Exercise 2: Scaling relations and universal properties

Assume the generalized homogeneous scaling form of the free energy near criticality

$$F_s(\lambda^{a_t} t, \lambda^{a_h} h) = \lambda F_s(t, h)$$

The number λ is a dimensionless constant of arbitrary value. All thermodynamical exponents can be computed from the above expression.

Let us compute one as example: First take a derivative with respect to h on both sides

$$\frac{\partial}{\partial h} F_s(\lambda^{a_t} t, \lambda^{a_h} h) = \lambda \frac{\partial}{\partial h} F_s(t, h)$$

We use the fact that the derivative of F_s with respect to h is the (negative of the) magnetization

$$\lambda^{a_h} M_s(\lambda^{a_t} t, \lambda^{a_h} h) = \lambda M_s(t, h)$$

and solve for $M_s(t, h)$ as follows

$$M_s(t, h) = \lambda^{a_h - 1} M_s(\lambda^{a_t} t, \lambda^{a_h} h)$$

Finally eliminate from the argument on the right hand side by taking $\lambda = t^{-\frac{1}{a_t}}$. We get the scaling form

$$M_s(t, h) = t^{\frac{1-a_h}{a_t}} M_s(1, t^{-\frac{a_h}{a_t}} h) = t^{\frac{1-a_h}{a_t}} M_s(1, t^{-\Delta} h) \equiv t^{\frac{1-a_h}{a_t}} g_M\left(\frac{h}{t^{\Delta}}\right)$$

The exponent β is defined from the scaling at zero magnetic field $M_s(t, 0) \sim t^{\beta}$, and the function g_M is regular by definition. This implies

$$\beta = \frac{1 - a_h}{a_t}$$

Part 1. Use the same procedure to determine the exponents α, γ and δ.

Part 2. Use the determined exponents to *check* the scaling relations (Rushbrooke's, Griffiths' and Widom's identities). Alternatively feel free to derive them as described in class.

In class we have seen that following hyperscaling arguments it is possible to introduce two new critical exponents ν and η of field-theoretical nature and relate them to α and γ (Josephson's and Fisher's identities).

Part 3. Use all identies to show

$$\alpha = 2 - \nu d \qquad\qquad \gamma = \nu(2 - \eta)$$
$$\beta = \tfrac{1}{2}(d - 2 + \eta)\nu \qquad\qquad \delta = \tfrac{d+2-\eta}{d-2+\eta}$$

(You do not have to derive hyperscaling identities here, feel free to bring them from the notes.)

The critical exponents are *universal* features of the phase transition. However, they are *not* the only universal features that one can compute. In general, a quantity such as the correlation length ξ behaves as

$$\xi = \begin{cases} \xi_0^+ t^{-\nu} & \text{for } \; t > 0 \; \text{ and } \; h = 0 \\ \xi_0^- (-t)^{-\nu} & \text{for } \; t < 0 \; \text{ and } \; h = 0 \end{cases}$$

We have introduced two overall coefficients $\xi_0^{\pm}$ known as **amplitudes** which normalize the powerlaw behavior and are related to the scaling form. Similar amplitudes can be introduced for the other quantities, but their value is not universal. However the ratio $\frac{\xi_0^+}{\xi_0^-}$, which is often called **amplitude ratio**, is a universal quantity that can be estimated (for example) with mean-field analysis.

Optional: Part 4. Using the Ginzburg-Landau approach of the notes with $r = r_0 \, |t|$, and the analysis of the two-point function show

$$\left(\frac{\xi_0^+}{\xi_0^-}\right)^2 = 2$$