Exercise 7: Vertex expansion

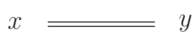
Consider the Wetterich equation

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\,\mathcal{G}_k k\partial_k \mathcal{R}_k$$

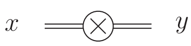
in which we denoted the **(modified) propagator** as

$$\mathcal{G}_k = \left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}$$

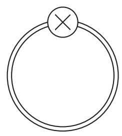
If you graphically represent the propagator as

$$\mathcal{G}_k(x,y) = \left(\frac{\delta^2\Gamma_k}{\delta\phi(x)\delta\phi(y)} + \mathcal{R}_k(x,y)\right)^{-1} = \Big[$$  $$\Big]$$

and the derivative of the cutoff as

$$k\partial_k\mathcal{R}_k(x,y) = \Big[$$  $$\Big]$$

then the RG flow equation has the following representation

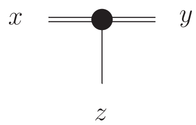
$$k\partial_k\Gamma_k[\phi] = \frac{1}{2}\Big[$$  $$\Big]$$

in which a closed loop means that we are taking the trace.

We want to construct the **vertex expansion** by acting with functional derivatives on the RG flow of the effective average action. We need to know how a derivative acts on the propagator

$$\frac{\delta}{\delta\phi(z)}\mathcal{G}_k(x,y) = -\int \mathrm{d}^d z_1 \mathrm{d}^d z_2\, \mathcal{G}_k(x,z_1)\,\frac{\delta^3\Gamma_k}{\delta\phi(z_1)\delta\phi(z_2)\delta\phi(z)}\,\mathcal{G}_k(z_2,y)$$

Graphically the action becomes very simple

$$\frac{\delta}{\delta\phi(z)}\mathcal{G}_k(x,y) = -\Big[$$  $$\Big]$$

in which we denoted with a black dot the vertex coming from the derivatives of $\Gamma_k[\phi]$. We distinguish internal from external lines by denoting the former with a double line and the latter with a standard line, so we remember to not attach propagators to the external ones.

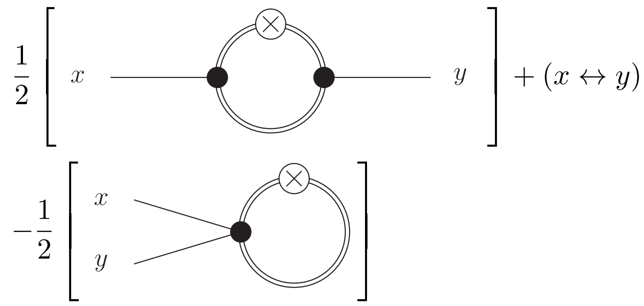
Part 1:
Give the graphical representation of the first derivative of the flow

$$\frac{\delta}{\delta\phi(x)} k\partial_k \Gamma_k[\phi]$$

This is the flow of the one-point function (which is the first vertex of the theory).

Part 2:
Show (graphically) that the following representation is correct

$$\frac{\delta^2}{\delta\phi(x)\delta\phi(y)} k\partial_k \Gamma_k[\phi] =$$



in which $(x \leftrightarrow y)$ repeats the preceeding term with an exchange of x and y. Assuming full symmetry in the exchange of the two coordinates $\{x, y\}$ how many "topologically" different diagrams do you have?

(Optional) Part 3:
Assuming full symmetry under the exchange of the three coordinates $\{x, y, z\}$, give a graphical representation of the third derivative of the flow

$$\frac{\delta^3}{\delta\phi(x)\delta\phi(y)\delta\phi(z)} k\partial_k \Gamma_k[\phi]$$

(Optional) Part 4:
Assuming full symmetry under the exchange of the four coordinates $\{x, y, z, w\}$, give a graphical representation of the fourth derivative of the flow

$$\frac{\delta^4}{\delta\phi(x)\delta\phi(y)\delta\phi(z)\delta\phi(w)} k\partial_k \Gamma_k[\phi]$$
