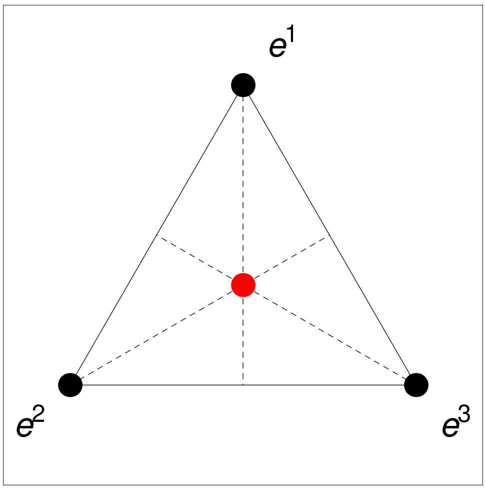
Exercise 5: Three states Potts model

Consider the vertices e_a^α for $a \in \{1,2\}$ and $\alpha \in \{1,2,3\}$ of the regular 2-simplex often known as "equilateral triangle"

$$e^1 = \{0, 1\} \qquad e^2 = \{-\sqrt{3}/2, -1/2\} \qquad e^3 = \{\sqrt{3}/2, -1/2\} \tag{1}$$



(Notice that we are using a normalization which is different from the notes.)

Introduce two scalar fields $\phi_a(x)$ and combine them in the field $\psi^\alpha = \sum_{a=1,2} e_a^\alpha \phi_a$. We define an action for the 3-states Potts model

$$S[\phi] = \int \mathrm{d}^d x \sum_{\alpha=1}^{3} \Big\{ \frac{1}{3} \partial_\mu \psi^\alpha \partial^\mu \psi^\alpha + g(\psi^\alpha)^3 \Big\} \tag{2}$$

The model is invariant under the group S_3 of the permutations of three objects: a permutation $p \in S_3$ acts on the vertices of the triangle as

$$p : \{e^1, e^2, e^3\} \to \{e^{p(1)}, e^{p(2)}, e^{p(3)}\}$$

Geometrically, the permutations of three elements correspond to the transformations that leave the triangle invariant. They are

- the identity transformation: $\{1,2,3\} \to \{1,2,3\}$,
- two rotational symmetries: $\{1,2,3\} \to \{2,3,1\}$ and $\{1,2,3\} \to \{3,1,2\}$ (for 120^{o} and 240^{o}),
- and three reflections: $\{1,2,3\} \to \{1,3,2\}$, $\{1,2,3\} \to \{3,2,1\}$ and $\{1,2,3\} \to \{2,1,3\}$.

Part 1:
Give an explicit expression of $S[\phi]$ in terms of ϕ_1 and ϕ_2. In particular show that the potential becomes

$$V(\phi_1, \phi_2) = \frac{3}{4}(\phi_2)^3 - \frac{9}{4}(\phi_1)^2\phi_2 \tag{3}$$

Part 2:
Consider the rotations with angles $\omega = \pm\frac{2\pi}{3}$ in the (ϕ_1, ϕ_2)-plane given by the matrices

$$R(2\pi/3) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} \qquad R(-2\pi/3) = R(2\pi/3)^T = R(2\pi/3)^{-1} \tag{4}$$

The rotations act on the fields as

$$\phi_a \to \phi'_a = R(\omega)_a{}^b\,\phi_b \tag{5}$$

Show that the potential $V(\phi_1, \phi_2)$ is invariant under those two rotations, $V(\phi_1, \phi_2) = V(\phi'_1, \phi'_2)$.

Part 3:
Why is the potential invariant under two rotations? How many more nontrivial symmetries do you expect the potential to have?

(Optional but very easy!) Part 4:
Consider a generic rotation $R(\alpha)$ of an angle α, and the reflection along the ϕ_2 axis

$$\mathcal{I} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

Can you write all transformations that leave the potential invariant as products of one specific rotation and the reflection $\mathcal{I}$? If yes, which angle α did you find convenient to use and why? *There are multiple answers and you should be able to solve this graphically.*

A bit more theory.
If instead we consider the action with quartic interaction there are in general two couplings

$$S[\phi] = \int \mathrm{d}^d x \Big\{ \frac{1}{3} \sum_{\alpha=1}^{3} \partial_\mu \psi^\alpha \partial^\mu \psi^\alpha + \lambda_1 \sum_{\alpha,\beta=1}^{3} (\psi^\alpha)^2 (\psi^\beta)^2 + \lambda_2 \sum_{\alpha=1}^{3} (\psi^\alpha)^4 \Big\} \tag{7}$$

but in the case $q = 3$ the potential is a function of only the combination $\lambda = \frac{9}{4}\lambda_1 + \frac{9}{8}\lambda_2$. The potential becomes

$$V(\phi_1, \phi_2) = \lambda\left((\phi_1)^2 + (\phi_2)^2\right) \tag{8}$$

In general the quartic interaction will have additional parity $\mathbb{Z}_2$ symmetries $\phi_i \to -\phi_i$ as compared to S_q, but, after direct inspection, in the case $q = 2$ the potential becomes a function of the norm of $\{\phi_1, \phi_2\}$ and therefore has symmetry enhanced to the full $O(2)$ group.