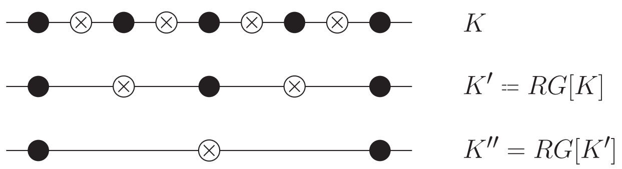
Exercise 3: Real space renormalization of the two-dimensional Ising model

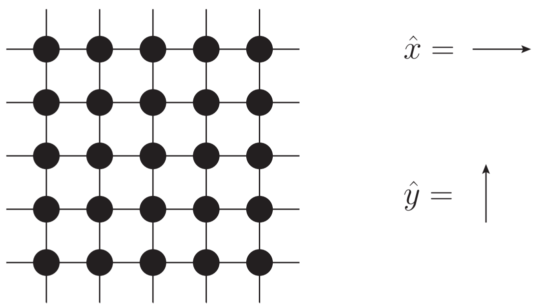
The objective of this exercise is to construct a real-space renormalization of the two dimensional Ising model using only the analysis that we have made for the one dimensional case. *Keep in mind that you should be able to infer the renormalization without doing any actual computation* besides the last numerical estimate that you can do with your favorite programming language.

First recall how we perform a renormalization step of the one dimensional model: we explicitly sum over half of the spins and we conclude that the Ising's coupling $K = \beta J$ of the original lattice is mapped to the coupling $K' = RG[K] = \frac{1}{2}\ln\cosh(2K)$ in the resulting lattice. The renormalization group flow is obtained by nesting this operation and it is easy to see that after few steps the flow leads the coupling to the (high temperature) fixed point $K^* = 0$. The figure below summarizes the procedure: circles correspond to spins, and lines to the bond interactions. Full black circles are spins that are not summed over, while circles with a cross are spins that we sum. The figure shows the first two steps of the blocking procedure.



The explicit RG transformation derived in class is repeated here $K' = RG[K] = \frac{1}{2}\ln\cosh(2K)$ and will be needed later.

Now we move to a two dimensional lattice. In class we discused how to sum an alternating sublattice, but for this exercise we are going to do something different. We are going to contruct the RG procedure in three steps. First let us introduce the unit vectors $\hat{x}$ and $\hat{y}$ as shown in the figure below.



A spin σ_i located in position $i = \{x, y\}$ couples with the four neigboring spins as follows

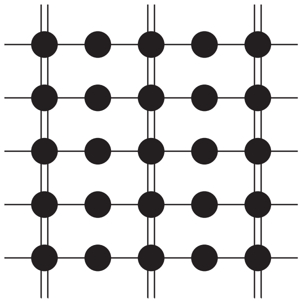
$$-\beta\mathcal{H} \sim K\Big\{\sigma_i\sigma_{i+\hat{x}} + \sigma_i\sigma_{i-\hat{x}} + \sigma_i\sigma_{i+\hat{y}} + \sigma_i\sigma_{i+\hat{y}}\Big\}$$

in which we are showing only the terms of the Hamiltonian which include σ_i. **The first procedure's step** is to allow the interaction K to be anisotropic as follows

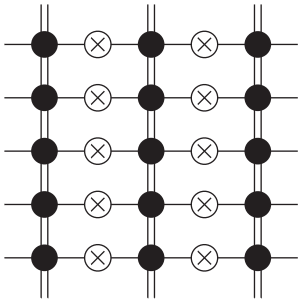
$$-\beta\mathcal{H} \sim K_x\Big\{\sigma_i\sigma_{i+\hat{x}} + \sigma_i\sigma_{i-\hat{x}}\Big\} + K_y\Big\{\sigma_i\sigma_{i+\hat{y}} + \sigma_i\sigma_{i+\hat{y}}\Big\}$$

Spins are now interacting with different strengths for different cardinal directions, but clearly requiring $K_x = K_y = K$ we can get to the original Hamiltonian!

The second procedure's step is to approximate the Hamiltonian by "sliding" half of the bond interactions in the $\hat{y}$ direction to the other half as shown in the figure blow. In practice: we neglect half of the vertical bond interactions, while the other half becomes *twice as strong*.



The third procedure's step is to sum over the spins which are interacting *only in the horizontal direction* as shown in the following figure.



Together the three procedure's steps form our initial RG step.

Question 1: Write down the effect of an RG step to the couplings: $\{K'_x, K'_y\} = RG_1[\{K_x, K_y\}]$.

From the answer to the first question it should be clear that it doesn't make sense to iterate this single RG step.

Question 2: Can you explain qualitatively why?

Now consider another RG step which we call $\{K'_x, K'_y\} = RG_2[\{K_x, K_y\}]$ which is of the same form but has x and y switched (so it is rotated by $\pi/2$).

Question 3: Consider the nested steps $\{K'_x, K'_y\} = RG_2[RG_1[\{K_x, K_y\}]]$. What are the fixed points for K_x and K_y? (Do it numerically.) Using either fixed point estimate the exponent α as done in class with scaling methods. Using Josephson's hyperscaling identity estimate the exponent ν too.

Optional question 4: Are these estimates better or worse than those obtained with the method shown in class? Can you give some qualitative argument why?