

## Exercise 7: Vertex expansion

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Consider the Wetterich equation

$$k\partial_k\Gamma_k[\phi] = \frac{1}{2} \text{Tr} \mathcal{G}_k k\partial_k \mathcal{R}_k$$

in which we denoted the **(modified) propagator** as

$$\mathcal{G}_k = \left( \Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1}$$

If you graphically represent the propagator as

$$\mathcal{G}_k(x, y) = \left( \frac{\delta^2 \Gamma_k}{\delta\phi(x)\delta\phi(y)} + \mathcal{R}_k(x, y) \right)^{-1} = \left[ x \text{ --- } y \right]$$

and the derivative of the cutoff as

$$k\partial_k \mathcal{R}_k(x, y) = \left[ x \text{ --- } \otimes \text{ --- } y \right]$$

then the RG flow equation has the following representation

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2} \left[ \text{loop with } \otimes \right]$$

in which a closed loop means that we are taking the trace.

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We want to construct the **vertex expansion** by acting with functional derivatives on the RG flow of the effective average action. We need to know how a derivative acts on the propagator

$$\frac{\delta}{\delta\phi(z)} \mathcal{G}_k(x, y) = - \int d^d z_1 d^d z_2 \mathcal{G}_k(x, z_1) \frac{\delta^3 \Gamma_k}{\delta\phi(z_1)\delta\phi(z_2)\delta\phi(z)} \mathcal{G}_k(z_2, y)$$

Graphically the action becomes very simple

$$\frac{\delta}{\delta\phi(z)} \mathcal{G}_k(x, y) = - \left[ x \text{ --- } \bullet \text{ --- } y \right] \begin{array}{c} | \\ z \end{array}$$

in which we denoted with a black dot the vertex coming from the derivatives of  $\Gamma_k[\phi]$ . We distinguish internal from external lines by denoting the former with a double line and the latter with a standard line, so we remember to not attach propagators to the external ones.

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**Part 1:**

Give the graphical representation of the first derivative of the flow

$$\frac{\delta}{\delta\phi(x)} k\partial_k\Gamma_k[\phi]$$

This is the flow of the one-point function (which is the first vertex of the theory).

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**Part 2:**

Show (graphically) that the following representation is correct

$$\frac{\delta^2}{\delta\phi(x)\delta\phi(y)} k\partial_k\Gamma_k[\phi] = \frac{1}{2} \left[ \begin{array}{c} \text{Diagram 1} \\ x \text{ --- } \bullet \text{ --- } \text{Diagram 1} \text{ --- } \bullet \text{ --- } y \end{array} \right] + (x \leftrightarrow y) - \frac{1}{2} \left[ \begin{array}{c} \text{Diagram 2} \\ x \text{ --- } \bullet \text{ --- } \text{Diagram 2} \\ y \text{ --- } \bullet \text{ --- } \text{Diagram 2} \end{array} \right]$$

The diagrams are:

- Diagram 1: A circle with a cross (⊗) at the top. Two black dots are on the horizontal diameter. A line from 'x' on the left connects to the left dot, and a line from 'y' on the right connects to the right dot.
- Diagram 2: A circle with a cross (⊗) at the top. A single black dot is on the horizontal diameter. Two lines from 'x' and 'y' on the left converge at this dot.

in which  $(x \leftrightarrow y)$  repeats the preceding term with an exchange of  $x$  and  $y$ . Assuming full symmetry in the exchange of the two coordinates  $\{x, y\}$  how many “topologically” different diagrams do you have?

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**(Optional) Part 3:**

Assuming full symmetry under the exchange of the three coordinates  $\{x, y, z\}$ , give a graphical representation of the third derivative of the flow

$$\frac{\delta^3}{\delta\phi(x)\delta\phi(y)\delta\phi(z)} k\partial_k\Gamma_k[\phi]$$

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**(Optional) Part 4:**

Assuming full symmetry under the exchange of the four coordinates  $\{x, y, z, w\}$ , give a graphical representation of the fourth derivative of the flow

$$\frac{\delta^4}{\delta\phi(x)\delta\phi(y)\delta\phi(z)\delta\phi(w)} k\partial_k\Gamma_k[\phi]$$

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