\,S^{(2)}) \tag{128}$$

The trace is an integral over all momenta

$$\mathrm{tr} \to \int \frac{\mathrm{d}^d q}{(2\pi)^d} \to \frac{1}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\int \mathrm{d}q^2 (q^2)^{\frac{d-2}{2}}$$

so the integral is just a Gaussian one. We find

$$\Gamma_{\mu,\Lambda}[\varphi] = S_\Lambda[\varphi] + \frac{1}{2}\frac{1}{(4\pi)^{d/2}}\int \mathrm{d}^d x\; V''(\varphi)^{d/2}\left\{\Gamma\left(-\frac{d}{2}, \frac{V''(\varphi)}{\mu^2}\right) - \Gamma\left(-\frac{d}{2}, \frac{V''(\varphi)}{\Lambda^2}\right)\right\} \tag{129}$$

in which we have defined the incomplete Gamma function $\Gamma(a,z) = \int_z^\infty t^{a-1}\mathrm{e}^{-t}\mathrm{d}t$.

The above formula is not particularly useful or interesting for our strategy. Let's perform the same steps to compute the renormalization group as in the previous section by setting $\mu = \lambda\Lambda$ and taking a derivative with respect to $t = \log\lambda$

$$\partial_t \Gamma_{\mu,\Lambda}[\varphi] = \frac{1}{(4\pi)^{d/2}}\mu^d \int \mathrm{d}^d x\; \mathrm{e}^{-V''(\varphi)/\mu^2}$$

We can use this result to obtain the **RG flow of the (dimensionful) potential**

$$\partial_t V(\varphi) = \frac{1}{(4\pi)^{d/2}} \mu^d \mathrm{e}^{-V''(\varphi)/\mu^2} = \frac{1}{(4\pi)^{d/2}} \lambda^d \Lambda^d \mathrm{e}^{-\lambda^{-2} V''(\varphi)/\Lambda^2} \tag{130}$$

From the analysis of the previous sections, however, we know that we also have to appropriately rescale the field and the couplings so that the theory after an RG step can be compared with itself before the step. The simplest thing to do is to measure everything in units of the scale μ: we define the **rescaled potential**

$$V_\lambda(\varphi) = \lambda^{-d} V(\lambda^{\frac{d-2}{2}} \varphi)$$

which has beta function

$$\beta_{V_\lambda} = -d V_\lambda(\varphi) + \frac{d-2}{2} \varphi V_\lambda'(\varphi) + \frac{1}{(4\pi)^{d/2}} \Lambda^d \mathrm{e}^{-V_\lambda''(\varphi)/\Lambda^2} \tag{131}$$

Using a polynomial expansion $V_\lambda = \frac{1}{2} m(\lambda)^2 \varphi^2 + \frac{1}{4!} g(\lambda) \varphi^4$ we can determine

$$\beta_{m(\lambda)^2} = -2m(\lambda)^2 - \frac{1}{(4\pi)^{d/2}} g(\lambda)\, \mathrm{e}^{-m(\lambda)^2/\Lambda^2} \Lambda^{d-2} \tag{132}$$

$$\beta_{g(\lambda)} = (d-4) g(\lambda) + 3 \frac{1}{(4\pi)^{d/2}} g(\lambda)^2\, \mathrm{e}^{-m(\lambda)^2/\Lambda^2} \Lambda^{d-4} \tag{133}$$

The fixed point is rather ugly

$$m^{*2} = \frac{1}{6}(d-4)\Lambda^2\,; \qquad g^* = \frac{1}{3}(4-d)(4\pi)^{d/2} \mathrm{e}^{(d-4)/6} \Lambda^{4-d} \tag{134}$$

but the critical exponents are not. In $d = 4 - \epsilon$

$$\theta_1 = 2 - \frac{\epsilon}{3}\,; \qquad \theta_2 = -\epsilon\,; \qquad \nu = \frac{1}{2} + \frac{\epsilon}{12} \tag{135}$$

5.4 Dimensionless quantities

Notice that while the fixed point does depend on Λ, the critical exponents do not! The dependence on Λ is actually representative of the dependence on the scheme in which the computation is done: it takes into account details of the microphysics, and of the way in which we decided to perform the momentum-shell integration (or in this case the propertime shell integration). Since it is not an important dependence, it is often chosen to remove it altogether. A simple way to achieve this removal is to define the **dimensionless potential**

$$v(\varphi) = \mu^{-d} V(\mu^{\frac{d-2}{2}} \varphi)$$

Rescaling the potential by the full RG scale μ is equivalent, for practical purposes, to rescaling it by just the weight λ. The dimensionless potential has a beta function that manifestly does not depend on Λ

$$\beta_v = -d v(\varphi) + \frac{d-2}{2} \varphi v'(\varphi) + \frac{1}{(4\pi)^{d/2}} \mathrm{e}^{-v''(\varphi)} \tag{136}$$

and simpler fixed points

$$m^{*2} = \frac{1}{6}(d-4)\,; \qquad g^* = \frac{1}{3}(4-d)(4\pi)^{d/2} \mathrm{e}^{(d-4)/6} \tag{137}$$

The manifest independence of the flow β_v on Λ explains the independence of the critical exponents on the method used to compute them.

5.5 Deconstructing the flow

Even though we have established that we can factor out the scale Λ let's try now to use it to infer some important property of the flow. Consider the expansion

$$\beta_{V_\lambda} = -dV_\lambda(\varphi) + \frac{d-2}{2}\varphi V'_\lambda(\varphi) - \frac{1}{(4\pi)^{d/2}}\Lambda^{d-2}V''_\lambda(\varphi) + \frac{1}{2(4\pi)^{d/2}}\Lambda^{d-4}V''_\lambda(\varphi)^2 + \mathcal{O}(\Lambda)^{d-6} \tag{138}$$

We have neglected a zero point energy contribution, which is not particularly interesting, and we have grouped all the terms that in dimension $d < 4$ are decreasingly less imporant for increasing size of the scale Λ. Let's specialize the flow to $d = 4 - \epsilon$:

$$\beta_{V_\lambda} = -4V_\lambda(\varphi) + \varphi V'_\lambda(\varphi) - \epsilon\left(V_\lambda(\varphi) + \frac{1}{2}\varphi V'_\lambda(\varphi)\right) - \frac{1}{(4\pi)^2}\Lambda^2 V''_\lambda(\varphi) + \frac{1}{2(4\pi)^2}V''_\lambda(\varphi)^2 + \dots \tag{139}$$

Our analysis seems to point to the fact that the only those two last terms are important.

To understand which term is actually important rewrite the one loop action as

$$\Gamma_{\mu,\Lambda}[\varphi] = S_\Lambda[\varphi] - \frac{1}{2}\,\mathrm{tr}\int_0^\infty \frac{\mathrm{d}s}{s}\rho_{\mu,\Lambda}(s)\exp(-s\,S^{(2)}) \tag{140}$$

in which the function $\rho_{\mu,\Lambda}(s)$ is a distribution that ensures the desired bounds on the integration. This is known as a different **scheme** for the computation of $\Gamma_{\mu,\Lambda}[\varphi]$. In the specific case of the previous sections

$$\rho_{\mu,\Lambda}(s) = \theta(s - \mu^{-2})\theta(\Lambda^{-2} - s)$$

Any distribution $\rho_{\mu,\Lambda}(s)$ is a good choice of **regularization** if in the limit $\mu \to 0$ it becomes $\rho_{0,\Lambda}(s) = \theta(\Lambda^{-2} - s)$. The dependence of the reseults on the distribution $\rho_{\mu,\Lambda}(s)$ is referred to as **scheme dependence**. If we repeat the computations of the previous sections and specialize to $d = 4 - \epsilon$ we get

$$\beta_{V_\lambda} = -4V_\lambda(\varphi) + \varphi V'_\lambda(\varphi) - \epsilon\left(V_\lambda(\varphi) + \frac{1}{2}\varphi V'_\lambda(\varphi)\right) - \frac{1}{(4\pi)^2}\Lambda^2\mathcal{I}[\rho]V''_\lambda(\varphi) + \frac{1}{2(4\pi)^2}V''_\lambda(\varphi)^2 + \dots \tag{141}$$

in which $\mathcal{I}[\rho]$ is an integral that depends only on the regularizing distribution. An explicit computation reveals that the critical exponents do not change for changing $\mathcal{I}[\rho]$, therefore they are **scheme independent** (sometimes referred to as **universal** too).

If our interest is to compute critical exponents in the ϵ-expansion it might thus be convenient to look for a scheme that evaluates all integrals like $\mathcal{I}[\rho]$ to zero. One such scheme would be the **minimal subtraction** one in **dimensional regularization**.

6 Functional renormalization in the local potential

We now graduate to a more modern method of renormalization. Consider the flow of the average effective action of the appendix

$$k\partial_k\Gamma_k[\overline{\varphi}] \;=\; \frac{1}{2}\,\mathrm{Tr}\left(\Gamma_k^{(2)}[\overline{\varphi}] + \mathcal{R}_k\right)^{-1} k\partial_k\mathcal{R}_k \tag{142}$$

We want to project the flow onto an operatorial truncation of the effective average action of the form

$$\Gamma_k[\varphi] = \int \mathrm{d}^d x\left\{\frac{Z_k}{2}\partial_\mu\varphi\partial^\mu\varphi + V_k(\varphi)\right\} \tag{143}$$

in which we have introduced a **wavefunction renormalization** Z_k that takes into account the fact that the RG might change the normalization of the kinetic term. The anomalous dimension is related to Z_k as $\eta = -k\partial_k Z_k/Z_k$.

We want to extract the RG running of the scale dependent potential $V_k(\varphi)$ and of Z_k. Noticing that

$$\Gamma_k[\varphi]|_{\varphi=\text{const.}} = \text{Vol}\, V_k(\varphi) \qquad\qquad p^2 Z_k + V_k''(\phi) = \left.\frac{\delta^2}{\delta\varphi_p \delta\varphi_{-p}}\Gamma_k[\varphi]\right|_{\varphi=\text{const.}} \tag{144}$$

we choose the following equations as prescriptions to extract the RG flows

$$\begin{aligned} k\partial_k V_k(\varphi) &= \frac{1}{\text{Vol}}\, k\partial_k \Gamma_k[\varphi]|_{\varphi=\text{const.}}\ , \\ \eta &= -\frac{1}{Z_k} k\partial_k \frac{\partial}{\partial p^2}\left.\frac{\delta^2\Gamma_k[\varphi]}{\delta\varphi_p\delta\varphi_{-p}}\right|_{\varphi=\text{const.}, p^2=0} \end{aligned} \tag{145}$$

Introducing a momentum space modified propagator

$$\mathcal{G}_k(q^2) \equiv \left(Z_k q^2 + V_k''(\varphi) + \mathcal{R}_k(q^2)\right)^{-1} \tag{146}$$

then the flows are

$$k\partial_k V_k(\varphi) = \frac{1}{2(2\pi)^d}\int \mathrm{d}^d q\, \mathcal{G}_k\, k\partial_k \mathcal{R}_k \tag{147}$$

$$\eta = -\frac{V^{(3)}(\varphi_0)^2}{(2\pi)^d Z_k}\int \mathrm{d}^d q \left(\mathcal{G}_k' + q^2\frac{2}{d}\mathcal{G}_k''\right)\mathcal{G}_k^2\, k\partial_k\mathcal{R}_k\Big|_{\varphi_0} \tag{148}$$

Notice that there is an implict freedom in how to evaluate η, specifically in which point to choose $\varphi \to \varphi_0$.

Now let us choose a specific "optimized" cutoff function that is easy to work with

$$\mathcal{R}_k(q^2) = Z_k(k^2 - q^2)\theta(k^2 - q^2) \tag{149}$$

and introduce **dimensionless renormalized quantities** in the usual way

$$\overline{\varphi} \equiv Z_k^{-1/2} k^{(2-d)/2}\varphi \qquad\qquad v_k(\overline{\varphi}) \equiv k^{-d} V_k(\varphi) \tag{150}$$

The flow becomes

$$k\partial_k v(\varphi) = -dv(\varphi) + \frac{d-2+\eta}{2}\varphi v'(\varphi) + c_d\frac{1-\frac{\eta}{d+2}}{1+v''(\varphi)} \tag{151}$$

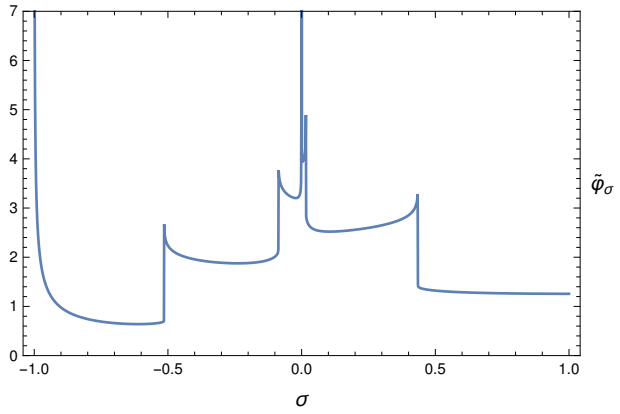
$$\eta = c_d\frac{v'''(\varphi_0)^2}{(1+v''(\varphi_0))^4} \tag{152}$$

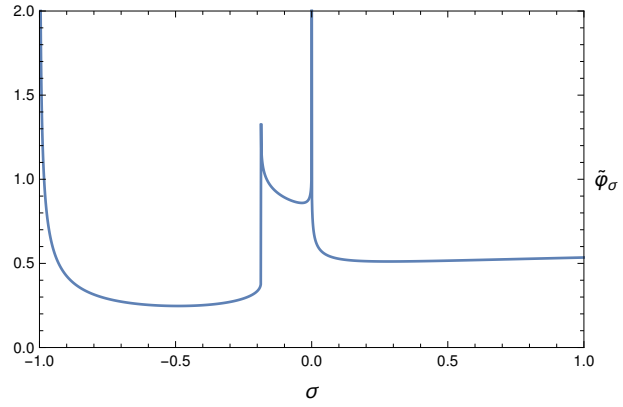
6.1 Numerical and scaling solutions

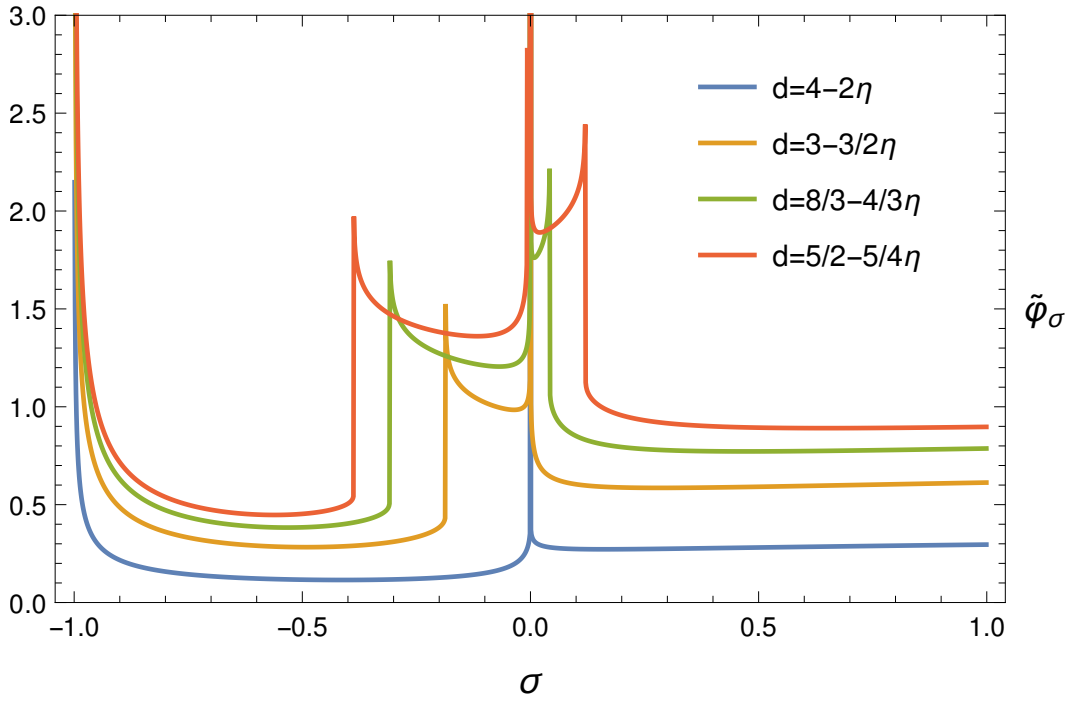
There are several methods with which one can study the fixed points of the flow numerically, including polynomial expansions or the study of scaling solutions. Here are some numerical results compared with other determinations:

	σ_{cr}	g_{cr}	η	ν
LPA	-0.1861	16.1912	0	0.6496
LPA' η	-0.1357	9.6098	0.1120	0.6453
LPA' η'	-0.1657	13.1693	0.0443	0.6473
best η	-0.1693	13.6776	0.0364	0.6477
P.&V.	–	–	0.0364	0.6301
Lattice	–	–	0.0363	0.6300
Bootstrap	–	–	0.036302	0.629977

The advantage of the functional approach is in the ability of working in any d (below are plots for $d = 2$ and $d = 3$ respectively) and providing solutions with global properties. Using a scaling solution approach solutions are seen as discontinuities of a special variable as shown below:







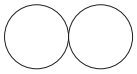
7 Perturbation theory

We can compute the effective action using an expansion in terms of the constant $\hbar$ (see appendix). The result is

$$\Gamma[\varphi] \;=\; S_{\rm B}[\varphi] + \sum_{L\geq 1} \hbar^L \Gamma_L[\varphi] \tag{153}$$

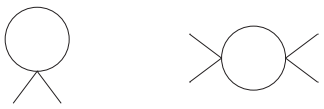
and the first few terms are

$$\Gamma_1[\varphi] \;=\; \frac{1}{2}\,{\rm Tr}\,\log S^{(2)}_{\rm B}[\varphi] \tag{154}$$

$$\Gamma_2[\varphi] \;=\; -\frac{1}{12}$$  $$+\,\frac{1}{8}$$  (155)

The above diagrams should be intended as background lines with full dependence on φ (through the vertices of the theory). Acting with derivatives with respect to φ one generates diagrams. In the example of a φ^4

theory two such diagrams generated by acting on Γ_1 are



and in principle infinitely many more with couples of external lines acting on the same vertex.

7.1 Superficial degree of divergence

Consider the interaction $V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{g}{4!}\varphi^4$. We have

$$\frac{1}{2}\quad = \frac{g}{2}\int \frac{d^4q}{(2\pi)^4}\frac{1}{q^2+m^2} \equiv \frac{g}{2}\int_q \frac{1}{q^2+m^2} \tag{156}$$

The integral is divergent because its interior grows with powers of q coming from the measure that are not balanced by those of the propagator. We have at large q

$$\frac{d^4q}{q^2+m^2} \sim \frac{d^4q}{q^2} \sim \frac{q^4}{q^2} \sim q^2$$

The exponent of the large q behavior is said to be the **degree of divergence** of the integral, which in this case is $D = 2$ and the integral is said to be **quadratically divergent**. Similarly the integral

$$\frac{1}{2}\quad = \frac{g^2}{2}\int_q \frac{1}{q^2+m^2}\frac{1}{(q+p)^2+m^2} \tag{157}$$

($p = p_1 + p_2$ is the sum of the incoming momenta entering from either the left or the right hand side) diverges at large q

$$\frac{d^4q}{q^2+m^2}\frac{1}{(q+p)^2+m^2} \sim \frac{d^4q}{q^4} \sim q^0 \sim \log q$$

with degree $D = 0$ and the integral is said to be **logarithmically divergent**. Diagrams with degree of divergence zero or higher do diverge, while diagrams with degree smaller than zero might converge. Generally, diagrams with degree smaller than zero might diverge because of subdiagrams which have degree higher or equal to zero. In the one loop case at hand all further diagrams obtained by attaching couples of external lines are finite.

Consider a general diagram with n vertices, E external lines, I internal lines and L loops in d dimensions. Such diagram will have L integration measures at the numerator and $I propagators$, resulting in

$$D = dL - 2I$$

Since there are I internal momenta and n momentum conservations (but one is already used for the overall diagram's conservation) there are $I - (n-1) = I - n + 1$ independent momenta in the diagram. However, the number of independent momenta inside the diagram is the number of loops, implying

$$L = I - n + 1$$

Furthermore, if the interaction is only ϕ^4 there can only be $4n$ legs entering the vertices with the external legs counted once, but the intenal ones counted twice because they enter two vertices. We have thus

$$4n = E + 2I$$

We use these two last formulas to determine

$$D = \frac{d-2}{2}E + d + n(d-4)$$

This implies the following

- For $d > 4$ the degree of divergence grows with n. At each order the theory has new divergence and the theory will not be renormalizable.
- For $d < 4$ the theory has a finite number of diverging diagrams, the theory will be super-renormalizable.
- For $d = 4$ the divergences do not depend on the number of verticesand:

$$D = 4 - E$$

As such there are infinitely many diverging diagrams, but only those with $E = 2$ and $E = 4$ can have $D \geq 0$. The theory will be perturbatively renormalizable.

7.2 Dimensional regularization: making the theory finite

The fact that the theory has different convergence properties for different d suggests that it could be useful to analytically continue the dimension to render the theory finite. Consider the Lagrangian in $d = 4$

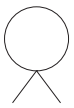
$$\mathcal{L}_4 = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{g}{4!}\varphi^4 \tag{158}$$

which has a dimensionless coupling g. We want to extend it to d dimensions, but do it in such a way that the properties of the theory do not change because of our extension. We choose

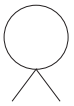
$$\mathcal{L}_d = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \mu^{4-d}\frac{g}{4!}\varphi^4 \tag{159}$$

in which we have introduced an *arbitrary reference mass scale* μ that allows us to maintain g as a dimensionless coupling. The limit $d \to 4$ in the above Lagrangian sees the dissapearance of μ as expected, although this will happen only at the classical level!

The strategy for computing the Feynman integrals will be to choose d such that they converge, then compute them and analytically continue the result. For example

$\frac{1}{2}$  $= \mu^{4-d}\frac{g}{2}\int \frac{d^{\mathrm{d}}q}{(2\pi)^4}\frac{1}{q^2+m^2} \equiv \mu^{4-d}\frac{g}{2}\int_q \frac{1}{q^2+m^2}$ (160)

converges in $0 < d < 2$ and we obtain

$\frac{1}{2}$  $= \frac{gm^2}{2(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{2-d/2}\Gamma(1-d/2)$ (161)

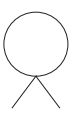
Using the continuation to $d = 4 - \epsilon$ and the expansion of the gamma function for a small parameter close to the poles at negative integers

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left(\frac{1}{\epsilon} + \psi_1(n+1) + \mathcal{O}(\epsilon)\right) \tag{162}$$

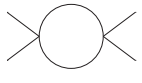
$$\psi_1(n+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \gamma \tag{163}$$

$$\gamma \simeq 0.577 \tag{164}$$

we get $\Gamma(1-d/2) = -\frac{2}{\epsilon} - 1 - \gamma + \mathcal{O}(\epsilon)$ and therefore

$\frac{1}{2}$  $= -\frac{gm^2}{(4\pi)^2}\frac{2}{\epsilon} + \text{finite}$ (165)

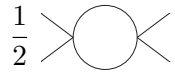
Similarly we analyse the other diagram.

$\frac{1}{2}$  $= \frac{g^2}{2}\mu^{2(4-d)}\int_q \frac{1}{q^2+m^2}\frac{1}{(q+p)^2+m^2}$ (166)

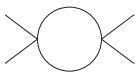
We use the formula

$$\frac{1}{ab} = \int_0^1 \mathrm{d}z \frac{1}{(za + (1-z)b)^2}$$

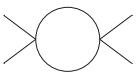
to combine the propagators and perform the change in the integrated momentum $q_\mu \to q'_\mu = q_\mu + (1-z)p_\mu$ which removes the linear dependence in q'_μ from the denominator. Combining everything we find

$\frac{1}{2}$  $= \frac{g^2}{2(4\pi)^2}\mu^{4-d}\int_0^1 \mathrm{d}z \left(\frac{m^2 + z(1-z)p^2}{4\pi\mu^2}\right)^{\frac{d-4}{2}} \Gamma(2-d/2)$ (167)

We need to obtain the divergence using the same formula as above $\Gamma(2-d/2) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$ and find

$\frac{1}{2}$  $= \frac{g^2}{(4\pi)^2} + \text{finite}$ (168)

The full form is

$\frac{1}{2}$  $= \frac{\mu^\epsilon g^2}{(4\pi)^2\epsilon} - \frac{\mu^\epsilon g^2}{2(4\pi)^2}\left\{F(p^2, m^2, \mu) + \gamma\right\}$ (169)

$$F(p^2, m^2, \mu) = \int_0^1 \mathrm{d}z \log\left(\frac{m^2 + z(1-z)p^2}{4\pi\mu^2}\right) \quad (170)$$

7.3 Making the theory finite at one loop

The diagrams are used to compute the effective action. We have just determined that up to one loop

$$\Gamma^{(2)}(p) = p^2 + m^2 - \frac{gm^2}{(4\pi)^2\epsilon} + \text{finite} + \text{higher loops} \quad (171)$$

$$= p^2 + m^2\left(1 - \frac{g}{(4\pi)^2\epsilon}\right) + \ldots \quad (172)$$

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = \mu^\epsilon g + 3\frac{\mu^\epsilon g^2}{(4\pi)^2\epsilon} - \frac{\mu^\epsilon g^2}{2(4\pi)^2}\left\{F(s, m^2, \mu) + F(t, m^2, \mu) + F(u, m^2, \mu) + 3\gamma\right\} \quad (173)$$

$$+\text{higher loops} \quad (174)$$

$$= \mu^\epsilon g\left(1 + 3\frac{g}{(4\pi)^2\epsilon}\right) + \text{finite} + \text{higher loops} \quad (175)$$

Let us define the **physical mass** and the **renormalized coupling** at one loop by requiring

$$\Gamma^{(2)}(p) \equiv p^2 + m^2_{L=1} \quad (176)$$

$$\Gamma^{(4)}(p_i = 0) \equiv g_{L=1} \quad (177)$$

Comparing with the regulated expressions we find

$$m^2_{L=1} = m^2\left(1 - \frac{g}{(4\pi)^2\epsilon}\right) \quad (178)$$

$$g_{L=1} = g\mu^\epsilon\left\{1 + 3\frac{g}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma - F(0, m_1^2, \mu)\right)\right\} \quad (179)$$

which can be inverted in terms of an expansion in small $g_{L=1}$ or g if we pretend not to see the divergence

$$m^2 = m^2_{L=1}\left(1 + \frac{g_{L=1}}{(4\pi)^2\epsilon}\right) + \mathcal{O}(g_1^2) \quad (180)$$

$$g = g_{L=1}\mu^{-\epsilon}\left\{1 - 3\frac{g_{L=1}\mu^{-\epsilon}}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma - F(0, m_1^2, \mu)\right)\right\} + \mathcal{O}(g_1^3) \quad (181)$$

We now assume that the new quantities (couplings) are finite, implying that the original ones are divergent. In terms of the new couplings we have that $\Gamma^{(2)}(p)$ is "finite" by construction, but interestingly also the fourth vertex is

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) \quad = \quad g_{L=1} - \frac{\mu^{-\epsilon} g_{L=1}^2}{2(4\pi)^2}\left\{F(s,m_1^2,\mu)+F(t,m_1^2,\mu)+F(u,m_1^2,\mu)-3F(0,m_1^2,\mu)\right\} \quad (182)$$

which verifies the condition $\Gamma^{(4)}(p_i=0)=g_1$. The physical renormalized mass satisfies $\Gamma^{(2)}(p^2=0)=m_1^2$ which is rather natural for a mass measured in the rest frame (hence the name "physical"). The condition on the fourth coupling is less straightforward, and in fact we could have chosen a different renormalization point. One good reason to do it is that the point $p_i=0$ is unphysical, in the sense that no scattering occurs for that momentum configuration. One more physical example could be to replace the point $p_i=0$ with some nonzero value for the Mandlestam variables. Choose, for example, the configuration $s=t=u=p^2$ for some momentum value p: in this case we could define

$$\Gamma^{(4)}(\text{config.}) \equiv g_1^{\text{phys.}}$$

In this latter example it is a scattering process determined by the chosen momentum configuration that determines the value (measure) of the coupling g_1. Repeating the above renormalization steps we can relate

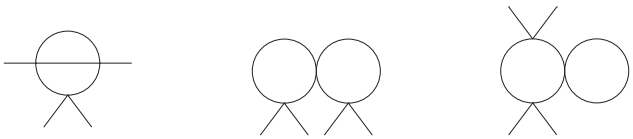
$$g_1^{\text{phys.}} = g_1 + A\cdot g_1^2 + \ldots$$

for some constant A which is uniquely determined by the **scheme** which we are adopting, that in this case depends on the chosen configuration too. This also shows how to experimentally determine the quantity g_1 by relating it to some physical measure output. This whole discussion also tells us that we cannot simply start with given mass and coupling in a QFT, but that it is the QFT itself that tells us which are the parameters that need experimental determination (in this case m_1^2 and g_1 in any given scheme).

7.4 Renormalization conditions at two loops and beyond

A complete list of the diagrams that diverge at two loops includes the following diagrams with two legs

and with four legs

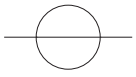


There are two notable features of going to two loops. The first feature is that new poles emerge:

$$\frac{1}{\epsilon^2}$$

The existence of such poles can be seen easily from the diagrams that are straight up products of two one-loop diagrams. The new poles are not really an issue at this stage of the discussion, but will become relevant later.

The second feature is that the diagram



has a pole $\frac{1}{\epsilon^2}$ proportional to the square of the incoming momentum p^2. This implies that the unrenormalized effective Lagrangian includes a term of the form

$$\frac{A}{\epsilon}\partial_\mu\varphi\partial^\mu\varphi$$

for some constant A determined by the diagram. All other diagrams can, in principle, be taken care of by the same technique of the previous section, but this one changes the normalization of the propagator. The simplest way to handle this new divergence is to perform a rescaling of the field to its renormalized counterpart

$$\varphi_{\rm R} = Z_\varphi \varphi_B$$

and require that the two point function of the renormalized field is correctly normalized

$$\Gamma^{(2)}_{\rm R}(p^2) \;=\; \frac{\delta^2\Gamma}{\delta\varphi_{\rm R}(p)\delta\varphi_{\rm R}(-p)} = p^2 + m^2_{L=2} + \Sigma(p^2) \tag{183}$$

We have introduced a new function $\Sigma(p^2)$ known as the **self-energy** which includes all terms of the two point function that are not expressible in terms of the previous two, and therefore by definition $\Sigma(0) = 0$. This new condition uniquely determines the renormalized field, but changes all other vertices of the theory by some power of Z_φ.

We can sum up all the **renormalization conditions** which can be used to make the theory finite including those of the previous section as

$$\Gamma^{(2)}_{\rm R}(p^2)\Big|_{p^2=0} = m^2 \qquad \frac{\partial\Gamma^{(2)}_{\rm R}(p^2)}{\partial p^2}\Big|_{p^2=0} = 1 \qquad \Gamma^{(4)}_{\rm R}(p_i)\Big|_{p_i=0} = g \tag{184}$$

These conditions *define* the renormalized mass m, the renormalized coupling g, and the normalization of the renormalized field $\varphi_{\rm R}$. (Of course the same discussion on the renormalization point for g applies as in the previous section substituting $p_i = 0$ with a more physical configuration.) In terms of these quantities it is possible to show that the predictions of the theory are finite at every loop, and therefore that the only two parameters to be determined experimentlly (for example with a collider) are m^2 and g (as long as the scattered particle states can be interpreted as one-particle states).

The discussion of the last two sections highlights a stategy for the computation of a QFT which in steps is:

1. Compute the regularized amplitudes in the bare field;
2. Impose renormalization conditions to express the bare quantities in terms of the renormalized ones;
3. Rewrite the regularized amplitudes in terms of the renormalized quantities.

The problem with this strategy is that we are forced to be "blind" of the physical quantities until the second step and then "backtrack" in the third step to obtain meaningful results. A solution to this problem is to make the handling of the divergences more systematic as we shall describe later.

7.5 Dimensional poles, logarithmic divergences and modified poles

Before proceeding let's discuss the relation between $\frac{1}{\epsilon}$ poles and logarithmic divergences of momentum cutoff regularization. Assuming an UV cutoff mass scale Λ, we assert the duality

$$\log\Lambda \quad\longleftrightarrow\quad \frac{1}{\bar\epsilon} = \frac{1}{\epsilon} + \frac{1}{2}\log\left(\frac{4\pi\mu^2}{m^2}\right) - \frac{1}{2}\gamma$$

while all other divergences do not have a correspondence in dimensional regularization

$$\Lambda^2, \Lambda^4, \ldots \quad\longleftrightarrow\quad 0$$

The variable $\bar\epsilon$ is a convenient way to take care of all factors coming from the expansion of $(2\pi)^d$ and the angular volume $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ (other ways to do this include the loop-by-loop rescaling $\hbar \to \hbar\cdot(2\pi)^\epsilon$).

Let's observe this relation in practice by testing two diagrams. We begin with a single propagator loop. The divergent part in dimensional regularization is

$$\mathbf{divp}\left[\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\mu^\epsilon}{q^2+m^2}\right] = -\frac{m^2}{8\pi^2}\frac{1}{\bar{\epsilon}}$$

while in momentum regularization

$$\mathbf{divp}\left[\int_{|q|<\Lambda} \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{q^2+m^2}\right] = -\frac{m^2}{8\pi^2}\log\left(\frac{\Lambda}{m}\right) + \frac{\Lambda^2}{16\pi^2}$$

Likewise for a single loop with two propagators the divergent part in dimensional regularization is

$$\mathbf{divp}\left[\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\mu^\epsilon}{(q^2+m^2)^2}\right] = \frac{1}{8\pi^2}\frac{1}{\bar{\epsilon}}$$

while in momentum regularization

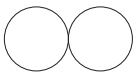
$$\mathbf{divp}\left[\int_{|q|<\Lambda} \frac{\mathrm{d}^4 q}{(2\pi)^4}\frac{1}{(q^2+m^2)^2}\right] = \frac{1}{8\pi^2}\log\left(\frac{\Lambda}{m}\right)$$

7.6 Subtraction method

A systematic strategy to avoid the unwanted backtracking in the steps to the renormalization that still allows to construct a theory with finite predictions is the method of subtractions via counterterms. We begin by postulating a bare local action

$$S_{\mathrm{B}}[\varphi_{\mathrm{B}}] = \int \mathrm{d}^d x \left\{\frac{1}{2}(\partial\varphi_{\mathrm{B}})^2 + \frac{1}{2}m_{\mathrm{B}}^2\varphi_{\mathrm{B}}^2 + \frac{1}{4!}g_{\mathrm{B}}\varphi_{\mathrm{B}}^4\right\}$$

which is function of the bare field and depends parametrically on the bare couplings. This action gives divergent answers for some predictions through the path integral and therefore it is not good for computations. In particular the divergences appear from the momentum loops of the effective action: in an expansion in orders of $\hbar$ it is

$$\Gamma_{\mathrm{B}}[\varphi_{\mathrm{B}}] = S_{\mathrm{B}}[\varphi_{\mathrm{B}}] + \frac{1}{2}\operatorname{Tr}\log S_{\mathrm{B}}^{(2)} - \frac{1}{12}$$  $$+ \frac{1}{8}$$  $$+ \ldots$$

and it diverges because it is the *wrong* field φ_{B} propagating in the loops.

We know that a renormalized action that gives finite answers must have different (renormalized) couplings, and also be a function of a different (renormalized) field. Let such field be defined as $\varphi_{\mathrm{R}} \equiv Z_\varphi^{-\frac{1}{2}}\varphi_{\mathrm{B}}$. We also define

$$S_{\mathrm{B}}[\varphi_{\mathrm{B}}] = S_{\mathrm{R}}[\varphi_{\mathrm{R}}] + S_{\mathrm{c.t.}}[\varphi_{\mathrm{R}}]$$

in which *we choose* $S_{\mathrm{R}}[\varphi_{\mathrm{R}}]$ to be of the same form as $S_{\mathrm{B}}[\varphi_{\mathrm{B}}]$ but with new renormalized couplings to be determined

$$S_{\mathrm{R}}[\varphi_{\mathrm{R}}] = \int \mathrm{d}^d x \left\{\frac{1}{2}(\partial\varphi_{\mathrm{R}})^2 + \frac{1}{2}m_{\mathrm{R}}^2\varphi_{\mathrm{R}}^2 + \frac{1}{4!}g_{\mathrm{R}}\varphi_{\mathrm{R}}^4\right\}$$

The above relations *define implicitly* the **counterterms** which we *parametrize* as

$$S_{\mathrm{c.t.}}[\varphi_{\mathrm{R}}] = \int \mathrm{d}^d x \left\{\frac{1}{2}(Z_\varphi - 1)(\partial\varphi_{\mathrm{R}})^2 + \frac{1}{2}(Z_{m^2} - 1)m_{\mathrm{R}}^2\varphi_{\mathrm{R}}^2 + \frac{1}{4!}(Z_g - 1)g_{\mathrm{R}}\varphi_{\mathrm{R}}^4\right\}$$

Using the definitions $S_{\mathrm{R}}[\varphi_{\mathrm{R}}]$ and $S_{\mathrm{c.t.}}[\varphi_{\mathrm{R}}]$ we can relate the bare and renormalized couplings using the new constants

$$m_{\mathrm{B}}^2 = \frac{Z_{m^2}}{Z_\varphi}m_{\mathrm{R}}^2 \qquad\qquad g_{\mathrm{B}} = \frac{Z_g}{Z_\varphi^2}g_{\mathrm{R}}$$

The new constants are assumed to have an expansion in $\hbar$ which becomes nontrivial from the first order (the classical theory does not diverge (well kind of)). If we momentarily restore $\hbar$ it looks something like

$$Z_{m^2} = 1 + \sum_{L\geq 1} \hbar^L \delta m_L^2 \qquad Z_g = 1 + \sum_{L\geq 1} \hbar^L \delta g_L \qquad Z_\varphi = 1 + \sum_{L\geq 1} \hbar^L \delta Z_{\varphi,\,L}$$

which at the level of the counterterm action looks like

$$S_{\text{c.t.}}[\varphi_{\text{R}}] = \sum_{L\geq 1} \hbar^L \delta S_{\text{c.t.}\,L}[\varphi_{\text{R}}] = \hbar\, \boxed{1} + \hbar^2\, \boxed{2} + \dots$$

in which the boxes represents the terms of the expansion. If we now use the new parametrization and compute the $\hbar$ expansion we get

$$\Gamma_{\text{R}}[\varphi_{\text{R}}] = S_{\text{R}}[\varphi_{\text{R}}] + \frac{1}{2}\,\text{Tr}\log S^{(2)}_{\text{R}} + \boxed{1} \tag{185}$$

$$-\frac{1}{12}\,\bigcirc\!\!\!\!-\!\!\!\!\bigcirc + \frac{1}{8}\,\bigcirc\!\!\bigcirc + \boxed{2} - \frac{1}{2}\,\bigcirc\!\!\!\!\boxed{1} + \dots \tag{186}$$

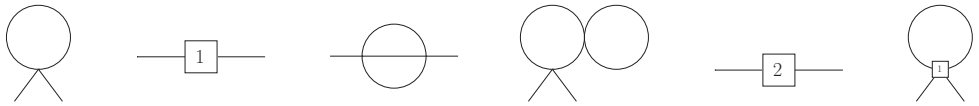
where we again set $\hbar = 1$.

Now the point is to use $S_{\text{c.t.}}[\varphi_{\text{R}}]$ to make the theory finite. This can always be done loop-by-loop: the first counterterm can be used to cancel the divergences of the trace-log, the second one of the two loop diagrams, and so on. What might go wrong is that $S_{\text{c.t.}}[\varphi_{\text{R}}]$ might contain new action monomials that are not included in the original action. If this happens one has to go back and include the new terms in $S_{\text{R}}[\varphi_{\text{R}}]$. If we have to go back infinitely many times because new and new counterterms are needed the theory is said to be non-renormalizable (example: Einstein-Hilbert gravity is not renormalizable in $d = 4$).

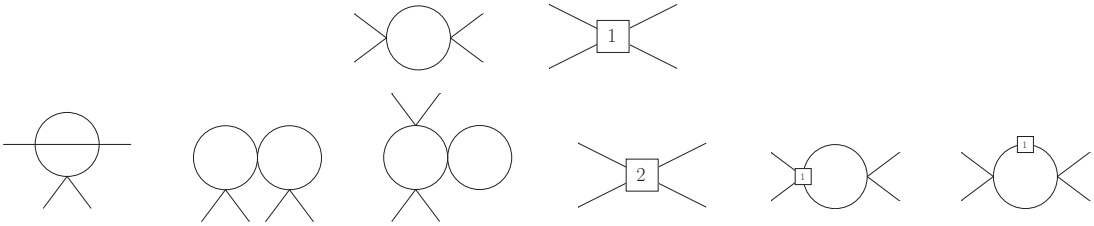
If the renormalization is successful then the constants have eliminated all divergences and Γ is finite. Therefore, given the correct renormalization constants the perturbative expansion of the theory that uses the renormalized field φ_{R} is and has always been finite! Magic!

7.7 Subdivergences and consistency of the ϵ-expansion beyond one loop

Having expanded using a new field there are several new diagrams for the two- and four-point functions. Neglecting the prefactors, the two point function is determined by



and the four point function by



The diagrams involving only the vertices of the counterterms straightforwardly cancel divergences in the corresponding vertex, while those involving a loop and a counterterm cancel a **subdivergence**. The cancellation is very delicate, nontrivial and important: the discussion on the degree of divergence is in fact not sufficient to ensure convergence of a diagram because of the possible subdivergces coming from lower loop diagrams inside the considered one that diverge. In practice, the presence of the one loop counterterm ensures that all two loop diagrams which have $D < 0$ but still diverge are actually finite!

Recalling that at two loop order new pole types emerge of the form $\frac{1}{\epsilon^2}$ something even less trivial happens in dimensional regularization: the new poles cancel. The two loop diagrams can have $\frac{1}{\epsilon}$ and $\frac{1}{\epsilon^2}$ poles and are multiplied by additional powers of μ^ϵ. In the case of φ^4 the higher poles coming from two loop are

$$A\,\frac{\mu^{2\epsilon}}{\epsilon^2}$$

for some A depending on the diagram. The poles with a loop and a counterterm can diverge as $\frac{1}{\epsilon^2}$ as

$$B\,\frac{\mu^{\epsilon}}{\epsilon^2}$$

for some B depending on the diagram. When everything is combined the poles combine in structures such as

$$\frac{\mu^{2\epsilon}}{\epsilon^2} - 2\frac{\mu^{\epsilon}}{\epsilon^2} = 0 + \mathcal{O}(\epsilon)$$

The cancellation of subdivergences (and specifically of these poles) is a fundamental *consistency check* when doing higher loop computations!

7.8 Renormalization group equations

Let us relate bare and renormalized correlators with all explicit dependences:

$$\begin{aligned} G^{(n)}_{\rm B}(x_1,\ldots,x_n;m_{\rm B},g_{\rm B},\epsilon) &= \langle\varphi_{\rm B}(x_1)\ldots\varphi_{\rm B}(x_n)\rangle & (187)\\ G^{(n)}_{\rm R}(x_1,\ldots,x_n;m_{\rm R},g_{\rm R},\mu,\epsilon) &= Z_\varphi(g(\mu),\epsilon)^{-\frac{n}{2}}\langle\varphi_{\rm B}(x_1)\ldots\varphi_{\rm B}(x_n)\rangle & (188)\\ &= Z_\varphi(g(\mu),\epsilon)^{-\frac{n}{2}}\,G^{(n)}_{\rm B}(x_1,\ldots,x_n;m_{\rm B},g_{\rm B},\epsilon) & (189)\end{aligned}$$

The proper vertices are amputated by the external legs and therefore have the opposite relation

$$\Gamma^{(n)}_{\rm R}(p_i;m_{\rm R},g_{\rm R},\mu,\epsilon) = Z_\varphi(g(\mu),\epsilon)^{\frac{n}{2}}\,\Gamma^{(n)}_{\rm B}(p_i;m_{\rm B},g_{\rm B},\epsilon) \quad (190)$$

All renormalized quantities are finite in the limit $\epsilon\to 0$.

The renormalized couplings and field are defined

$$\varphi_{\rm R} = Z_\varphi^{-1}(g(\mu),\epsilon)\,\varphi_{\rm B} \qquad m^2_{\rm R} = \frac{Z_\varphi(g(\mu),\epsilon)}{Z_m^2(g(\mu),\epsilon)}\,m^2_{\rm B} \qquad g_{\rm R}(\mu) = \mu^{-\epsilon}\frac{Z^2_\varphi(g(\mu),\epsilon)}{Z_g(g(\mu),\epsilon)}\,g_{\rm B}$$

and show that everything depends on μ either from the argument of $g(\mu)$ or from the insertion of μ^ϵ of promoting the theory to d dimensions.

Using the fact that

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(n)}_{\rm B}(p_i;m_{\rm B},g_{\rm B},\epsilon) = 0 \quad (191)$$

one derives (we omit the subscript R)

$$\left(-n\mu\,\frac{\partial}{\partial\mu}\log Z^{\frac12}_\varphi\Big|_{\rm B} + \mu\,\frac{\partial g}{\partial\mu}\Big|_{\rm B}\frac{\partial}{\partial g} + \mu\,\frac{\partial m}{\partial\mu}\Big|_{\rm B}\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu}\right)\Gamma^{(n)}_{\rm R}(p_i;m_{\rm R},g_{\rm R},\mu,\epsilon) = 0 \quad (192)$$

This equation tells us how the vertex transform under a RG step $(\mu,g,m)\to(\mu',g',m')$.

We define the RG functions

$$\begin{aligned}\gamma(m,g,\mu) &= \mu\,\frac{\partial}{\partial\mu}\log Z^{\frac12}_\varphi\Big|_{\rm B} & (193)\\ \beta_g(m,g,\mu) &= \mu\,\frac{\partial g}{\partial\mu}\Big|_{\rm B} & (194)\\ \gamma_m(m,g,\mu) &= \frac{\mu}{m}\,\frac{\partial m}{\partial\mu}\Big|_{\rm B} & (195)\end{aligned}$$

so that

$$\left(-n\gamma+\beta_g\frac{\partial}{\partial g}+\gamma_m m\frac{\partial}{\partial m}+\mu\frac{\partial}{\partial\mu}\right)\Gamma^{(n)}_{\rm R}(p_i;m_{\rm R},g_{\rm R},\mu,\epsilon)=0 \tag{196}$$

By knowing the RG functions we know the RG behavior of the vertices too. These equations are however very hard to solve, so any simplification helps. We will see that in dimensional regularization the RG functions do not depend on the mass, which simplifies a lot.

7.9 Intermezzo: criticality

The usual scaling arguments can be used to write the vertex at criticality in the scaling form. Suppose that $\Gamma(p_i;m,g,\mu,\epsilon)$ scales with an overall exponent α_n correcting the naive $-nd^0_\varphi+d=-n(d-2)/2+d$ at criticality. Requiring scale invariance

$$\Gamma(p_i;m,g,\mu,\epsilon)=\mu_0^{-nd^0_\varphi+d+\alpha_n}\Gamma(p_i/\mu_0;m/\mu_0,g\mu_0^{4-d},\mu/\mu_0,\epsilon)$$

and taking $\mu_0=\mu$ and $m=0$ (infinite corelation length)

$$\Gamma(p_i;0,g,\mu,\epsilon)=\mu^{-nd^0_\varphi+d+\alpha_n}f(p_i/\mu_0;g\mu^{4-d},\epsilon)$$

Comparing with the solution of the RG equation in the same limits and iff $\beta_g=0$

$$\Gamma(p_i;0,g,\mu,\epsilon)=\mu^{-nd^0_\varphi+d+n\gamma}f(p_i/\mu_0;g\mu^{4-d},\epsilon)=\mu^{-nd_\varphi+d}f(p_i/\mu_0;g\mu^{4-d},\epsilon)$$

with $d_\varphi=d^0_\varphi+\gamma$ implying $\alpha_n=-n\gamma$. The naive Gaussian scaling is substituted by one in which the field has dimension d_φ and in particular anomalous dimension $\eta=2\gamma$. Therefore we can identify the two quantities *at the fixed points*. From here one can construct the full analogy between the analysis of the previous sections and this one.

7.10 RG system and ϵ-expansion

The final RG system is conveniently written in terms of the rescaled coupling $\overline{g}=\frac{g}{(4\pi)^2}$. At two loops

$$\gamma = \frac{1}{12}\overline{g}^2 \tag{197}$$

$$\beta_{\overline{g}} = -\epsilon\overline{g}+3\overline{g}^2-\frac{17}{3}\overline{g}^3 \tag{198}$$

$$\gamma_m = \frac{1}{2}\overline{g}-\frac{5}{12}\overline{g}^2 \tag{199}$$

The calculation of critical exponents in the ϵ-expansion proceeds as in the exercises!

8 Heat kernel: representations

8.1 Let's move to curved space first

This section is dedicated to motivating and introducing the heat kernel methods for field theory. Having in mind some future applications, we graduate to field theories in curved spaces which are equipped with geometrical structures such as a metric $g_{\mu\nu}$, a Clifford algebra Γ_a for spinors, bundles for internal indices, etc. We will be always working in (torsionless) Euclidean space unless explicitly mentioned.

Some recurring matter fields are the scalar field, the Dirac field, the Proca field (which is a massive vector), the Maxwell field (which is a massless vector) or more generally the Yang-Mills field. Typical quadratic actions are for scalars:

$$S_{\rm s}[\phi] \;=\; \int {\rm d}^d x\sqrt{g}\left\{\frac{1}{2}(\partial\phi)^2+\frac{1}{2}m^2\phi^2+\frac{\xi}{2}R\phi^2\right\} \tag{200}$$

For spinors

$$S_{\rm f}[\psi] \;=\; \int {\rm d}^d x\sqrt{g}\,\overline{\psi}\left\{\not{D}+m\right\}\psi \tag{201}$$

$$\overline{\psi}\equiv\psi^\dagger\gamma^0\,;\qquad \not{D}\equiv\gamma^\mu D_\mu \tag{202}$$

$$D_\mu=\partial_\mu+\Gamma_\mu\,;\qquad \Gamma_\mu=-\frac{\rm i}{4}\omega_\mu{}^{ab}\sigma_{ab}\,;\qquad \sigma_{ab}=\frac{\rm i}{2}\left[\gamma_a,\gamma_b\right] \tag{203}$$

For Proca:

$$S_{\rm p}[B] \;=\; \int {\rm d}^d x\sqrt{g}\left\{-\frac{1}{4}B_{\mu\nu}B^{\mu\nu}+\frac{1}{2}B_\mu B^\mu\right\} \qquad\qquad B_{\mu\nu}\equiv\nabla_\mu B_\nu-\nabla_\nu B_\mu \tag{204}$$

For Maxwell:

$$S_{\rm ga}[A] \;=\; -\frac{1}{4}\int {\rm d}^d x\sqrt{g}\,F_{\mu\nu}F^{\mu\nu} \qquad\qquad F_{\mu\nu}\equiv\nabla_\mu A_\nu-\nabla_\nu A_\mu \tag{205}$$

and likewise the latter for Yang-Mills A^a_μ upon the replacement $F_{\mu\nu}\to F^a_{\mu\nu}=\nabla_\mu A^a_\nu-\nabla_\nu A^a_\mu+{\rm i}f^a{}_{bc}A^b_\mu A^c_\nu$ and the inclusion of an additional internal trace.

Actions which are just like their flat space counterpart except for the inclusion of $\sqrt{g}$ in the volume element and the replacement $\partial\to\nabla, D$ are defined to describe a **minimal** coupling to the curved geometry. If there is an explicit coupling to curvatures, like ξ in the scalar case, the action describes a **non-minimal** coupling.

8.2 How general is the "trace-log" formula?

We can compute the one loop effective actions in the traditional way

$$\Gamma_{\rm s}[g] \;=\; \frac{1}{2}\,{\rm Tr}_{\rm s}\ln\left(\Delta+\xi R+m^2\right) \tag{206}$$

$$\Gamma_{\rm f}[g] \;=\; -\,{\rm Tr}_{\rm f}\ln\left(\not{D}+m\right) \tag{207}$$

$$\Gamma_{\rm p}[g] \;=\; \frac{1}{2}\,{\rm Tr}_{\rm v}\ln\left(\delta^\nu_\mu\Delta+\nabla_\mu\nabla^\nu+R_\mu{}^\nu+\delta^\nu_\mu m^2\right) \tag{208}$$

with $\Delta=-\nabla^2$ the **Laplacian operator** in curved space. We have already manipulated the scalar action often, it is the famous "trace-log" formula in which we have to compute the trace of the logarithm of an operator of Laplace-type, which means it is the sum of a Laplacian and an additional local term (known in math as **endomorphism)** which is this case is ξR.

Notice that in the absence of further interactions, or if interactions can be neglected, these are the full effective actions. The three actions above seem to have almost nothing in common, but let's perform a couple of manipulations first. Consider the square of the Diract operator

$$\left(\not{D} + m_{\rm f}\right)^2 = \Delta + \frac{R}{4} + m^2$$

which you can use to rewrite

$$\Gamma_{\rm f}[g] = -\frac{1}{2}\,{\rm Tr}_{\rm f}\ln\left(\Delta + \frac{R}{4} + m^2\right) \tag{209}$$

Similar manipulations can be done for the Proca case

$$\Gamma_{\rm p}[g] = \frac{1}{2}\,{\rm Tr}_{\rm v}\ln\left(\delta^\nu_\mu\Delta + R_\mu{}^\nu + \delta^\nu_\mu m^2\right) - \frac{1}{2}\,{\rm Tr}_{\rm s}\ln\left(\Delta_g + m^2\right)\,. \tag{210}$$

The spinor effective action looks like, modulo the overall sign and factor, the one of a scalar (four scalar actually) but with a different endomorphism, and similar considerations apply to the Proca case.

In other words we can generically write all traces as

$$\Gamma[g] = \frac{1}{2}\,{\rm Tr}\log\left(\mathcal{O} + m^2\right) \qquad\qquad \mathcal{O} = -\nabla^2 + E$$

for specific endomorphisms E. To make sense of the above formula we introduce $\mathcal{H}(s) \equiv {\rm e}^{-s\mathcal{O}}$ which is known as the **heat kernel** of the operator $\mathcal{O}$ and represent the trace as

$$\Gamma[g] = -\frac{1}{2}\,{\rm Tr}\int\frac{{\rm d}s}{s}\,{\rm e}^{-sm^2}\,\mathcal{H}(s)$$

The trace is performed by giving a representation to $\mathcal{H}(s)$. In coordinate space it is a matrix which is formally written as $\mathcal{H}(s;x,x') \equiv \langle x|{\rm e}^{-s\mathcal{O}}|x'\rangle$ in which $|x\rangle$ are normalized $\int {\rm d}^dx\sqrt{g}\langle x'|x\rangle = 1$ and therefore

$$\Gamma[g] = -\frac{1}{2}\,{\rm tr}\int\frac{{\rm d}s}{s}\int{\rm d}^dx\sqrt{g}\,{\rm e}^{-sm^2}\,\mathcal{H}(s;x,x)$$

8.3 How about beyond the "trace-log" formula?

Feynman diagrams are constructed by integrating products of propagators, and loosely speaking the propagator of any of the actions above is the inverse of some operator of the form $\left(-\nabla^2 + E\right)^{-1} = \mathcal{O}^{-1}$. Let's define the **Green function** of $\mathcal{O}$ from

$$\mathcal{O}_x G(x,x') = \delta^{(d)}(x,x') \tag{211}$$

The Dirac delta is covariant $\delta^{(d)}(x,x') = \langle x'|x\rangle$. We can think at the Green function as being the inverse in coordinate space

$$G(x,x') = \langle x|\mathcal{O}^{-1}|x'\rangle \tag{212}$$

Following this logic we give a representation of the Green function using the heat kernel

$$G(x,x') = \int_0^\infty {\rm d}s\,\mathcal{H}(s;x,x') \tag{213}$$

In which we used that for $a > 0$ the integral of the exponentisal function gives

$$\int_0^\infty {\rm d}s\,{\rm e}^{-sa} = \frac{1}{a}$$

(so the spectrum of the operator should at least be bounded).

8.4 What about any trace?

We have seen in an earlier exercise that the Laplace transform method can be used to cast any trace as the trace of the heat kernel

$$\frac{1}{2}\operatorname{Tr} f\left(\mathcal{O}\right)=\frac{1}{2}\int_0^\infty \mathrm{d}s\,\mathcal{L}^{-1}[f](s)\operatorname{Tr}\mathrm{e}^{-s\mathcal{O}}$$

in which $\mathcal{L}^{-1}[f](s)$ is the inverse Laplace transform of the function $f(x)$.

We now know that several functional traces as well as loop computations can be cast as products of integrals of the trace of the heat kernel. We also know that several heat kernel operators are similar, and ideally computing the heat trace of the most general case will allow us to benefit from a result that can be used in several contexts! And so we do it.

9 Heat kernel: methods

9.1 Heat kernel from the sum of the eigenvalues

Very trivially, if $\mathcal{O}$ is diagonalized by a basis of eigenfunctions

$$\mathcal{O}_x\phi_n(x)=\lambda_n\phi_n(x)$$

and the basis is orthonormal and complete

$$\int_x \phi_n(x)\phi_m(x)=\delta_{nm}\qquad\qquad\sum_n \phi_n(x)\phi_n(y)=\delta(x,y)$$

then we can use the notation $\phi_n(x)=\langle x|n\rangle$ to show

$$\mathcal{H}(s;x,x')=\sum_n \mathrm{e}^{-s\lambda_n}\phi_n(x)\phi_n(x')$$

The trace is perticularly simple

$$\operatorname{Tr}\mathcal{H}(s)=\sum_n \mathrm{e}^{-s\lambda_n}$$

If you know the eigenvalues of a certain manifold, this is the best way to go.

9.2 Perturbative expansion of the heat kernel

Let's first condense the notation to save some space

$$\mathcal{H}(s;x,x')\to\mathcal{H}^s_{x,x'}$$

The heat kernel is understood as the solution to the following differential equation

$$\partial_s\mathcal{H}^s_{x,x'}+\mathcal{O}_x\mathcal{H}^s_{x,x'}=0\qquad\qquad\mathcal{H}^0_{x,x'}=\delta_{x,x'}$$

It can be verified that $\langle x|\mathrm{e}^{-s\mathcal{O}}|x'\rangle=\mathrm{e}^{-s\mathcal{O}_x}\langle x|x'\rangle$ is a formal solution to the above. The heat kernel can be understood as describing the propagation of heat from a point source at $x=x'$ in the time s, which is often known as **heat kernel time** or **propertime**.

Now consider another operator $\mathcal{O}'=\mathcal{O}+V$ and its heat kernel $\mathcal{H}'(s;x,x')$, where V is an arbitrary differential operator. We want to relate the two heat kernels using an expansion in V. We define

$$\mathcal{U}^s_{xy}=\int_z \mathcal{H}^{-s}_{xz}\mathcal{H}'^s_{zy}$$

which has time derivative

$$\partial_s \mathcal{U}^s_{xy} = -\int_z \mathcal{H}^{-s}_{xz} V_z \mathcal{H}'^s_{zy} = -\int_{zw} \mathcal{H}^{-s}_{xz} V_z \mathcal{H}^s_{zw} \mathcal{U}^s_{wy}$$

This equation's right hand side is like a matrix acting on $\mathcal{U}$. It is formally solved by the **Dyson series**

$$\mathcal{U}^s_{xy} = \mathcal{T} \exp\left\{-\int_0^s \mathrm{d}t \int_z \mathcal{H}^{-t}_{xz} V_z \mathcal{H}^t_{zy}\right\} = \mathcal{T} \exp\left\{-s\int_0^1 \mathrm{d}t \int_z \mathcal{H}^{-st}_{xz} V_z \mathcal{H}^{st}_{zy}\right\}$$

(Formally in the sense that the above formula is defined as the solution of that differential equation.) The symbol T means that the time parameter s must be ordered, to see that notice that the solution can be constructed by iteration of

$$\mathcal{U}^s_{xy} = \delta_{x,y} - s\int_0^s \mathrm{d}t_1 \tilde{V}^{t_1}_{xw} \mathcal{U}^s_{wy}$$

with $\tilde{V}^t_{xy} = \int_z \mathcal{H}^{-t}_{xz} V_z \mathcal{H}^t_{zy}$. Any new "time" parameter such as t_1 must be ordered as $0 < t_n < \cdots < t_1 < s$. Now use the Dyson series to show

$$\mathcal{H}'^s_{xy} = \int_w \mathcal{H}^s_{xw} \mathcal{T} \exp\left\{-s\int_0^1 \mathrm{d}t \int_z \mathcal{H}^{-st}_{wz} V_z \mathcal{H}^{st}_{zy}\right\}$$

which in practice means

$$\mathcal{H}'^s_{xy} = \sum_{n\geq 0} (-s)^n \int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n \, \mathcal{H}^{s(1-t_1)}_{xx_1} V_{x_1} \mathcal{H}^{s(t_1-t_2)}_{x_1x_2} V_{x_2} \mathcal{H}^{s(t_2-t_3)}_{x_2x_3} \ldots V_{x_n} \mathcal{H}^{st_n}_{x_nx} \quad (214)$$

The above formula can be used to obtain the heat kernel of any operator $\mathcal{O}'$ if the one of $\mathcal{O}$ is known. The heat kernel of the flat space Laplacian is known exactly thanks to momentum transformation. Let $\mathcal{O} = -\partial^2$, we have in coordinate space

$$\mathcal{H}^s_{xy} = \frac{1}{(4\pi s)^{d/2}} \mathrm{e}^{-\frac{|x-y|^2}{4s}}$$

and in momentum space

$$\tilde{\mathcal{H}}^s(p) = \mathrm{e}^{-sp^2}$$

One strategy for the computation of the heat kernel is therefore to use an expansion around flat space. To expand in flat space take $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$ and pretend that $h_{\mu\nu}$ is "small". Imagine thus that $\mathcal{O}' = -\partial^2 + V$ in which V includes everything of the operator $\Delta + E$ which is not of the form $-\partial^2$. We have that

$$V = E + \mathcal{O}(h)$$

and the infinitely many terms $\mathcal{O}(h)$ have in general a complicate differential structure. In one exercise we are working out the second order in E, but the computations of the further orders in $h_{\mu\nu}$ is also possible. This expansion in fluctuations of the metric can be cast in one-to-one correspondence with an expansion in curvatures and obtain

$$\begin{aligned}
\int_x \mathcal{H}(s;x,x) &= \frac{1}{(4\pi s)^{d/2}} \int \mathrm{d}^d x \sqrt{g}\, \mathrm{tr}\Big\{\mathbf{1} - sE + s\frac{R}{6} + s^2 R F_R(s\Delta) R && (215)\\
&\quad + s^2 R^{\mu\nu} F_{Ric}(s\Delta) R_{\mu\nu} + s^2 E F_E(s\Delta) E + s^2 E F_{RE}(s\Delta) R && (216)\\
&\quad + s^2 \Omega^{\mu\nu} F_\Omega(s\Delta) \Omega_{\mu\nu}\Big\} + \mathcal{O}\left(\mathcal{R}^3\right), && (217)
\end{aligned}$$

with $\Omega_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$ and **form-factors**

$$\begin{aligned}
F_{Ric}(x) &= \frac{1}{6x} + \frac{f(x)-1}{x^2} && (218)\\
F_R(x) &= -\frac{7}{48x} + \frac{f(x)}{32} + \frac{f(x)}{8x} - \frac{f(x)-1}{8x^2} && (219)\\
F_{RE}(x) &= -\frac{f(x)}{4} - \frac{f(x)-1}{2x} && (220)\\
F_E(x) &= \frac{f(x)}{2} && (221)\\
F_\Omega(x) &= -\frac{f(x)-1}{2x} && (222)
\end{aligned}$$

All the functions depend on the so-called basic form factor

$$f(x) \;=\; \int_0^1 d\alpha\, \mathrm{e}^{-\alpha(1-\alpha)x} \tag{223}$$

9.3 Seeley-de Witt expansion of the heat kernel

Recall the flat space Laplacian's the heat kernel

$$\mathcal{H}_0(s;x,x') = \frac{1}{(4\pi s)^{d/2}} \mathrm{e}^{-\frac{|x-y|^2}{4s}}$$

The results of the previous section tell us that we can imagine the heat kernel of an operator in curved space as resulting from an appropriate expansion of $\mathcal{H}_0$. Furthermore, we can imagine that expanding in curvatures *and* derivatives this expansion will be in powers of the parameter s (starting from $s^{-d/2}$).

The problem is that the flat space heat kernel is not covariant one a general curved space. In the previous section this problem was solved by requiring asymptotic flatness, which practically grants a fiducial asymptotic metric and allows for the use of momentum space. Here we follow the strategy of making $\mathcal{H}_0(s;x,x')$ covariant and *then* computing its corrections.

We begin by introducing **Synge's world function**

$$\sigma(x,x')$$

which is defined as half of the square of the geodesic distance between the points x and x'. Strictly speaking $\sigma(x,x')$ and all quantities like $\mathcal{H}(s;x,x')$ that depend on two spacetime points (and transform covariantly) are known as **bi-tensors** and in particular Synge's function is a **bi-scalar**. In flat space we have that

$$\sigma(x,x')\big|_{\text{flat}} = \frac{1}{2}\left|x-x'\right|^2$$

It should be clear that $\sigma(x,x')$ serves the purpose of giving a covariant version of formulas involving the distance $|x-x'|$. One important property that Synge's function satisfies is the relation

$$\sigma_\mu \sigma^\mu = 2\sigma \tag{224}$$

which is very easy to check in flat space, while in curved space requires the use of **Riemann normal coordinates** to prove it.

It is customary to introduce another bi-tensor known as the **van Vleck determinant**

$$\Delta(x,x') = (g(x)g(x'))^{-1/2} \det\left(-\partial_\mu \partial_{\nu'}\sigma\right)$$

which, while being not strictly necessary, helps maintain the **density weight** of the heat kernel when transitioning from flat to curved space. In flat space

$$\Delta(x,x')\big|_{\text{flat}} = 1$$

The van Vleck determinant also satisfies a covariant relation

$$\Delta^{1/2}\sigma_\mu{}^\mu + 2\sigma^\mu \nabla_\mu \Delta^{1/2} = d\Delta^{1/2}\,, \tag{225}$$

With the above definitions in mind we argue that, if the heat kernel comes as a covariant correction of its flat space counterpart, then

$$\mathcal{H}(s;x,x') = \frac{\Delta(x,x')^{1/2}}{(4\pi s)^{d/2}}\; \mathrm{e}^{-\frac{\sigma(x,x')}{2s}}\,(1+\dots)$$

and that the corrections hidden in ... must be of order s or higher (notiec that 1 should be replaced by the number of degrees of freedom if the field is not a scalar). We therefore make the ansatz known as **Seeley-de Witt expansion**

$$\mathcal{H}(s;x,x') = \frac{\Delta(x,x')^{1/2}}{(4\pi s)^{d/2}}\ \mathrm{e}^{-\frac{\sigma(x,x')}{2s}} \sum_{k\geq 0} a_k(x,x')\ s^k \tag{226}$$

An important result of the mathematical literature is that the above expansion exists and converges in some limits. (Practically it is an asymptotic expansion that converges for small-s, but recall that small-s is very useful for the renormalization being it related to the UV).

The **coefficients of the SdW expansion** are also bi-tensors

$$a_k(x,x')$$

are the coefficients of the asymptotic expansion and contain the geometrical information of the operator $\mathcal{O}$, which includes curvatures, connections and interactions expressed in a bi-local way.

Recall now that functional traces are related to the diagonal part $x = x'$ of the heat kernel. We therefore define the **coincidence limit** of any bi-tensor $B(x,x')$ as

$$[B] = \lim_{x'\to x} B(x,x') \tag{227}$$

After the limit $[B]$ becomes a standard tensor with indices in the vector spaces of both its parts. In the case of the heat kernel coefficients this limit will reveal only (local) curvatures. Notice that covariant derivatives do not generally commute with the coincidence limit $\nabla[B] \neq [\nabla B]$!

The coincidence limits of the bitensors $\sigma(x,x')$ and $\Delta(x,x')$ and their derivatives can be obtained by knowing that

$$[\sigma] = 0 \qquad\qquad [\Delta] = 1$$

and then by repeated differentiation of the *crucial* relations

$$\sigma_\mu \sigma^\mu = 2\sigma \qquad\qquad \Delta^{1/2}\sigma_\mu{}^\mu + 2\sigma^\mu \nabla_\mu \Delta^{1/2} = d\Delta^{1/2}$$

The oincidence limits of the coefficients $a_k(x,x')$ can be obtained by differentiating and inductively using

$$k a_k + \sigma^\mu \nabla_\mu a_k + \Delta^{-1/2}\mathcal{O}(\Delta^{1/2} a_{k-1}) = 0 \tag{228}$$

with the boundary condition $\sigma^\mu \nabla_\mu a_0 = 0$. In the relevant example of a simple scalar field the first coefficient is trivial $a_0(x,x') = 1$, because the Seeley-de Witt expansion solves the diffusion equation in flat space. The first two nontrivial coincidence limits for the expansion of the (scalar) operator $\mathcal{O} = -\nabla^2 + E$ are

$$\begin{aligned}
[a_0] &= 1 \\
[a_1] &= \frac{R}{6} - E \\
[a_2] &= \frac{1}{72}R^2 - \frac{1}{6}RE + \frac{1}{2}E^2 - \frac{1}{6}\nabla^2\left(E - \frac{1}{6}R\right) + \frac{1}{180}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}\right)
\end{aligned}$$

which are often referred to as **heat kernel coefficients** because the coincidence limit of the heat kernel is

$$\mathcal{H}(s;x,x) = \frac{1}{(4\pi s)^{d/2}} \sum_{k\geq 0} [a_k]\, s^k \tag{229}$$

10 Heat kernel: applications

We now apply some of the results of the previous section.

10.1 General considetion: s-integration and dimensional poles

One general strategy to complete the integration of the effective action

$$\Gamma[g] = -\frac{1}{2}\,\mathrm{tr}\int\frac{\mathrm{d}s}{s}\int \mathrm{d}^d x\sqrt{g}\,\mathrm{e}^{-sm^2}\,\mathcal{H}(s;x,x)$$

is to leave the s-integration last. All momentum and momentum-like integrations are finite thanks to the exponential, therefore it is the s integral that has to reproduce the standard poles of dimensional regularization. We have established that $\mathcal{H}(s;x,x)$ has an expansion that starts with $s^{-d/2}$, therefore expanding the interior of the integral we realize that the relevant integration is

$$\mathcal{I}_{d,n}(m^2) = \int\frac{\mathrm{d}s}{s}\,\frac{1}{s^{d/2}}\,\mathrm{e}^{-sm^2}$$

The variable s is dual to an energy scale, meaning that qualitatively $s=\infty$ is the IR and $s=0$ is the UV. As one can expect, the IR is finite thanks to the presence of the square mass in the decreasing exponential e^{-sm^2} that makes the integral converge at infinity. The UV is not equally lucky, in fact the integral does not converge for $s\to 0$ if $d\geq 2n$.

Let's analytically continue the integral in d

$$\mathcal{I}_{d,n}(m^2) = m^{d-2n}\Gamma\left(n-\frac{d}{2}\right)$$

The expression has poles for all values of $d=2(n+q)$ in which $q\in\mathbb{N}$ (is a natural number). For example if $d=2n-\epsilon$:

$$\mathcal{I}_{2n-\epsilon,n}(m^2) = \frac{2}{\epsilon}+\ldots$$

If $d=2n+2-\epsilon$:

$$\mathcal{I}_{2n+2-\epsilon,n}(m^2) = -\frac{2}{\epsilon}m^2+\ldots$$

If $d=2n+4-\epsilon$:

$$\mathcal{I}_{2n+4-\epsilon,n}(m^2) = \frac{1}{\epsilon}m^4+\ldots$$

For different increasing values of d (say 2, 4, 6, etc.) we can have different poles for an increasing number of values of n, but only one of them is the "logarithmic" pole $\frac{1}{\epsilon}$ in which the integral is dimensionless.

Suppose $d=2$, then the relevant poles are $n=1$ and $n=0$, with $n=1$ being the logarithmic. For $d=4$ the relevant poles are $n=2$, $n=1$ and $n=0$ with $n=2$ being the logarithmic. Recall that the coefficient $[a_n]$ multiplies these poles, therefore it is the coefficient $[a_n]$ with $d=2n$ that provides the logarithmic pole. Further coefficients multiply finite integrals, and therefore are not required for renormalization. This means that, as long as we are interested in the $d=2$ and $d=4$ cases, then the coefficients given in the previous section are enough!

Odd dimensionalities (such as $d=3$ or $d=5$) are *odd* in the sense that they present no poles (if we are concerned with the one loop effective action, they do present genuine poles at two or more loops). This does not mean that the result is always finite, it simply means that the analytic continuation can be carried to odd dimensionalities without the production of a pole (sometimes this is linked to the concept of superrenormalizability which we did not dive into in these lectures).

10.2 Vacuum effective action in two-dimensional spacetime

Conside a non-minimally coupled non-self-interacting scalar field in $d=2$. Keep in mind that in two dimensions all curvatures can be reduced to the Ricci scalar and the metric, so for example $R_{\mu\nu}=\frac{R}{2}g_{\mu\nu}$. If the scalar field does not have a self interaction then the operator is $\mathcal{O}=-\nabla^2+\xi R$, which does not

depend on the field φ. This means that the final effective action does not depend on the field, but only (parametrically) on the metric (because the metric is not a quantum field yet). In general, for a weakly interacting field the effective action depends only on the metric and it is often called **vacuum effective action**.

Using the considerations of the heat kernel of the previous sections, we can establish that divergences come from the first two heat kernel coefficients. We also know a nonlocal expression for the heat kernel, which in two dimensions and for our operator can only have nonlocalities between two scalar curvatures. Putting everything together we can argue that the effective action will look like

$$\Gamma[g] \;=\; \Gamma_{\rm loc}[g] + \Gamma_{\rm non-loc}[g] \tag{230}$$

with the local action $\Gamma_{\rm loc}[g]$ including the contributions from the divergences $n = 0, 1$ (multiplying $[a_0]$ and $[a_1]$) which are proportional to the volume element and the Ricci scalar. The non-local part will have a form factor between two copies of R.

For our case we find

$$\Gamma_{\rm loc}[g] \;=\; \frac{1}{4\pi}\int {\rm d}^2x\sqrt{g}\left\{\left(\frac{1}{\bar\epsilon}+\frac{1}{2}\right)m^2 + \left(\xi-\frac{1}{6}\right)\frac{1}{\bar\epsilon}R\right\} \tag{231}$$

using $\frac{1}{\bar\epsilon} = \frac{1}{\epsilon} + \frac{1}{2}\ln\left(\frac{4\pi\mu^2}{m^2}\right) - \frac{\gamma}{2}$. This teaches us that in curved space we need two new counterterms to get rid of divergences: one for the volume element and one for the Ricci scalar. In practice, the (local part of the) renormalized action must include counterterms for the cosmological constant and the Newton's constant. It could be of the form:

$$\Gamma_{\rm ren}[g] \;=\; \int {\rm d}^2x\sqrt{g}\,\left\{\Lambda - \frac{1}{G}R\right\} \tag{232}$$

This happens even though these terms were not in the original action, but they cause no harm because φ is the only propagating field. Clearly we have two new beta functions

$$\beta_\Lambda = -\frac{m^2}{4\pi} \qquad\qquad \beta_{1/G} = \left(\xi - \frac{1}{6}\right)$$

and we are forced to consider an Einstein-Hilbert action even if we did not originally plan to!

The finite non-local part is also interesting. Let's use the notation

$$z = \frac{\Delta}{m^2} \qquad a = \sqrt{\frac{4z}{4+z}} \qquad Y = 1 - \frac{1}{a}\log\left|\frac{1+a/2}{1-a/2}\right|$$

then

$$\Gamma_{\rm non-loc}[g] = -\frac{1}{96\pi}\int {\rm d}^2x\sqrt{g}\,R\,\frac{C(z)}{\Delta}\,R \tag{233}$$

with

$$C(z) \;=\; -\frac{1}{2} - \frac{6Y}{a^2} - 12\left(\xi-\frac{1}{4}\right)Y + 6\left(\xi-\frac{1}{4}\right)^2(1-Y) \tag{234}$$

The limit $m^2 \to 0$ corresponds to taking $z \to \infty$ in $C(z)$. We have that $C(\infty) = 1 - 12\xi\,(1 - \xi\ln(z))$ and iff we take $\xi = 0$ the nonlocal action reduces to the **Polyakov action**

$$\Gamma_{\rm non-loc}[g] = \Gamma_{\rm Poly}[g] = -\frac{1}{96\pi}\int {\rm d}^2x\sqrt{g}\,R\,\frac{1}{\Delta}\,R \tag{235}$$

10.3 Interacting theory in four-dimensional spacetime

Simple dimensional analysis tells us that the marginal operators are: ϕ^4, $\phi^2 R$, R^2, $R_{\mu\nu}^2$, $R_{\mu\nu\alpha\beta}^2$ and $\nabla^2 R$. We take the action

$$S[\phi] = \int \mathrm{d}^d x \sqrt{g} \Big\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) + F(\phi) R - a\mathcal{F} - b\mathcal{G} - cR^2 - e\nabla^2 R \Big\} , \tag{236}$$

with

$$\begin{aligned} \mathcal{F} &= \frac{2}{(d-2)(d-1)} R^2 - \frac{4}{d-2} R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\theta} R^{\mu\nu\rho\theta} , \\ \mathcal{G} &= R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\theta} R^{\mu\nu\rho\theta} . \end{aligned} \tag{237}$$

The (integrated) invariant $\mathcal{G}$ is topological, while $\mathcal{F}$ is the square of the Weyl tensor

$$C_{\mu\nu\rho\theta} = R_{\mu\nu\rho\theta} + \frac{1}{d-2} \left(g_{\mu\theta} R_{\nu\rho} - g_{\mu\rho} R_{\nu\theta} + g_{\nu\rho} R_{\mu\theta} - g_{\nu\theta} R_{\mu\rho} \right) + \frac{1}{(d-1)(d-2)} R \left(g_{\mu\rho} g_{\nu\theta} - g_{\mu\theta} g_{\nu\rho} \right)$$

We define

$$\begin{aligned} U(\phi, R) &= V(\phi) + F(\phi) R - a\mathcal{F} - b\mathcal{G} - cR^2 - e\nabla^2 R , \\ \hat{U}(\phi, R) &= U(\phi, R) - \frac{1}{12} R \phi^2 . \end{aligned} \tag{238}$$

At one loop the counterterm to $U(\phi, R)$ can be obtained by the heat kernel $a_2(x, x)$ coefficient:

$$-\frac{1}{(4\pi)^2 \, \epsilon} \int \Big\{ \frac{1}{2} \partial_\phi^2 \hat{U}(\phi, R)^2 + \frac{1}{120} \mathcal{F} - \frac{1}{360} \mathcal{G} \Big\} , \tag{239}$$

while the wavefunction renormalization is analog to the flat space. The beta functions are

$$\begin{aligned} \beta_U &= \frac{1}{(4\pi)^2} \Big\{ \frac{1}{2} \partial_\phi^2 \hat{U}(\phi, R)^2 + \frac{1}{120} \mathcal{F} - \frac{1}{360} \mathcal{G} \Big\} , \\ \beta_Z &= -\frac{1}{6(4\pi)^4} V^{(4)}(\phi)^2 . \end{aligned} \tag{240}$$

Returning to the original functions we find the functional beta functions

$$\begin{aligned} &\beta_V = \frac{1}{2(4\pi)^2} V''(\phi)^2 , \qquad \beta_Z = -\frac{1}{6(4\pi)^4} V^{(4)}(\phi)^2 , \\ &\beta_F = -\frac{1}{(4\pi)^2} \Big\{ \frac{1}{6} - F''(\phi) \Big\} V''(\phi) . \end{aligned} \tag{241}$$

as well as the beta functions for the higher derivative couplings

$$\begin{aligned} \beta_a &= -\frac{1}{120(4\pi)^2} , & \beta_c &= \frac{1}{2(4\pi)^2} \Big\{ \frac{1}{6} - F''(\phi) \Big\}^2 , \\ \beta_b &= \frac{1}{360(4\pi)^2} , & \beta_e &= -\frac{1}{6(4\pi)^2} \Big\{ \frac{1}{5} - F''(\phi) \Big\} . \end{aligned} \tag{242}$$

The function $F(\phi) = \frac{1}{2} \xi \phi^2$ has fixed point $\xi = \frac{1}{6}$. The nonminimal coupling is a special case of

$$\xi_c = \frac{d-2}{4(d-1)} = \frac{1}{6} + \dots . \tag{243}$$

in $d = 4 - \epsilon$. ξ_c is known as the **conformal (or Weyl) coupling** because

$$S[\phi] = \int \mathrm{d}^d x \left\{ \frac{1}{2} (\partial \phi)^2 + \frac{d-2}{4(d-1)} \phi^2 R \right\} \tag{244}$$

is invariant under **Weyl transformations**. Weyl transformations generalize conformal transformations to curved space

$$\phi \to \phi' = \Omega^w \phi = \phi , \qquad g_{\mu\nu} \to g'_{\mu\nu} = \Omega^{-2} g_{\mu\nu} \tag{245}$$

and are local $\Omega = \Omega(x)$. Good reasons to promote scaling $\to$ conformal $\to$ Weyl invariance is that curved space gives access to more information than flat space $T^{\mu\nu} \propto \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$!

10.4 Interacting theory in four-dimensional spacetime and the Sine-Gordon model

The computation of the leading counterterms and beta functions necessitates only the use of the coefficient $a_1(x,x)$. We find the leading counterterm at one loop

$$\frac{1}{4\pi\ \epsilon}\int \partial_\phi^2 \hat{U}(\phi,R) \tag{246}$$

and deduce the very simple RG beta functional

$$\beta_U = -\frac{1}{4\pi}\partial_\phi^2 \hat{U}(\phi,R) \tag{247}$$

Notice that there is no anomalous dimension renormalization coming from our leading order computation.

Introduce the dimensionless potential $u(\varphi,R) = \mu^{-2}U(\varphi,\mu^2 R)$. Using the boundary conditions $u(\varphi,R) = u(-\varphi,R)$ and $\partial_\phi^2 U(\phi,R)|_{\phi=0} = m^2$, at the fixed point in $d=2$ we find

$$u(\varphi,R) = -\frac{m^2}{8\pi}\cos\Big(\sqrt{8\pi}\varphi\Big) + \frac{R}{48\pi}\,. \tag{248}$$

with an implicit dependence on R might in principle be hidden in the mass $m^2 = m^2(R)$. This solution generalizes the **Sine-Gordon model** to curved space. The phase $\sqrt{8\pi}$ is known as **Coleman phase**.

10.5 $2d$ quantum gravity

Consider several matter fields coupled to gravity ($c \gg 1$)

$$\begin{aligned} S[g,\phi] = \int \mathrm{d}^d x\sqrt{g}\Big\{-\frac{1}{G}R \\ + \frac{1}{2}\sum_i\Big(\partial_\mu\phi^i\partial^\mu\phi^i + \xi_c\phi^i\phi^i R\Big)\Big\}\,. \end{aligned} \tag{249}$$

In two dimensions all possible metrics are related by a Weyl transformation, and therefore only their conformal mode is allowed to fluctuate. Close to two dimensions, instead, it is customary to parametrize the metric $g_{\mu\nu} \to (\epsilon/8)^{2/\epsilon}\psi^{4/\epsilon}g_{\mu\nu}$ into a conformal mode ψ and a metric $g_{\mu\nu}$ which is not allowed to fluctuate in its trace part The gauge group $Diff^*$ of the formulation comes from the breaking of a semidirect product of diffeomorphisms and Weyl transformations which is itself isomorphic to the diffeomorphisms group

$$Diff \ltimes Weyl \to Diff^* \simeq Diff$$

but acts on ψ and $g_{\mu\nu}$ in a nonstandard way.

After the decomposition

$$\begin{aligned} S[g,\psi,\phi] = \int \mathrm{d}^d x\sqrt{g}\Big\{-\frac{1}{G}L(\psi,\phi_i)R \\ - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi + \frac{1}{2}\sum_i \partial_\mu\phi^i\partial^\mu\phi^i\Big\}\,. \end{aligned} \tag{250}$$

and renormalize it such that the function $L(\psi,\phi_i)$ respects the conformal coupling. Notice the wrong overall sign of ψ, a problem known as **conformal mode instability**.

Assuming that the instability of ψ can be cured by opportunely Wick rotating the theory

$$\psi \to \mathrm{i}\psi$$

we assume that for large-c the loops of ϕ_i dominate. We can use

$$-\frac{\mu^{-\epsilon}}{G}L(\phi_i)R = U(\phi_i, R) \tag{251}$$

for the dimensionless versions of L and G.

Now we use the beta function of U to determine the renormalization group flow of the renormalized G and $L(\phi_i)$ imposing $L(0) = 1$ along the flow. We find

$$\begin{aligned}\beta_G &= -\epsilon G + \frac{c}{24\pi}G^2 + \frac{c}{4\pi}G\,L''(0)\\ \beta_L &= -\frac{c}{24\pi}G\Big\{1 - L(\phi)\Big\} + \frac{c}{4\pi}\Big\{L(\phi)\;L''(0) - L''(\phi)\Big\}.\end{aligned} \tag{252}$$

and substituing further

$$\beta_G = -\epsilon G - \frac{c}{24\pi}G^2 \tag{253}$$

with $A = -\frac{c}{24\pi}$. This result agrees with the large-c limit of the exact leading result of $2d$ quantum gravity

$$\beta_G = -\epsilon G + \frac{25-c}{24\pi}G^2 \tag{254}$$

11 Invitation to quantum gravity

11.1 History

Problems: conformal mode instability and renormaliability...

- t'Hooft & Veltman 1974. One loop divergences of (Einstein-Hilbert)

  $$S = M_{\rm Pl}^2\int \mathrm{d}^d x\sqrt{g}R$$

  quantum gravity in $d = 4 - \epsilon$. They found the counterterm

  $$\frac{1}{\epsilon}\sqrt{g}\left(\frac{1}{120}R^2 + \frac{7}{20}R_{\mu\nu}R^{\mu\nu}\right)$$

  which is zero on-shell for pure gravity. It can be eliminated through a redefinition $\delta g_{\mu\nu} \propto \frac{1}{\epsilon}$ and dissappears from amplitudes. Less known fact: adding a scalar makes the counterterm nonzero even on-shell, so *gravity coupled to matter is non renormalizable.*

- Goroff & Sagnotti 1985; van de Ven 1991. Two loop divergences contain

  $$\frac{209}{2880\epsilon}\sqrt{g}R_{\mu\nu}{}^{\rho\theta}C_{\rho\theta\alpha\beta}C^{\alpha\beta}{}_{\mu\nu}$$

  which cannot be eliminated on-shell. *Pure gravity is non renormalizable starting from two loops.*

Alternatives that work?

- Weinberg 1976. Gravity might be **asymptotically safe**.

- Stelle 1977; Julve & Tonin 1978; Barth & Christensen 1983. Higher derivative gravity in $d = 4 - \epsilon$

$$S = \int \mathrm{d}^d x \sqrt{g} \left\{ \alpha R^2 + \beta R_{\mu\nu}^2 + \gamma R_{\mu\nu\rho\theta}^2 \right\}$$

  is perturbatively renormalizable and **asymptotically free**! However it is non-unitary: the problem is intrinsic to the higher derivative propagator $\Box^2$ but also manifests through either a **propagator's ghost** and/or a **tachion**

$$\frac{1}{p^4 + \lambda M_{\mathrm{Pl}}^2 p^2} = \frac{1}{\lambda M_{\mathrm{Pl}}^2} \frac{1}{p^2} - \frac{1}{\lambda M_{\mathrm{Pl}}^2} \frac{1}{p^2 + \lambda M_{\mathrm{Pl}}^2}$$

- Kawai & Ninomiya 1989; Kawai, Kitazawa & Ninomiya 1993. Einstein-Hilbert gravity is renormalizable in $d = 2 + \epsilon$.

Ideas: rethink how things work in either **string theory** or **loop quantum gravity** or even "weirder" formalisms.

- Concentrate on background (in)dependence and covariance? LQG, spin-foams etc.
- Concentrate on unification? String theory and friends.

More traditional/minimal solutions:

- Make work higher derivative gravity in $d = 4 - \epsilon$ as unitary theory in some way (proposals range from J&T 1976 to some in 2018). According to proposal you trade renormalizability with (micro)causality, CPT invariance etc.
- Reuter 1996. Extend $d = 2 + \epsilon$ to $\epsilon = 2$ and make work standard Einstein-Hilbert gravity as asymptotically safe theory. Here perturbation theory does not work! Using nonpertubative methods enhances difficulties associated to background and non-universal approximations.

A Standard vs background field path integral

A.1 Standard path integral

We follow the conventions that the path integral is

$$Z[J] \;=\; \int D\varphi\, \mathrm{e}^{-S[\varphi]+J\cdot\varphi} \tag{255}$$

with correlators

$$\langle\varphi(x_1)\dots\varphi(x_n)\rangle = \left.\frac{\delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)}\right|_{J=0} \tag{256}$$

The generator of the connected correlators is

$$\mathrm{e}^{W[J]} \;=\; Z[J] = \int D\varphi\, \mathrm{e}^{-S[\varphi]+J\cdot\varphi} \tag{257}$$

with the connected correlators being

$$\langle\varphi(x_1)\dots\varphi(x_n)\rangle_c = \left.\frac{\delta^n W[J]}{\delta J(x_1)\dots\delta J(x_n)}\right|_{J=0} \tag{258}$$

As a function of the source we can define

$$\overline{\varphi}(x_1) = \langle\varphi(x_1)\rangle_J = \frac{\delta W[J]}{\delta J(x_1)} \tag{259}$$

sometimes known as **classical** or **average field**.

The effective action is defined as the Legendre transform

$$\Gamma[\overline{\varphi}] = \sup_J \left(\int \mathrm{d}^d x\, \overline{\varphi}(x) J(x) - W[J] \right) \tag{260}$$

When taking the extremum among all sources we find one such that $J = J[\overline{\phi}]$ and use it to express the functional in terms of the average field. The effective action generates the 1PI vertices of the theory. Notice the compact definition

$$\mathrm{e}^{-\Gamma[\overline{\varphi}]} \;=\; \int D\varphi\, \mathrm{e}^{-S[\varphi]+\frac{\delta\Gamma}{\delta\overline{\varphi}}\cdot(\varphi-\overline{\varphi})} \tag{261}$$

A.2 Background field path integral

In the background approach one follows the strategy of decomposing the field into a background and fluctuations $\varphi \to \varphi + \chi$ and integrating over the fluctuations χ. We define the path integral

$$Z[\varphi; J] \;=\; \int D\chi\, \mathrm{e}^{-S[\varphi+\chi]+J\cdot\chi} \tag{262}$$

The dependence on the background is kept parametrically. All other functionals are derived in the same way, but manipulating χ. The connected generator is

$$\mathrm{e}^{W[\varphi;J]} \;=\; Z[\varphi; J] = \int D\chi\, \mathrm{e}^{-S[\varphi+\chi]+J\cdot\chi} \tag{263}$$

Then we define

$$\overline{\chi}(x_1) = \langle\chi(x_1)\rangle_{\varphi;J} = \frac{\delta W[\varphi; J]}{\delta J(x_1)} \tag{264}$$

that is used in the effective action

$$\Gamma[\varphi;\overline{\chi}] = \sup_J \left(\int \mathrm{d}^d x\, \overline{\chi}(x) J(x) - W[\varphi;J] \right) \tag{265}$$

The compact definition is

$$\mathrm{e}^{-\Gamma[\varphi;\overline{\chi}]} \;=\; \int D\chi\, \mathrm{e}^{-S[\varphi+\chi]+\frac{\delta\Gamma}{\delta\overline{\chi}}\cdot(\chi-\overline{\chi})} \tag{266}$$

Why did we apply the background field method? For two intimately connected reasons. First consider the obvious fact that if we supplement the condition $\overline{\chi}=0$ then

$$\mathrm{e}^{-\Gamma[\varphi;0]} \;=\; \int D\chi\, \mathrm{e}^{-S[\varphi+\chi]+\frac{\delta\Gamma}{\delta\overline{\chi}}\cdot\chi} \tag{267}$$

and thus we have a functional of a single field, $\Gamma[\varphi;0]=\overline{\Gamma}[\varphi]$, which would be computed from a zero point function. Also, if there is a symmetry associated to φ then this symmetry is manifest already at this level!

The question is if this functional has actually any physical meaning. To understand this let's notice that our "unphysical" split of the original field $\varphi \to \varphi+\chi$ has introduced a new symmetry known as **split symmetry**

$$\varphi \to \varphi - B \qquad\qquad \chi \to \chi + B \tag{268}$$

By construction $S[\varphi+\chi]$ is invariant under this transformation. This is equivalent to say that

$$\frac{\delta S}{\delta\varphi} = \frac{\delta S}{\delta\chi} \tag{269}$$

Assuming that the measure is invariant under translations (this is a much more sensitive point than you would imagine!), one can perform a variation wrt $\varphi\to\delta\varphi$ of the path integral on the left

$$\delta \mathrm{e}^{-\Gamma[\varphi;\overline{\chi}]} = -\mathrm{e}^{-\Gamma[\varphi;\overline{\chi}]}\frac{\delta\Gamma}{\delta\varphi}\delta\phi \tag{270}$$

and on the right do the same but also compensate with the change of the measure $\chi\to\chi-\delta\varphi$ (assuming $D\chi\to D\chi$) to get

$$\delta\int D\chi\, \mathrm{e}^{-S[\varphi+\chi]+\frac{\delta\Gamma}{\delta\overline{\chi}}\cdot(\chi-\overline{\chi})} = -\mathrm{e}^{-\Gamma[\varphi;\overline{\chi}]}\frac{\delta\Gamma}{\delta\overline{\chi}}\delta\phi \tag{271}$$

Together they imply the **split Ward identity**

$$\frac{\delta\Gamma}{\delta\varphi} = \frac{\delta\Gamma}{\delta\overline{\chi}} \tag{272}$$

We use it to argue that the background field effective action is a function of the sum $\varphi+\overline{\chi}$ because it is invariant under the shift

$$\Gamma[\varphi;\overline{\chi}] = \Gamma[\varphi+\overline{\chi},\overline{\chi}-\overline{\chi}] = \Gamma[\varphi+\overline{\chi},0] = \overline{\Gamma}[\varphi+\overline{\chi}] \tag{273}$$

and therefore making the identification that $\overline{\varphi}=\varphi+\overline{\chi}$ we prove that the background effective action and the original effective action are in form the same

$$\overline{\Gamma}[\overline{\varphi}] = \Gamma[\overline{\varphi}] \tag{274}$$

A.3 Symmetries and the background path-integral

Suppose that $\varphi \to \varphi' = F[\varphi]$ is a symmetry of the action $S[\varphi]$:

$$S[\varphi] = S[\varphi'] \tag{275}$$

and suppose that this symmetry is not anomalous, so that it is preserved at the level of the path-integral and becomes a symmetry of the effective action

$$\Gamma[\varphi] = \Gamma[\varphi'] \tag{276}$$

If we now try to construct the background path-integral $\varphi \to \varphi + \chi$, how do we best treat the symmetry? We have two possibilities:

- **background symmetry:** the background changes as would the original field $\varphi \to \varphi' = K[\varphi]$ and the fluctuation changes linearly for small χ, $\chi \to \chi' = K[\varphi + \chi] - K[\varphi] \simeq \frac{\delta K[\varphi]}{\delta \varphi}\chi$
- **full (quantum) symmetry:** the background remains invariant $\varphi \to \varphi$ and the fluctuation takes over the full transformation $\chi \to \chi' = K[\varphi + \chi] - \varphi$

The second one is the most important one, because it is equivalent to the original symmetry. However, when constructing the background path-integral one generally expands in (as little as possible) powers of χ, which implies that there is no control on the full symmetry. What typically happens is that in the background method we are left with the manifest linearized background symmetry, which is not equivalent to the full one. Instead of probing higher vertices, the simple solution to this problem is to use the Ward identities for the split shown before. *In fact if both the split identity and the background symemtry are true, then the full symmetry is also true.*

B Average effective action

Define the modified path-integral

$$Z_k[J] = \int D\varphi \, \mathrm{e}^{-S[\varphi] - \Delta S_k[\varphi] + J\cdot\varphi} \tag{277}$$

with an IR cutoff

$$\Delta S_k[\varphi] = \frac{1}{2}\int d^dx \, d^dy \, \varphi(x)\mathcal{R}_k(x-y)\varphi(y) \tag{278}$$

$$= \frac{1}{2}\, \varphi \cdot \mathcal{R} \cdot \varphi \tag{279}$$

The functional $\Delta S_k[\varphi]$ is a cutoff that slows the propagation of the IR modes, so that the high modes are intergated in the path integral. The deriving functionals therefore are natural effective functionals for the high modes. In momentum space we choose

$$\Delta S_k[\varphi] = \frac{1}{2}\int d^dq \, \varphi_{-q} R_k(q^2) \varphi_q \tag{280}$$

The IR cutoff is required to satisfy:

- $R_{k=0}(q^2) = 0$ which ensures $Z_{k=0}[J] = Z[J]$;

- $R_{k\to\infty}(q^2) \to \infty$, so no mode is propagating for high values of k;
- $R_k(q^2) \simeq 0$ for $q^2 \geq k$, so the rapid modes are unaffected;
- additionally we may want $R_k(q^2) \sim k^2$ for $q^2 \leq k$, so the slow modes are cut off by the scale k^2.

We follow the standard definitions almost straightforwardly

$$\mathrm{e}^{W_k[J]} = Z_k[J] \tag{281}$$

and

$$\overline{\varphi}(x_1) = \langle \varphi(x_1) \rangle_{k;J} = \frac{\delta W_k[J]}{\delta J(x_1)} \tag{282}$$

The average effective action is defined as

$$\Gamma_k[\overline{\varphi}] = \sup_J \left(\int \mathrm{d}^d x \, \overline{\varphi}(x) J(x) - W_k[J] - \Delta S_k[\overline{\varphi}] \right) \tag{283}$$

and the usual path integral definition holds with a slight modification

$$\mathrm{e}^{-\Gamma_k[\overline{\varphi}]} = \int D\varphi \, \mathrm{e}^{-S[\varphi] - \Delta S_k[\varphi - \overline{\varphi}] + \frac{\delta \Gamma_k}{\delta \overline{\varphi}} \cdot (\varphi - \overline{\varphi})} \tag{284}$$

(which uses explicitly the fact that the cutoff is quadratic).

What are the properties of this new effective action? In the ultraviolet, which we define $k \to \Lambda \simeq \infty$, we can argue that if $R_k(q^2) \to \infty$ for all q^2 the integral can be approximated by saddle point. In other words recall the limiting definition of the Dirac delta

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-x^2/(2\sigma^2)} = \lim_{k \to 0} \frac{k}{\sqrt{2\pi}} \mathrm{e}^{-k^2 x^2/2} \tag{285}$$

which implies

$$\mathrm{e}^{-\Delta S_k[\varphi - \overline{\varphi}]} = \mathrm{e}^{-(\varphi - \overline{\varphi}) \cdot \mathcal{R}_k \cdot (\varphi - \overline{\varphi})} \tag{286}$$

$$\simeq \mathcal{N} \delta[\varphi - \overline{\varphi}] \tag{287}$$

Using this in the path integral we argue

$$\mathrm{e}^{-\Gamma_\Lambda[\overline{\varphi}]} = \int D\varphi \, \mathrm{e}^{-S[\varphi] - \Delta S_k[\varphi - \overline{\varphi}] + \frac{\delta \Gamma_\Lambda}{\delta \overline{\varphi}} \cdot (\varphi - \overline{\varphi})} \tag{288}$$

$$\simeq \int D\varphi \, \delta[\varphi - \overline{\varphi}] \, \mathrm{e}^{-S[\varphi] + \frac{\delta \Gamma_\Lambda}{\delta \overline{\varphi}} \cdot (\varphi - \overline{\varphi})} \tag{289}$$

$$= \mathrm{e}^{-S[\overline{\varphi}]} \tag{290}$$

which means that the ultraviolet limit of the average action is the bare action itself (notice that this manipulation is only formal, in the actual $\Lambda \to \infty$ limit there might be divergences related to $\mathcal{N}$ that must be handled with more care). In the infrared we know that $\Delta S_{k=0}[\varphi] = 0$ and therefore by construction

$$\Gamma_{k=0}[\overline{\varphi}] = \Gamma[\overline{\varphi}] \tag{291}$$

it becomes the full effective action. Following the **sliding scale** k the average effective action interpolates between the bare and the effective action, that's a nice property!

It should be clear and rather straightforward that this method can be paired with the background field technique too to obtain a background effective average action.

B.1 Wetterich equation

All the k-dependent functionals satisfy meaningful flow equations wrt the scale k. Take a scale derivative of the path integral

$$\begin{aligned}
\partial_k \mathrm{e}^{-\Gamma_k[\overline{\varphi}]} &= \partial_k \int D\varphi\, \mathrm{e}^{-S[\varphi]-\Delta S_k[\varphi-\overline{\varphi}]+\frac{\delta\Gamma_k}{\delta\overline{\varphi}}\cdot(\varphi-\overline{\varphi})} && (292)\\
-\partial_k\Gamma_k[\overline{\varphi}]\mathrm{e}^{-\Gamma_k[\overline{\varphi}]} &= -\int D\varphi \left\{\partial_k\Delta S_k[\varphi-\overline{\varphi}]-\partial_k\frac{\delta\Gamma_k}{\delta\overline{\varphi}}\cdot(\varphi-\overline{\varphi})\right\}\mathrm{e}^{-S[\varphi]-\Delta S_k[\varphi-\overline{\varphi}]+\frac{\delta\Gamma_k}{\delta\overline{\varphi}}\cdot(\varphi-\overline{\varphi})} && (293)\\
\partial_k\Gamma_k[\overline{\varphi}] &= \mathrm{e}^{\Gamma_k[\overline{\varphi}]}\int D\varphi\,\partial_k\Delta S_k[\varphi-\overline{\varphi}]\,\mathrm{e}^{-S[\varphi]-\Delta S_k[\varphi-\overline{\varphi}]+\frac{\delta\Gamma_k}{\delta\overline{\varphi}}\cdot(\varphi-\overline{\varphi})} && (294)\\
&= \langle\partial_k\Delta S_k[\varphi-\overline{\varphi}]\rangle && (295)
\end{aligned}$$

Now study

$$\begin{aligned}
\langle\partial_k\Delta S_k[\varphi-\overline{\varphi}]\rangle &= \frac{1}{2}\langle(\varphi-\overline{\varphi})\cdot\partial_k\mathcal{R}_k(\varphi-\overline{\varphi})\rangle && (296)\\
&= \frac{1}{2}\langle\varphi\cdot\mathcal{R}_k\cdot\varphi\rangle-\frac{1}{2}\langle\varphi\rangle\cdot\mathcal{R}_k\cdot\langle\varphi\rangle && (297)
\end{aligned}$$

Introduce a collective index for coordinate and eventual internal indices $\varphi=\varphi^i$. The above object is composed by the connected two point function

$$\langle\varphi^i\varphi^j\rangle-\langle\varphi^i\rangle\langle\varphi^j\rangle=\frac{\delta^2W_k[J]}{\delta J^i\delta J^j} \tag{298}$$

The second derivative of $W_k[J]$ and of $\Gamma_k[\overline{\varphi}]$ are related (the relation is modified by the subtraction of the cutoff that we have made before)

$$\begin{aligned}
\langle\varphi^i\varphi^j\rangle-\langle\varphi^i\rangle\langle\varphi^j\rangle &= \frac{\delta^2W_k[J]}{\delta J^i\delta J^j} && (299)\\
&= \left(\frac{\delta^2}{\delta\overline{\varphi}^i\delta\overline{\varphi}^j}\left(\Gamma_k[\overline{\varphi}]+\Delta S_k[\overline{\varphi}]\right)\right)^{-1} && (300)\\
&= \left(\Gamma_k^{(2)}[\overline{\varphi}]+\mathcal{R}_k\right)^{-1}_{ij} && (301)
\end{aligned}$$

We use it to evaluate

$$\begin{aligned}
\langle\partial_k\Delta S_k[\varphi-\overline{\varphi}]\rangle &= \frac{1}{2}\left(\Gamma_k^{(2)}[\overline{\varphi}]+\mathcal{R}_k\right)^{-1}_{ij}(\partial_k\mathcal{R}_k)_{ij} && (302)\\
&= \frac{1}{2}\,\mathrm{Tr}\left(\Gamma_k^{(2)}[\overline{\varphi}]+\mathcal{R}_k\right)^{-1}\partial_k\mathcal{R}_k && (303)
\end{aligned}$$

This implies the Wetterich equation

$$k\partial_k\Gamma_k[\overline{\varphi}] = \frac{1}{2}\,\mathrm{Tr}\left(\Gamma_k^{(2)}[\overline{\varphi}]+\mathcal{R}_k\right)^{-1}k\partial_k\mathcal{R}_k \tag{304}$$

B.2 Relation with the one-loop improved flow

Consider the one loop effective action

$$\Gamma_{1-\mathrm{loop}} = S[\varphi]+\frac{1}{2}\,\mathrm{Tr}\log S^{(2)} \tag{305}$$

The above expression can be understood in a renormalized sense, so that $S[\varphi]$ contains the renormalized couplings and the counterterms to cancel the divergences of the trace. Now modify the right hand side with the introduction of the IR cutoff

$$\Gamma_{1-\mathrm{loop};k} = S[\varphi]+\frac{1}{2}\,\mathrm{Tr}\log\left(S^{(2)}+\mathcal{R}_k\right) \tag{306}$$

Take a scale derivative

$$k\partial_k\Gamma_{1-\text{loop};k} = \frac{1}{2}\,\text{Tr}\left(S^{(2)}+\mathcal{R}_k\right)^{-1} k\partial_k\mathcal{R}_k \tag{307}$$

One of the first ways in which several nonperturbative RG equations were originally derived was through the process of **RG improvement**, that is by replacing $S \to \Gamma$ on the right hand side. The result of this improvement is the Wetterich equation; the interesting fact is of course that it is an exact equation that does not require the improvement!

B.3 Wetterich equation and the background field

The quadratic cutoff of the average action is well suited if we are trying to preserve linear symmetries. For example, suppose that our field φ is complex and the action is invariant under a $U(1)$ transformation $\varphi \to \mathrm{e}^{\mathrm{i}\alpha}$. Then the quadratic cutoff $\Delta S_k = \frac{1}{2}\varphi^* \cdot \mathcal{R}_k \cdot \varphi$ is invariant, which can be used to show that the average action Γ_k will be invariant too. If for some reason the symmetry is realized nonlinearly (for example it is genuinely nonlinear or it is broken), however, the cutoff is not invariant. Imagine a transformation like $\varphi \to \varphi + \alpha\varphi^2$: there is no way that a quadratic cutoff is invariant under it. The strategy to preserve these kind of symmetries is to resort to the background field.

It should be clear that the ideas of this section and those of the background method can be combined to have a background effective average action. Given the fluctuation χ, the average fluctuation $\overline{\chi}$, and the background φ, we have the result

$$\mathrm{e}^{-\Gamma_k[\varphi,\overline{\chi}]} = \int D\chi\, \mathrm{e}^{-S[\varphi+\chi]-\Delta S_k[\varphi;\chi-\overline{\chi}]+\frac{\delta\Gamma_k}{\delta\overline{\chi}}\cdot(\chi-\overline{\chi})} \tag{308}$$

We have introduced a cutoff action $\Delta S_k[\varphi;\chi-\overline{\chi}]$ which is quadratic in $\overline{\chi}$ but depends parametrically on the background so that it is possible to preserve arbitrary nonlinear background versions of a symmetry. We can use the path integral to derive the background version of the flow equation

$$k\partial_k\Gamma_k[\varphi,\overline{\chi}] = \frac{1}{2}\,\text{Tr}\left(\Gamma_k^{(0,2)}[\varphi,\overline{\chi}]+\mathcal{R}_k[\varphi]\right)^{-1} k\partial_k\mathcal{R}_k[\varphi] \tag{309}$$

Recall that the usefulness of the background method was that we could just compute $\Gamma_k[\varphi,0]$ and essentially identify it with the full effective action thanks to the split Ward identities. It is tempting to assume that

$$k\partial_k\Gamma_k[\varphi,0] \simeq \frac{1}{2}\,\text{Tr}\left(\Gamma_k^{(2,0)}[\varphi,0]+\mathcal{R}_k[\varphi]\right)^{-1} k\partial_k\mathcal{R}_k[\varphi] \tag{310}$$

but this, as displayed, is an approximation even when the measure is invariant.

To see this let us derive the split Ward identities from the path integral: take an arbitrary change of the background field

$$\delta\mathrm{e}^{-\Gamma_k[\varphi,\overline{\chi}]} = \delta\int D\chi\, \mathrm{e}^{-S[\varphi+\chi]-\Delta S_k[\varphi;\chi-\overline{\chi}]+\frac{\delta\Gamma_k}{\delta\overline{\chi}}\cdot(\chi-\overline{\chi})} \tag{311}$$

The left hand side works in the same way as the standard path integral

$$\delta\mathrm{e}^{-\Gamma_k[\varphi,\overline{\chi}]} = -\mathrm{e}^{-\Gamma_k[\varphi,\overline{\chi}]}\frac{\delta\Gamma_k}{\delta\varphi}\delta\varphi \tag{312}$$

while the right hand side has one additional contribution coming from the parametric dependence of the cutoff

$$\delta\int D\chi\, \mathrm{e}^{-S[\varphi+\chi]-\Delta S_k[\varphi;\chi-\overline{\chi}]+\frac{\delta\Gamma_k}{\delta\overline{\chi}}\cdot(\chi-\overline{\chi})} = -\mathrm{e}^{-\Gamma_k[\varphi,\overline{\chi}]}\frac{\delta\Gamma_k}{\delta\overline{\chi}}\delta\varphi - \frac{1}{2}\mathrm{e}^{-\Gamma_k[\varphi,\overline{\chi}]}\langle(\chi-\overline{\chi})\cdot\frac{\delta\mathcal{R}_k}{\delta\varphi}\cdot(\chi-\overline{\chi})\rangle\delta\varphi \tag{313}$$

We manipulate it and equating to the lhs

$$\frac{\delta\Gamma_k}{\delta\varphi} = \frac{\delta\Gamma_k}{\delta\overline{\chi}} + \frac{1}{2}\,\text{Tr}\left(\Gamma_k^{(0,2)}[\varphi,\overline{\chi}]+\mathcal{R}_k[\varphi]\right)^{-1}\frac{\delta\mathcal{R}_k}{\delta\varphi} \tag{314}$$

This is known as **modified split Ward identity**.

B.4 Loop expansion from functional renormalization

Disclaimer: this is a non-standard derivation of perturbation theory from functional renormalization and should be taken for what it is. I do believe that it teaches a lot of the meaning of the process of renormalization. Star by restoring $\hbar$ in the flow equation

$$k\frac{\partial}{\partial k}\Gamma_k[\varphi] = \frac{\hbar}{2}\,\mathrm{Tr}\,(G_k k\partial_k R_k) = \frac{\hbar}{2}\,\mathrm{Tr}\frac{k\frac{\partial}{\partial k}R_k}{\Gamma_k^{(2)}+R_k} \tag{315}$$

and perform a "loop" expansion of the effective average action (it will become clear soon why this is a loop-expansion)

$$\Gamma_k[\varphi] = S_{\mathrm{B}}[\varphi] + \sum_{L\geq 1}\hbar^L\Gamma_{L,\,k}[\varphi] \tag{316}$$

By inserting the expansion into the flow equation

$$\hbar k\partial_k\Gamma_{1,\,k}[\varphi] + \hbar^2 k\partial_k\Gamma_{2,\,k}[\varphi] + \cdots = \frac{\hbar}{2}\,\mathrm{Tr}\frac{k\partial_k R_k}{S_{\mathrm{B}}^{(2)}[\varphi] + R_k + \hbar\Gamma_{1,\,k}^{(2)}[\varphi] + \hbar^2\Gamma_{2,\,k}^{(2)}[\varphi] + \ldots} \tag{317}$$

One can get flows for each order of the expansion using the generating formula

$$k\partial_k\Gamma_{L,\,k}[\varphi] = \frac{1}{L!}\,\frac{\partial^L}{\partial\hbar^L}k\partial_k\Gamma_k[\varphi]\bigg|_{\hbar=0} \tag{318}$$

The first few flows are

$$k\partial_k S_{\mathrm{B}}[\varphi] = 0 \tag{319}$$

$$k\partial_k\Gamma_{1,\,k}[\varphi] = \frac{1}{2}\,\mathrm{Tr}\,(G_{\mathrm{B},\,k}k\partial_k R_k) \tag{320}$$

$$k\partial_k\Gamma_{2,\,k}[\varphi] = \frac{1}{2}\,\mathrm{Tr}\left(\Gamma_{1,\,k}^{(2)}[\varphi]\,k\partial_k G_{\mathrm{B},\,k}\right) \tag{321}$$

with $G_{\mathrm{B},\,k} \equiv \left(S_{\mathrm{B}}^{(2)}[\varphi] + R_k\right)^{-1}$. Notice that the flows in this expansion feature a modificed propagator $G_{\mathrm{B},\,k}$ which includes only the action $S_{\mathrm{B}}[\varphi]$ instead of the full average action $\Gamma_k[\varphi]$.

It is easy to see that the "bare" action S_{B} is k-independent, however all next orders are not. It is possible to integrate the further orders in k. For example the first one comes from observing that

$$k\partial_k\Gamma_{1,\,k}[\varphi] = \frac{1}{2}\,\mathrm{Tr}\,(G_{\mathrm{B},\,k}k\partial_k R_k) \tag{322}$$

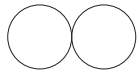
$$= \frac{1}{2}\,\mathrm{Tr}\,k\partial_k\left(S_{\mathrm{B}}^{(2)}[\varphi] + R_k\right) \tag{323}$$

The general integration of both sides will produce

$$\Gamma_{1,\,k}[\varphi] = \frac{1}{2}\,\mathrm{Tr}\log\left(S_{\mathrm{B}}^{(2)}[\varphi] + R_k\right) \tag{324}$$

which is generally a divergent result in quantum field theory. Before going forward with the analysis of the divegences let's notice that we can keep integrating term by term in the perturbative expansion. Neglecting the boundary conditions

$$\Gamma_{1,\,k}[\varphi] = \frac{1}{2}\,\mathrm{Tr}\log\left(S_{\mathrm{B}}^{(2)}[\varphi] + R_k\right) \tag{325}$$

$\Gamma_{2,\,k}[\varphi] =$  (326)

By taking $k = 0$ and integrating further orders you can reprouce the loop expansion as given in textbooks.

How do we interpret this result?

- If we are working in statistical mechanics, $S_{\rm B}$ is a microscopic action which represents the starting point of our RG flow to the IR while the fluctuations give contributions

$$\Gamma_{1,\,k}[\varphi] = \frac{1}{2}\,{\rm Tr}_{q^2\lesssim\Lambda^2}\log\left(S^{(2)}_{\rm B}[\varphi]+R_k\right)$$

  in which we explicitly restricted to physics below a given UV momentum scale Λ.

- If instead we are working in quantum field theory we would like to have a result that can be extended to arbitrary UV scales $\Lambda\to\infty$. In this case we perform two manipulations: we first replace the trace Tr with a certain regulated version of it $\rm Tr_{reg}$ which always gives finite results (examples would be momentum regularization, but also dimensional regularization). We thus have

$$\Gamma_{1,\,k}[\varphi] = \frac{1}{2}\,{\rm Tr}_{\rm reg}\log\left(S^{(2)}_{\rm B}[\varphi]+R_k\right)$$

  and $\Gamma_{1,\,k}[\varphi]$ happens to diverge for some parameter

$$\Gamma^{\rm div}_{L=1,\,k} = {\rm DivP}\left[\frac{1}{2}\,{\rm Tr}_{\rm reg}\log\left(S^{(2)}_{\rm B}[\varphi]+R_k\right)\right]$$

This situation can be iteratively seen loop-by-loop.

On the statistical mechanical side we don't really need any further step: we started in the UV and moved to the IR a little bit. On the QFT side we have made $\Gamma_{1,\,k}[\varphi]$ finite, but we have to deal with the boundary condition $\Gamma_{1,\,k}[\varphi]$. If $\Gamma_{1,\,k}[\varphi]$ is local and has a finite number of terms that coincide with the bare action we can cancel the divergences by performing a further expansion of the bare action in *local* terms

$$S_{\rm B}[\varphi] = S_{\rm R}[\varphi] + \sum_{L\geq 1}\hbar^L\delta S_L[\varphi], \tag{327}$$

where the δS_L are known as **counterterms**. We choose the counterterms to cancel the divergent part of the renormalized $\Gamma_{1,\,k}$ at $k=0$, and more generally they cancel cancel the divergent part of $\Gamma_{L,\,k}$ at $k=0$

$$\delta S_L = -\Gamma^{\rm div}_{L,\,k} = -\Gamma^{\rm div}_{L,\,k=0} \tag{328}$$

So the effective action was always finite in the first place:

$$\begin{aligned}
\Gamma_k[\varphi] &= S_{\rm B}[\varphi] + \sum_{L\geq 1}\hbar^L\Gamma_{L,\,k}[\varphi] && (329)\\
&= S_{\rm R}[\varphi] + \sum_{L\geq 1}\hbar^L\left(\Gamma_{L,\,k}[\varphi] - \Gamma^{\rm div}_{L,\,k=0}\right) && (330)\\
&\equiv S_{\rm R}[\varphi] + \sum_{L\geq 1}\hbar^L\Gamma^{\rm ren}_{L,\,k}[\varphi] && (331)
\end{aligned}$$

Magic! The price to pay for finiteness is the following: if at the beginning we knew which bare action we inputted in the path integral, after subtracting the counterterms we are left with a *different* object $S_{\rm R}$ which we do not know a priori. One can define a theory to be **renormalizable** if $S_{\rm R}$ *is local and has a finite number of parameters to be determined.*

C Schwinger-Dyson equations

Consider the path integral

$$Z[J] = \int D\varphi\,{\rm e}^{-S[\varphi]+J\cdot\varphi} \tag{332}$$

and relabel the integration to be over $\varphi' = \varphi + \delta\varphi$ so that $\delta\varphi$ is infinitesimal. Assuming that $D\varphi = D\varphi'$

$$\begin{aligned} Z[J] &= \int D\varphi' \, \mathrm{e}^{-S[\varphi']+J\cdot\varphi'} && (333)\\ &= \int D\varphi \, \mathrm{e}^{-S[\varphi]+J\cdot\varphi} \left\{ 1 + \left(J - \frac{\delta S}{\delta\varphi} \right) \cdot \delta\varphi \right\} && (334)\\ &= Z[J] + \langle \left(J - \frac{\delta S}{\delta\varphi} \right) \delta\varphi \rangle_J && (335) \end{aligned}$$

We argue that

$$\langle \left(J - \frac{\delta S}{\delta\varphi} \right) \delta\varphi \rangle_J = 0 \tag{336}$$

generates infinitely many relations. For example taking $J = 0$ we have

$$\langle \frac{\delta S[\varphi]}{\delta\varphi(x)} \rangle = 0 \tag{337}$$

Taking one derivative with respect to J and the taking $J = 0$ we have

$$\langle \frac{\delta S[\varphi]}{\delta\varphi(x)} \varphi(y) \rangle = -\delta(x-y) \tag{338}$$

The right hand side is a **contact term**, if the two points are separate it is zero. Following the same strategy one can prove that for any local operator $\mathcal{O}[\varphi; y]$ constructed with the field φ we have the **Schwinger-Dyson equations**

$$\langle \frac{\delta S[\varphi]}{\delta\varphi(x)} \mathcal{O}[\varphi; y] \rangle = 0 \qquad \text{if} \quad x \neq y \tag{339}$$

D Field theory of the q-states Potts model

The microscopic degrees of freedom of the Potts model are lattice variables which can take q distinct values/states: $\alpha_i = \{1, \ldots, q\}$ with interaction

$$\mathcal{H} = -J \sum_{\langle lr \rangle} \delta_{\alpha_l, \alpha_r} \tag{340}$$

The Kronecker delta

$$\delta_{\alpha,\alpha'} = \begin{cases} 0 & \text{if} \quad \alpha \neq \alpha' \\ 1 & \text{if} \quad \alpha = \alpha' \end{cases} \tag{341}$$

D.1 Relation with the Ising model

Let's first see that this is a generalization of Ising because the Ising Hamiltonian appears for $q = 2$. Consider the two tables of values

$\delta_{\alpha,\alpha'}$	$\alpha = 1$	$\alpha = 2$
$\alpha' = 1$	1	0
$\alpha' = 2$	0	1

$\sigma \cdot \sigma'$	$\sigma = 1$	$\sigma = -1$
$\sigma' = 1$	1	-1
$\sigma' = -1$	-1	1

Notice that you can rewrite the second table to look like the first

$(\sigma\cdot\sigma'+1)/2$	$\sigma=1$	$\sigma=-1$
$\sigma'=1$	1	0
$\sigma'=-1$	0	1

Make the assignment $\alpha=\{1,2\}\leftrightarrow\sigma=\{1,-1\}$, under this assignment

$$\delta_{\alpha,\alpha'}=(\sigma\cdot\sigma'+1)/2$$

and therefore

$$-J_{\text{Potts}}\sum_{\langle r,l\rangle}\delta_{\alpha_r,\alpha_l}=-J_{\text{Potts}}\sum_{\langle r,l\rangle}(\sigma_r\cdot\sigma_l+1)/2=-\frac{J_{\text{Potts}}}{2}\sum_{\langle r,l\rangle}\sigma_r\sigma_l+E_0$$

Neglecting an unimportant zero point energy, we can write down the explicit relation between the couplings of the two models $J_{\text{Ising}}=J_{\text{Potts}}/2$.

D.2 Field theory

The Hamiltonian of the Potts model is invariant under the action of the group S_q of permutations of q objects which acts globally on the set of q lattice states. We are looking for an S_q covariant way to generalize the table based analysis.

Consider an N-simplex embedded in $\mathbb{R}^N$ for $N=q-1$ and label each of its vertices by a value of α. If the vertices of the simplex e^α are normalized as $e^\alpha\cdot e^\alpha=\sum_{i=1}^N e_i^\alpha e_i^\alpha=N=q-1$ then they satisfy

$$e^\alpha\cdot e^{\alpha'}=\sum_{i=1}^N e_i^\alpha e_i^{\alpha'}=(N+1)\delta_{\alpha,\alpha'}-1 \tag{342}$$

$$\sum_{\alpha=1}^{N+1}e_i^\alpha=0\,,\qquad\sum_{\alpha=1}^{N+1}e_i^\alpha e_j^\alpha=(N+1)\delta_{ij}\,. \tag{343}$$

and are completely determined modulo an overall $O(N)$ rotation.

We can use the vectors e^α to find a representation of the Kronecker delta

$$\delta_{\alpha,\alpha'}\;=\;\frac{1+e^\alpha\cdot e^{\alpha'}}{q} \tag{344}$$

and with the same logic as in the $q=2$ case we know that the first term only contributes to the zero point energy. We thus use the product

$$(\alpha,\alpha')=e^\alpha\cdot e^{\alpha'}=(e^\alpha,e^{\alpha'}) \tag{345}$$

instead of the Kronecker delta.

Now consider the general path integral

$$Z=\sum_{\{\alpha\}}\mathrm{e}^{\frac{1}{2}\sum_{i,j}J_{ij}(e^{\alpha_i},e^{\alpha_j})+E_0+\sum_i(h_i,e^{\alpha_i})} \tag{346}$$

and introduce the auxiliary integral over the fields $\phi_i=\{\phi_i^1,\ldots,\phi_i^N\}$ which are assembled as $\psi^\alpha=e^\alpha\cdot\phi=\sum_{i=1}^N e_i^\alpha\phi^i$

$$\int D\phi\,\mathrm{e}^{\frac{1}{2}\sum_{i,j}\left(J^{-1}\right)_{ij}(\psi_i,\psi_j)+\sum_i(\psi_i,e^{\alpha_i})}=\sum_{\{\alpha\}}\mathrm{e}^{\frac{1}{2}\sum_{i,j}J_{ij}(e^{\alpha_i},e^{\alpha_j})} \tag{347}$$

From now on the manipulation goes almost exactly like the Ising case: insert the auxiliary integral, sum explicitly over the states, and shift the magnetic field.

Interactions can be determined much like in the Ising case. The important thing is that we now know the key object to use to "construct" the possible interactions: $\psi^\alpha(x)$. We have that by summing over α any power of ψ^α you have a full S_q invariant interaction. The simplest action involving ψ would be

$$S[\phi] = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \phi^2 g \sum_\alpha (\psi^\alpha)^3 \right\} \tag{348}$$

$$= \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + g \sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha \phi^i \phi^j \phi^k \right\} \tag{349}$$

If this action controls the critical point then we expect a perturbative model in $d = 6 - \epsilon$. This action does in fact control the critical point, but it is clearly unbounded from below being cubic, and it also does not conform with the findings for the $q = 2$ Ising's field theory.

The first problem is solved easily: the Hamiltonian is bounded by construction, so the microscopic field theory must be. In fact the field theory arising from the path integral will include more terms that make the action bounded, for example

$$S_\Lambda[\phi] = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial\phi)^2 + \frac{m^2(\Lambda)}{2}\phi^2 + g(\Lambda) \sum_\alpha (\psi^\alpha)^3 + \lambda_1(\Lambda)\phi^4 + \lambda_2(\Lambda) \sum_\alpha (\psi^\alpha)^4 + \dots \right\} \tag{350}$$

for Λ the microscopic energy scale. An important question is: since the critical point is (or can be) unbounded, is the RG preserving the boundedness of the microscopic action? We will disuss this point later.

D.3 Field theory example: $N = 1$, $q = 2$

The vectors for the case $q = 2$ are $e^1 = -1$ and $e^2 = 1$ (the one dimensional simplex is a segment). It is easy to see that

$$\sum_\alpha (\psi^\alpha)^2 = 2\phi^2 \qquad \sum_\alpha (\psi^\alpha)^3 = 0 \tag{351}$$

$$\lambda_1\phi^4 + \lambda_2 \sum_\alpha (\psi^\alpha)^4 = (\lambda_1 + 2\lambda_2)\phi^4 \tag{352}$$

which means that the cubic interaction is not the simplest because it is identically zero, and that the actual simples is a single quartic interaction because the two possibilities are degenerate. We are back to the Ising case with a coupling λ.

D.4 Field theory example: $N = 2$, $q = 3$

The vectors are

$$e^1 = \sqrt{2}\{0, 1\} \qquad e^2 = \sqrt{2}\{-\sqrt{3}/2, -1/2\} \qquad e^3 = \sqrt{2}\{\sqrt{3}/2, -1/2\} \tag{353}$$

and this time the cubic interaction is nontrivial

$$\sum_\alpha (\psi^\alpha)^2 = (\phi_1)^2 + (\phi_2)^2 \tag{354}$$

$$\sum_\alpha (\psi^\alpha)^3 = \frac{3}{\sqrt{2}}\phi_2((\phi_2)^2 - 3(\phi_1)^2) \tag{355}$$

$$\lambda_1\phi^4 + \lambda_2 \sum_\alpha (\psi^\alpha)^4 = \left(\lambda_1 + \frac{9}{2}\lambda_2\right)((\phi_1)^2 + (\phi_2)^2)^2 \tag{356}$$

but there is still only one quartic interaction.

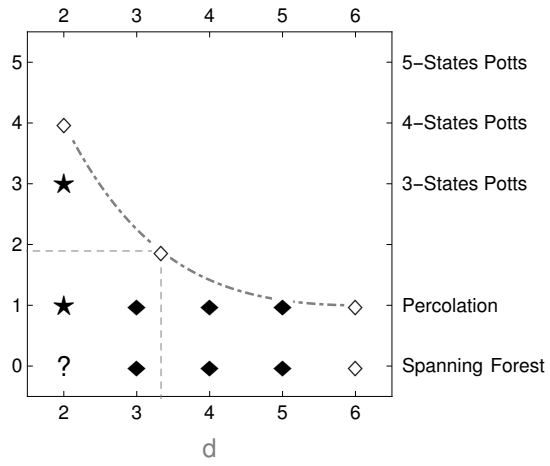
You can imagine constructing a ϕ^3 theory in $d = 6-\epsilon$ dimensions (this is the universality class of the 3-states Potts model), or alternatively imposing the parities $\mathbb{Z}_2 : \phi_i \to -\phi_i$ to eliminate the cubic interactions and construct a generalization of the ϕ^4 theory in $d = 4 - \epsilon$ dimensions with enhanced $O(2)$ symmetry).

D.5 Other examples

For $q > 3$ you can have models with genuine S_q symmetry, or restricted symmetry.

The analytic continuation to the cases $q = 0$ and $q = 1$ are also important. The first one is believed to be related to random a cluster model known as **Spanning Forest** (a spanning forest is a collection of clusters of links that covers an entire lattice) and more generally to other random cluster models like the spanning tree if the limit to $q \to 0$ is taken in different ways, while the second one is known to be related to **Percolation models**.

An interesting question is for which values of the pair (d, q) the model has a second order phase transition. One such possibility is below the curve:



E ϵ-expansion of the LPA and multicriticality (Unfinished)

See exercise 6 which we generalize for φ^{2n} in $d = \frac{2n}{n-2}$. Use Sturm-Liouville for a more general derivation. Assume:

$$v(\varphi) \propto \epsilon\,, \qquad\qquad \eta \propto \epsilon^2 \tag{357}$$

Under these assumptions it is always possible to expand the flow

$$\begin{aligned} k\partial_k v(\varphi) &= -dv + \frac{1}{2}(d-2+\eta)\varphi v' + \mathcal{F}(v'') && (358)\\ &= -dv + \frac{1}{2}(d-2+\eta)\varphi v' + f_0 - f_1 v'' + f_2 v''^2 - f_3 v''^3 + \dots\,, && (359)\\ \eta &= (v''')^2 \mathcal{G}(v'') = g_0 v'''^2 - g_1 v'' v'''^2 + \dots \end{aligned}$$

where we introduced a set of numerical coefficients that carry the information about the cutoff integrals

$$\begin{aligned} f_i &\equiv (-1)^i i!\, \partial^i \mathcal{F}(y)/\partial y^i|_{y=0}\,, \\ g_i &\equiv (-1)^i i!\, \partial^i \mathcal{G}(y)/\partial y^i|_{y=0}\,. \end{aligned} \tag{360}$$

All the coefficients shown are positive. We shall also drop $f_0 = 0$ since it contributes only to an ininfluent change in the zero point energy.

We now perform a useful rescaling of the dimensionless renormalized field

$$x = \frac{1}{2}\sqrt{\frac{d-2+\eta}{f_1}}\varphi\,, \tag{361}$$

and by abuse of notation indicate the functions $v(x)$ and $z(x)$ with the same symbols after the field has been rescaled. In the new variable become

$$\begin{aligned} -\frac{2k\partial_k v(x)}{d-2+\eta} &= \frac{2d}{d-2+\eta}v(x) - xv' + \frac{v''(x)}{2} + \tilde{f}_2 v''(x)^2 - \tilde{f}_3 v''(x)^3 + \dots , \\ \eta &= +\tilde{g}_0 v'''^2 - \tilde{g}_1 v'' v''''^2 + \dots , \end{aligned} \tag{362}$$

with new coefficients

$$\begin{aligned} \tilde{f}_i &\equiv 2^{1-2i}(d-2+\eta)^{i-1} f_i/(f_1)^i , \\ \tilde{g}_i &\equiv 4^{-2-i}(d-2+\eta)^{i+3} g_1/(f_1)^{i+3} . \end{aligned} \tag{363}$$

The rescaling has been engineered so that the fixed point equation for $v(x)$ is (modulo an overall coefficient and at the linear order in the potential itself) a second order ODE in which the second derivative is weighted by $\tilde{f}_1 = 1/2$, while the first derivative is multiplied by $-x$.

We now explicitly use the fact that $d = d_n - \epsilon$ and expand the fixed point equation to order ϵ recalling the ansatz. Using the fact that

$$\frac{2d}{d-2+\eta} = 2n + \mathcal{O}(\epsilon) , \tag{364}$$

the leading order of the differential equation for the stationary solutions becomes a linear ODE

$$\mathcal{D}_{2n} v(x) \equiv 2nv(x) - xv'(x) + v''(x)/2 = 0 , \tag{365}$$

for which we defined the second order derivative operator $\mathcal{D}_{2n}$ acting on functions whose argument is x. The ODE admits a two parameter family of solutions, but only one of these is polynomially bounded at large values of x: solutions must be proportional to the $2n$-th Hermite polynomial $H_{2n}(x)$. The solution must be of the form

$$v(x) = \epsilon\, c_n H_{2n}(x) . \tag{366}$$

The coefficient c_n still cannot be determined due to the linear nature of the ODE.

In order to compute c_n and the corrections to the next orders it is useful to embed the solution in the space of function equipped with the standard norm for Hermite polynomials

$$\int \mathrm{d}x\, \mathrm{e}^{-x^2} H_p(x) H_q(x) = 2^p \pi^{1/2}\, p!\, \delta_{p,q} . \tag{367}$$

We can use this norm to project on any desired polynomial. The operator $\mathcal{D}_{2n}$ can only generate corrections which are orthogonal to the function $H_{2n}(x)$. It is thus possible to use the first order solution the fixed point, expand it to order ϵ^2, and project it onto $H_{2n}(x)$. At this order the interplay between the first and the last monomials of the first line of the fixed point equation determines a quadratic equation for c_n

$$c_n \left(2^{n+2} n (2n)!^2 \tilde{f}_2 c_n - (n-1) n!^3 \right) = 0 , \tag{368}$$

In deriving this formula we used

$$\int \mathrm{d}x\, \mathrm{e}^{-x^2} H_p(x) H_q(x) H_r(x) = \frac{2^s \pi^{1/2}\, p! q! r!\, \delta_{s \bmod 1,0}}{(p-q)!(q-r)!(r-p)!} . \tag{369}$$

Simply put, the solution $c_n = 0$ corresponds to the Gaussian fixed point, while for $n \geq 2$ the nontrivial solution

$$c_n = \frac{(n-1)n!^3}{2^{n+2} n (2n!)^2} \frac{1}{\tilde{f}_2} . \tag{370}$$

is the fixed point that we are looking for. It is clear that c_n depends on the cutoff as evidenced by the last factor depending on $\tilde{f}_2$, but this dependence is to be expected since it is not a physical quantity such as a critical exponent.

Tthe improvement to order ϵ^2 of the solution needs to take into account the anomalous dimension η and a new ansatz

$$\begin{aligned} v(x) &= \epsilon\, c_n H_{2n}(x) + \epsilon^2\, c_n^2 \sum_p a_{p,n} H_{2p}(x) \\ \eta &= \epsilon^2\, c_n^2\, \eta_{n,2} \end{aligned} \tag{371}$$

The coefficients $a_{p,n}$ can be determined with similar strategies as c_n.

The coefficient $\eta_{n,2}$ appearing in the parametrization of the anomalous dimension is determined by projecting the second equation onto the simplest Hermite polynomial $H_0(x) = 1$. The projection over the constant polynomial gives an interesting formula

$$\eta = \pi^{-1/2} \int \mathrm{d}x\, \mathrm{e}^{-x^2} \Big\{ g_0 v'''^2 - g_1 v'' v'''^2 + \dots \Big\}, \tag{372}$$

which when expanded to ϵ^2 can be used to determine $\eta_{n,2}$. We get

$$\eta_{n,2} = 2^{2n+5}(n-1)n(2n-1)(2n)!\tilde{g}_0\,. \tag{373}$$

and therefore

$$\eta = \frac{2(n-1)^3(2n-1)n!^6}{n(2n)!^3}\, \frac{\tilde{g}_0}{(\tilde{f}_2)^2}\, \epsilon^2\,. \tag{374}$$

Notice how this differs from the universal perturbative result

$$\eta = \frac{4(n-1)^2 n!^6}{(2n)!^3} \epsilon^2 \tag{375}$$

The spectrum of deformations around the critical solutions can be computed by linearizing the flow through the replacement

$$v(x) \to v(x) + \delta v(x)\, (k/k_0)^{-\lambda}\ , \tag{376}$$

It proves convenient to define

$$\hat{\lambda} \equiv \frac{2(\lambda - d)}{d - 2 + \eta}\,, \tag{377}$$

so that the linearized equation for the fluctuations takes the simple form

$$\mathcal{D}_{\hat{\lambda}} \delta v(x) = 2\tilde{f}_2 v'' \delta v''(x) - 3\tilde{f}_3 v''^2 \delta v''(x) + \dots\,, \tag{378}$$

A spectrum of countably many deformations $\{\delta v(x)\}$ can be obtained if, as boundary condition, the deformations are chosen to be polynomially bounded at large values of x like the solution is. This leads to a quantization of the exponent λ or analogously $\hat{\lambda}$.

As for the solution, the spectrum can be computed order by order in ϵ: the leading ϵ^0 part becomes

$$\mathcal{D}_{\hat{\lambda}} \delta v(x) = 0\,. \tag{379}$$

The polynomially bounded spectrum of this system is well known

$$\delta v_m(x) = H_m(x)\,, \qquad \hat{\lambda}_m = m \in \mathbb{N}\,. \tag{380}$$

Further corrections to the spectrum can be computed with the standard methods of (quantum mechanical) perturbation theory as

$$\hat{\lambda}_m = m + \frac{2(n-1)^2 m! n!}{(m-n)!(2n)!}\epsilon\,. \tag{381}$$

We obtain at first order

$$\lambda_m = \frac{2n-m}{n-1} + \left\{ \frac{k-2}{2} - 2\frac{(n-1)m!n!}{(m-n)!(2n)!} \right\}\epsilon\,. \tag{382}$$

F Perturbative renormalization in the local potential (Unfinished)

F.1 Analytic continuation to d dimensions

Start with the basic action in four dimensions

$$S[\phi] = \int \mathrm{d}^4x \left\{ \frac{1}{2}(\partial\phi)^2 + V(\phi) \right\} \tag{383}$$

and the background path integral

$$Z = \int D\chi\, \mathrm{e}^{-S[\varphi+\chi]} \tag{384}$$

We also assume that the potential has a finite number of interactions, and in particular it is a polynomial of the fourth order. For example

$$V(\phi) = \frac{g}{4!}\phi^4 + \frac{g_2}{2}\phi^2 \tag{385}$$

In four dimensions dimensional analysis says that

$$[V(\phi)] = M^4 \qquad\qquad [\phi] = M^1 \tag{386}$$

Now we want to promote the action $S[\phi]$ to $d = 4-\epsilon$ dimensions. Naively:

$$S[\phi] \to S[\phi] = \int \mathrm{d}^dx \left\{ \frac{1}{2}(\partial\phi)^2 + V'(\phi') \right\} \tag{387}$$

We have included a prime label because clearly these new field and potential in d dimensions cannot be the original ones in $d = 4$. This is evident from the canonical analysis of their mass dimensions

$$[V'(\phi')] = M^d \qquad\qquad [\phi'] = M^{d/2-1} \tag{388}$$

The simplest way to preserve the dimensionality (and hence the scaling) of the fields and the couplings during the continuation to d dimensions is to introduce a reference scale μ and solve for the mass dimensions below

$$V'(\phi') = \mu^A V(\mu^B\phi) \qquad\qquad \phi' = \mu^B\phi \tag{389}$$

It is easy to see that $A = -\epsilon$ and $B = \epsilon/2$ which implies

$$V'(\phi') = \mu^{-\epsilon} V(\mu^{\epsilon/2}\phi) \tag{390}$$

and therefore the action is promoted to d dimensions to

$$S[\phi] \to S[\phi] = \int \mathrm{d}^dx \left\{ \frac{1}{2}(\partial\phi)^2 + \mu^{-\epsilon} V(\mu^{\epsilon/2}\phi) \right\} \tag{391}$$

The explicit form of the promoted potential is

$$\mu^{-\epsilon} V(\mu^{\epsilon/2}\phi) = \frac{g}{4!}\mu^\epsilon\phi^4 + \frac{g_2}{2}\phi^2 \tag{392}$$

F.2 Vertex expansion

We now temporarily work without the rescaling to avoid cumbersome notation. First we expand the theory around the free massless action

$$S_0[\phi] = \frac{1}{2}\int \mathrm{d}^d x (\partial\phi)^2 \tag{393}$$

which has exact propagator

$$G_0(x) = \langle \phi(x)\phi(0)\rangle \tag{394}$$

The propagator solves $-\partial^2 G_0(x) = \delta^{(d)}(x)$ and for general d (specifically away from $d=2$) it is

$$G_0(x) = \frac{1}{4\pi}\frac{\Gamma(v)}{\pi^v}\frac{1}{|x|^{2v}} \qquad \text{with} \quad v = \frac{d}{2}-1 \tag{395}$$

Now we expand the path integral, neglect insertions of $-\partial^2\chi$ thanks to the Schwinger-Dyson equations, and absorb the terms independent on χ in a normalization

$$Z = \int D\chi\, \mathrm{e}^{-S_0[\varphi]-S_0[\chi]} \exp\left\{-\int \mathrm{d}^d x \sum_{r\geq 0}\frac{1}{r!}V^{(r)}(\varphi(x))\chi(x)^r\right\} \tag{396}$$

$$= \mathcal{N}\sum_{n\geq 0}\frac{(-1)^n}{n!}\int D\chi\, \mathrm{e}^{-S_0[\chi]}\prod_{i=1}^{n}\int \mathrm{d}^d x_i \sum_{r_i\geq 0}\frac{1}{r_i!}V^{(r_i)}(\varphi(x_i))\chi(x_i)^{r_i} \tag{397}$$

We assume that for $x=0$ the propagator satisfies $G_0(0)=0$, which is actually true only for $d<2$. The first order of the expansion of the exponential contains only fluctuations that are evaluated at one coordinate and therefore cannot contribute. The second order instead contributes

$$\frac{1}{2}\int D\chi\, \mathrm{e}^{-S_0[\chi]}\int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2 \sum_{r_1\geq 0}\sum_{r_1\geq 0}\frac{1}{r_1! r_2!}V^{(r_1)}(\varphi(x_i))V^{(r_2)}(\varphi(x_i))\chi(x_1)^{r_i}\chi(x_2)^{r_2} \tag{398}$$

$$= \frac{1}{2}\int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2 \sum_{r_1\geq 0}\sum_{r_1\geq 0}\frac{1}{r_1! r_2!}V^{(r_1)}(\varphi(x_1))V^{(r_2)}(\varphi(x_2))\langle\chi(x_1)^{r_1}\chi(x_2)^{r_2}\rangle \tag{399}$$

Recall that we are interested in the connected irreducible vertices (because all others descend from them). There need to be as many r_1 as r_2, otherwise closed loops go to zero because of $G_0(0)=0$; once $r=r_1=r_2$ is taken, then there are $r!$ ways to connect $\chi(x_1)^{r_1}$ with $\chi(x_2)^{r_2}$. The correction to the generator of the proper vertices is thus

$$\frac{1}{2}\int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2 \sum_{r\geq 0}\frac{1}{r!}V^{(r)}(\varphi(x_1))V^{(r)}(\varphi(x_2))\langle\chi(x_1)\chi(x_2)\rangle^r \tag{400}$$

$$= \frac{1}{2}\int \mathrm{d}^d x_1 \int \mathrm{d}^d x_2 \sum_{r\geq 0}\frac{1}{r!}V^{(r)}(\varphi(x_1))V^{(r)}(\varphi(x_2))G(x_1-x_2)^r \tag{401}$$

Having to deal with one or more propagators connecting the same coordinates is a standard situation of renormalization thorugh standard perturbation theory. Take for a moment $x = x_1 - x_2$, to understand how $G(x)^r$ behaves it is useful to perform its Fourier transform

$$\int \mathrm{d}^d x\, \mathrm{e}^{\mathrm{i}p\cdot x}\, G(x)^r = \frac{1}{(4\pi)^r}\frac{\Gamma(v)^r}{\Gamma(rv)}\Gamma\left(1-(r-1)v\right)\left(\frac{p^2}{4\pi}\right)^{-1+(r-1)v} \tag{402}$$

The question is where are the poles of the above integral? The gamma function $\Gamma(n)$ has poles for $n = 0,-1,-2,\ldots$. This implies that $\Gamma\left(1-(r-1)v\right)$ has poles for $(r-1)v = 1,2,3,\ldots$. We are interested to

$v = d/2 - 1 \simeq 1$, so r can take the values $2, 3, 4, \ldots$. We can truncate the number of terms knowing that $V(\phi)$ is a polynomial of the fourth order and therefore $r \leq 4$. Expanding the transform above close to the value $d = 4 - \epsilon$ and retaining only the divegences we get

$$G_0(x)^2 \sim \frac{2}{(4\pi)^2\epsilon}\delta^{(d)}(x) \tag{403}$$

$$G_0(x)^3 \sim \frac{1}{2(4\pi)^4\epsilon}\partial_x^2\delta^{(d)}(x) \tag{404}$$

$$G_0(x)^4 \sim \frac{1}{18(4\pi)^6\epsilon}(\partial_x^2)^2\delta^{(d)}(x) \tag{405}$$

Now we use these estimates for the divergences in the correction to the generator of the connected diagrams. We get that the divergent part of the generator is

$$\frac{1}{2}\int \mathrm{d}^d x \left\{ \frac{1}{(4\pi)^2\epsilon}V^{(2)}(\phi(x))^2 + \frac{1}{6(4\pi)^4\epsilon}V^{(3)}(\phi(x))\partial^2 V^{(3)}(\phi(x)) + \frac{1}{216(4\pi)^6\epsilon}V^{(4)}(\phi(x))(\partial^2)^2 V^{(4)}(\phi(x)) \right\}$$

The above correction must be balanced by the counterterms (it essentially is the counter terms!) However, first we have to cast it in the appropriate form that resembles the original action $S[\varphi]$. The first term is a polynomial of the fourth order, while the second term must be manipulated by integrating by parts

$$\frac{1}{2}\int \mathrm{d}^d x \frac{1}{6(4\pi)^4\epsilon}V^{(3)}(\phi(x))\partial^2 V^{(3)}(\phi(x)) \tag{406}$$

$$= -\frac{1}{2}\int \mathrm{d}^d x \frac{1}{6(4\pi)^4\epsilon}\partial_x V^{(3)}(\phi(x))\partial_x V^{(3)}(\phi(x)) \tag{407}$$

$$= -\frac{1}{2}\int \mathrm{d}^d x \frac{1}{6(4\pi)^4\epsilon}(\partial\phi)^2 V^{(4)}(\phi)^2 \tag{408}$$

Finally, it is easy to see that if the interaction is a polynomial of the fourth rank the third term is zero automatically. We have the divergences

$$\Gamma_{\mathrm{div}}[\phi] = \frac{1}{2}\int \mathrm{d}^d x \left\{ \frac{1}{(4\pi)^2\epsilon}V^{(2)}(\phi(x))^2 - \frac{1}{6(4\pi)^4\epsilon}(\partial\phi)^2 V^{(4)}(\phi)^2 \right\} \tag{409}$$

which can be reabsorbed through the bare action with a local potential. (Notice that we indicate $V^{(4)}(\phi)$ with full field dependence, but for a potential of the fourth order we have that by construction $V^{(4)}(\phi) = V^{(4)}(0)$.)

Recall the discussion on the dimensionality of the previous section. We introduce the reference scale

$$\Gamma_{\mathrm{div}}[\phi] = \frac{1}{2}\int \mathrm{d}^d x \left\{ \frac{1}{(4\pi)^2\epsilon}V^{(2)}(\phi(x))^2 - \frac{1}{6(4\pi)^4\epsilon}(\partial\phi)^2 V^{(4)}(\phi)^2 \right\} \tag{410}$$

The counter terms are the negative of the divergences

$$S_{\mathrm{c.t.}}[\phi] = -\Gamma_{\mathrm{div}}[\phi] \tag{411}$$

Let us switch to Lagrangians

$$S_{\mathrm{B}}[\phi] = \int \mathrm{d}^4 x\, \mathcal{L}_{\mathrm{B}} \qquad S_{\mathrm{R}}[\phi] = \int \mathrm{d}^d x\, \mathcal{L}_{\mathrm{R}} \tag{412}$$

$$S_{\mathrm{c.t.}}[\phi] = \int \mathrm{d}^d x\, \mathcal{L}_{\mathrm{c.t.}} \tag{413}$$

We introduce the reference scale μ as a scale that allows us to keep track of the change of dimensionality from $d = 4$ to $d = 4 - \epsilon$. The bare action, the renormalized one, and the counter terms are related

$$\mu^{-\epsilon}\left\{S_{\mathrm{R}}[\phi] - S_{\mathrm{c.t.}}[\phi]\right\} = S_{\mathrm{B}}[\phi] \tag{414}$$

The beta functions are determined by the requirement that the renormalized action