Exercise 6: Dimensionless variables and critical properties

Consider a scale k dependent potential $V_k(\phi)$ with renormalization group flow:

$$k\partial_k V_k(\phi) = k^d F\left(\frac{V''(\phi)}{Z_k k^2}\right) \tag{1}$$

Assume that the function F is nowhere singular. The above notation means that the potential is a function of ϕ that depends also on k: the derivative ∂_k only acts on the explicit dependence of V_k and not on the argument ϕ. In other words, the renormalization group flow of the potential $V_k(\phi)$ is *the logarithmic derivative with respect to k at fixed field ϕ.*

The scale dependent constant Z_k has renormalization group flow $k\partial_k Z_k = -\eta Z_k$ and it normalizes the kinetic term $\frac{Z_k}{2}(\partial\phi)^2$. Define the *dimensionless renormalized field*

$$\varphi \equiv k^{-d/2+1} Z_k^{1/2} \phi \tag{2}$$

and the dimensionless potential

$$v_k(\varphi) \equiv k^{-d} V_k(\phi) = k^{-d} V_k(k^{d/2-1} Z_k^{-1/2} \varphi) \tag{3}$$

The dimensionless renormalized field has canonically normalized kinetic term in units of the scale k.

Part 1:
Show that the renormalization group flow of the dimensionless potential (that is the logarithmic scale derivative at fixed φ) is

$$k\partial_k v_k(\varphi) = -dv(\varphi) + \frac{1}{2}(d-2+\eta)\varphi v_k'(\varphi) + F(v_k''(\varphi)) \tag{4}$$

Some definitions: the first few terms of the scale derivative depend only on the rescaling

$$-dv(\varphi) + \frac{1}{2}(d-2+\eta)\varphi v_k'(\varphi)$$

and generally are referred to as **scaling part** to distinguish it from $F(v_k''(\varphi))$ which carries the quantum or statistical effects.

Part 2:
Approximate the potential to two couplings $v_k(\varphi) = \frac{1}{2}g_2\varphi^2 + \frac{1}{4!}g_4\varphi^4$ and project the flow as

$$k\partial_k v_k(\varphi) = \frac{1}{2}\beta_{g_2}\varphi^2 + \frac{1}{4!}\beta_{g_4}\varphi^4 \tag{5}$$

Give explicit expressions for the beta functions β_{g_2} and β_{g_4} and identitfy their scaling parts.

(Optional) Part 3:
Take $\eta = Bg_4^2$ and $F(x) = Ax^2$ for A and B two positive constants. Expand in $d = 4 - \epsilon$ and find the fixed points of $\beta_{g_2} = 0 = \beta_{g_4}$ and the eigenvalues of the stability matrix *at the leading order in* ϵ.

Does the result depend on A and B? Why?

Some comments that will be discussed during the standard or the exercise class:

Assuming in general that $\eta \sim g_4^2$, it is possible to construct the leading orders in the ϵ expansion of both fixed points and eigenvalues for a general function F, making the above result universal.

From $\beta_{g_4} = 0$ you can find $g_4 = \epsilon/(3F''(g_2))$, and substituting it in $\beta_{g_2} = 0$ you find $g_2 = \epsilon F'(g_2)/(6F''(g_2))$. Using the fact that both g_4 and g_2 are proportional to ϵ and the regularity of $F(x)$ in zero we get that to $\mathcal{O}(\epsilon)$ the nontrivial fixed point is

$$g_2^* = \frac{\epsilon}{6}\frac{F'(0)}{F''(0)} \qquad\qquad g_4^* = \frac{\epsilon}{3F''(0)}$$

The stability matrix at this fixed point becomes

$$\begin{bmatrix} -2 + \frac{\epsilon}{3} & \left(1 + \frac{\epsilon}{6}\right) F'(0) \\ 0 & \epsilon \end{bmatrix}$$

and the eigenvalues can be found trivially because it is a triangular matrix. The negative of these eigenvalues are (related to) the critical exponents.

Consider the eigenvector of the critical exponent $\theta = 2 - \frac{\epsilon}{3}$. In the above basis it is $\{1, 0\}$ which corresponds to the operator φ^2. This means that close to the fixed point we can deform

$$v(\varphi) = v^*(\varphi) + \left(\frac{k}{k_0}\right)^\theta \varphi^2$$

with k_0 an arbitrary reference mass scale.

Now identify $v^*(\varphi)$ with the critical point of a system: the reduced temperature is thus $\frac{T-T_c}{T_c} \sim (k/k_0)^\theta$. (We do not use the symbol t for the reduced temperature to not confuse it with the logarithm of the scale.)

Finally perform a rescaling of the system. The momentum scale transforms as $k \to \lambda \cdot k$, which implies that $\frac{T-T_c}{T_c} \sim \lambda^\theta$. Recalling that the exponent ν is related to the scaling of $\frac{T-T_c}{T_c}$ as $\frac{T-T_c}{T_c} \sim \lambda^{1/\nu}$ under the hyperscaling hypothesis, we deduce that $\nu = 1/\theta = 1/2 + \epsilon/12$.
