Exercise 4: The effective action

Consider a free massive scalar field in d dimensions with action

$$S[\varphi] = \frac{1}{2}\int \mathrm{d}^d x \left\{(\partial\varphi)^2 + m^2\varphi^2\right\} = \frac{1}{2}\int \mathrm{d}^d x\, \varphi\left(-\partial^2 + m^2\right)\varphi$$

and the path integral

$$\mathrm{e}^{-\Gamma[\varphi]} = \int D\chi\, \mathrm{e}^{-S[\varphi+\chi]}$$

The physical interpretation of the path integral above goes as follows: the field χ are **fluctuations** over a **background field** φ. By integrating the fuctuations we are obtaining an effective action for the field configuration φ which is valid at $\langle\chi\rangle = 0$.

Part 1:
Use the path-integral to show that the effective action is

$$\Gamma[\varphi] = S[\varphi] + \frac{1}{2}\operatorname{Tr}\log\mathcal{O}$$

in which $\mathcal{O} = -\partial^2 + m^2$.

Hints for part 1:
The action is free and therefore the dependences on χ and φ are separable (exactly like when we did momentum shell RG in class). Use the Gaussian integral formula for the functional determinant.

Part 2:
Apply the Laplace transform method to show

$$\frac{1}{2}\operatorname{Tr}\log\mathcal{O} = \frac{1}{2}\int_0^\infty \mathrm{d}s\, \mathcal{L}^{-1}[f](s)\operatorname{Tr}\mathrm{e}^{-s\mathcal{O}}$$

in which $\mathcal{L}^{-1}[f](s)$ is the inverse Laplace transform of the function $f(x) = \log x$.

Hints for part 2:
Given a function $f(x)$, the relation with its inverse Laplace transform is

$$f(x) = \int_0^\infty \mathrm{d}s\, \mathcal{L}^{-1}[f](s)\mathrm{e}^{-sx}$$

Part 3:
Show that the inverse Laplace transform of the logarithm is

$$\mathcal{L}^{-1}[\log](s) = -\frac{1}{s}$$

and therefore

$$\frac{1}{2}\,\mathrm{Tr}\log\mathcal{O} = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}s}{s}\;\mathrm{Tr}\,\mathrm{e}^{-s\mathcal{O}}$$

Hints for part 3:
The relation of the inverse Laplace transform with the original function is

$$\mathcal{L}^{-1}[f](s) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}z\, f(z)\mathrm{e}^{sz}$$

with $\gamma \in \mathbb{R}$ such that all the poles z_i of $f(z)$ lie on the left part of the complex plane with respect to γ: $\gamma > Re(z_i)\ \forall\, i$. You might want to use the following property of the inverse Laplace transform:

$$\mathcal{L}^{-1}[f'](s) = -s\mathcal{L}^{-1}[f](s)\,,$$

with f' the derivative of f.

Part 4:
Use the momentum space representation of the operator

$$\mathrm{e}^{-s\mathcal{O}}\,|p\rangle = \mathrm{e}^{-s(p^2+m^2)}\,|p\rangle$$

for a normalized state $|p\rangle$, and of the trace

$$\mathrm{Tr}(...) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\langle p|\,(...)\,|p\rangle$$

to find an explicit formula for

$$\frac{1}{2}\,\mathrm{Tr}\log\mathcal{O}$$

(integrate first in p and then in s). What are the assumptions that you have to make on d and m^2 for the integral to be convergent? When do you have to make these assumptions? What happens if $m^2 = 0$?

Hints for part 4:
Recall the integral form of the Euler Gamma function:

$$\Gamma(z+1) = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t}\,t^{z-1}\,,$$

which is convergent for $z > 0$.