# Exercise 8: Projection of the vertex expansion

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In this exercise we want to evaluate the vertex expansion of Exercise 7 for an action truncated to the local potential approximation

$$\Gamma_k[\phi] = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial\phi)^2 + V_k(\phi) \right\}$$

We want to evaluate the renormalization group flow of each vertex at constant field $\phi$ in momentum space. You can begin to familiarize with some steps of the procedure:

- The propagator is a function of one momentum conjugate to the difference of the coordinates

  $$\mathcal{G}_k(x, y) = \int_q G_k\left(q^2\right) \, \mathrm{e}^{iq\cdot(x-y)}$$

  We choose the Fourier transform to be a function of $q^2$ using rotational and translational invariance.

- The cutoff and its derivative are also functions of $q^2$ for the same reasons

  $$\mathcal{R}_k(x, y) = \int_q R_k\left(q^2\right) \, \mathrm{e}^{iq\cdot(x-y)} \,, \qquad k\partial_k \mathcal{R}_k(x, y) = \int_q k\partial_k R_k\left(q^2\right) \, \mathrm{e}^{iq\cdot(x-y)}$$

  The functions $G_k(q^2)$ and $R_k(q^2)$ are obviously related

  $$G_k(q^2) = \left(q^2 + V_k''(\phi) + R_k(q^2)\right)^{-1}$$

- The vertices with three or more legs become local because the potential is local. For example

  $$\frac{\delta^3 \Gamma_k[\phi]}{\delta\phi_{p_1}\delta\phi_{p_2}\delta\phi_{p_3}} = V_k^{(3)}(\phi) \, (2\pi)^d \, \delta(p_1 + p_2 + p_3)$$

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**Part 1:**
The evaluation of the zero point function gives

$$k\partial_k \Gamma[\phi]\big|_{\phi=\text{const.}} = \frac{1}{2} \int \mathrm{d}^d x \int_q G_k\left(q^2\right) k\partial_k R_k\left(q^2\right)$$

Evaluate the right hand side of the above formula for the cutoff $R_k\left(q^2\right) = \left(k^2 - q^2\right)\theta\left(k^2 - q^2\right)$ with $\theta(x)$ the Heaviside theta function. Deduce the flow of $V_k(\phi)$.

**Hints:**
The momentum integral is $\int_q \equiv \int \frac{\mathrm{d}^d q}{(2\pi)^d}$ and can immediately be changed to polar coordinates. Furthermore, you might find the following property useful

$$f\left(A\theta(x - x_0) + B\theta(x_0 - x)\right) = f(A)\theta(x - x_0) + f(B)\theta(x_0 - x)$$

(Heaviside thetas are projectors over the space of functions.)

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**Part 2:**
The evaluation of the two point function gives

$$
\begin{aligned}
k\partial_k \frac{\delta^2}{\delta\phi_p\delta\phi_{-p}}\Gamma[\phi]\Big|_{\phi=\text{const.}} &= V^{(3)}(\phi)^2 \int_q G_k\left((q+p)^2\right) G_k\left(q^2\right)^2 k\partial_k R_k\left(q^2\right) \\
&\quad -\frac{1}{2}V^{(4)}(\phi)\int_q G_k\left(q^2\right)^2 k\partial_k R_k\left(q^2\right)
\end{aligned}
$$

Show that upon Taylor expanding the above formula in $p_\mu$, the coefficient of $p^2$ of the right hand side of the above formula is

$$
V^{(3)}(\phi)^2 \int_q \left(G_k'\left(q^2\right) + q^2\frac{2}{d}G_k''\left(q^2\right)\right) G_k\left(q^2\right)^2 k\partial_k R_k\left(q^2\right)
$$

**Hints:**
Use the following properties that can be proved through invariance under rotational symmetry

$$
\int_q q_\mu f(q^2) = 0\,, \qquad \int_q q_\mu q_\nu f(q^2) = \frac{1}{d}g_{\mu\nu}\int_q q^2 f(q^2)
$$

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**(Optional) Part 3:**
Evaluate

$$
V^{(3)}(\phi)^2 \int_q \left(G_k'\left(q^2\right) + q^2\frac{2}{d}G_k''\left(q^2\right)\right) G_k\left(q^2\right)^2 k\partial_k R_k\left(q^2\right)
$$

for the cutoff $R_k\left(q^2\right) = \left(k^2-q^2\right)\theta\left(k^2-q^2\right)$.

**Hints:**
The derivative of the Heaviside function is the Dirac delta and they satisfy the following properties

$$
\int \mathrm{d}x\,\theta(x-x_0)\theta(x_0-x)f(x) = 0\,, \qquad \int \mathrm{d}x\,\theta(x-x_0)\delta(x-x_0)f(x) = \frac{1}{2}f(x_0)
$$

(Delta and Heaviside functions are distributions: a distribution is defined as the limit of a sequences of functions in the space of all functions. The formula on the right assumes that the elements of limiting sequence for the Dirac delta are the derivatives of the elements of the limiting sequence of the Heviside theta.)

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**(Optional) Part 4:**
We have not introduced a wavefunction normalization $Z_k$ in the truncation to the local potential of $\Gamma_k[\phi]$, but it should be clear that it is generated by the flow. Assuming that we start from $Z_k = 1$, after one infinitesimal RG step it becomes

$$
Z_{k+\delta k} = Z_k + \delta Z_k = Z_k + \frac{\partial Z_k}{\partial k}\delta k = Z_k\left(1-\eta\,\delta k\right) = 1 - \eta\,\frac{\delta k}{k}
$$

in which we used $\eta = -\partial\log Z_k/\partial\log k$. Can you estimate the anomalous dimension $\eta$ from the result of Part 3? What value does it have for $\phi = 0$ if the potential is symmetric?

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