Exercise 9: Perturbative RG and the renormalized potential

In this exercise we want to give a functional representation based on a renormalized potential to our lecture's results based on perturbation theory for small coupling.

Recall that during the lecture we promoted the standard ϕ^4 Lagrangian in $d = 4$ dimensions

$$\mathcal{L}_4 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}g_2\phi^2 + \frac{1}{4!}g_4\phi^4$$

to a d-dimensional action by introducing a scale μ as

$$\mathcal{L}_d = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}g_2\phi^2 + \frac{1}{4!}\mu^{4-d}g_4\phi^4$$

The "promotion" was done in such a way that the field ϕ has always canonical dimension, but the couplings g_2 and g_4 have maintained the same dimension when $4 \to d$. The potential is for now restricted to be a polynomial of the fourth order.

Part 1:
Consider the generalization of $\mathcal{L}_4$ to a full functional potential

$$\mathcal{L}_4 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)$$

The original Lagrangian can be recovered by expanding $V(\phi) = \frac{1}{2}g_2\phi^2 + \frac{1}{4!}g_4\phi^4$ and the couplings are the Taylor coefficients of this expansion. Find the generalization of the potential that is needed to promote the Lagrangian to $\mathcal{L}_d$ and leaves the dimension of the couplings invariant.

Hint:
You may try the ansatz

$$\mathcal{L}_d = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \mu^A V(\mu^B\phi)$$

and determine the unknown constants A and B by comparing with the coupling's Lagrangian.

After 1) dimensional regularization, 2) minimal subtraction of the $\frac{1}{\epsilon}$ poles and 3) an unimportant rescaling of factors of 4π, the beta function of the coupling g_4 at two loops in $d = 4$ is

$$\beta_{g_4} = 3g_4^2 - \frac{17}{3}g_4^3$$

and the anomalous dimension is

$$\eta = \frac{1}{6}g_4^2$$

Part 2:
Assume that in the functional representation the potential has beta function

$$\beta_V = C_1\,\left(V''(\phi)\right)^2 + C_2\,V''(\phi)\left(V'''(\phi)\right)^2 + \frac{1}{2}\,\eta\,\phi\,V'(\phi)$$

and the anomalous dimension η has the form

$$\eta = C_3\,\left(V^{(4)}(0)\right)^2$$

Determine the constants C_1, C_2 and C_3.

Hint:
There is no need to compute them from diagrams if you substitute $V(\phi) = \frac{1}{4!}g_4\phi^4 + \dots$ and interpret β_V as the generator of the beta functions $\beta_V = \frac{1}{4!}\beta_{g_4}\phi^4 + \dots$. It is probably convenient to determine the constant C_3 of the anomalous dimension first.

Now we are interested in computing the critical exponents *below* four dimensions using the beta functions and the perturbative expansion of $d = 4$.

One trick to do this is to "trade" the expansion in g_4 for a $\varepsilon = 4 - d$ expansion. We use the fact that $V(\phi)$ and ϕ are canonically normalized in d dimensions and switch to a dimensionless potential $v(\varphi) = \mu^{-d}V(\mu^{d/2-1}\varphi)$. (Notice that the field ϕ is already renormalized and the contribution of η is already inside β_V above so there is no need to rescale by a further wavefunction.)

We are purposedly distinguishing between $d = 4 - \varepsilon$ (which parametrizes any dimension below the upper critical one for statistical field theory) and $d = 4 - \epsilon$ (which analytically continued the theory to make it finite). *After the new additional rescaling all couplings are dimensionless* and the beta function is

$$\begin{aligned}
\beta_v &= -dv(\varphi) + \frac{d-2}{2}\varphi v'(\varphi) + \beta_V|_{V\to v} \\
&= -dv(\varphi) + \frac{d-2+\eta}{2}\varphi v'(\varphi) + \frac{1}{2}\left(v''(\varphi)\right)^2 - \frac{1}{2}v''(\phi)\left(v'''(\phi)\right)^2 \\
\eta &= = \frac{1}{6}(v^{(4)}(0))^2
\end{aligned}$$

(This expression has a familiar scaling term that should remind you of another exercise. It should also suggest you the correct values of C_1, C_2 and C_3 as a check of part 2!) The fixed point solutions of $\beta_v = 0$ are of the form $v^*(\varphi) = \frac{1}{4!}\lambda_4^*(\varepsilon)\varphi^4$.

(Optional) Part 3:

- Find all the fixed points $\lambda_4^*(\varepsilon)$ up to order ε^2. How many are they and why?
- Which fixed point is the one that we need for the ε expansion and why?
- Find a simple way to determine the critical exponent η and ν to order ε^2. (The results should be $\eta = \frac{\varepsilon^2}{54}$ and $\nu = \frac{1}{2} + \frac{\varepsilon}{12} + \frac{7\varepsilon^2}{162}$.)
