Exercise 12: Coincidence limits

This is going to be the most tedious exercise so far. Do not get intimidated: it took me less than four A4 pages and I am very bad at computations!

Consider the crucial relations of the Synge's function $\sigma(x, x')$

$$\sigma_\mu \sigma^\mu = 2\sigma \tag{1}$$

and the (root of the) van Vleck's determinant $\Delta(x, x')^{1/2}$

$$\Delta^{1/2} \sigma_\mu{}^\mu + 2\sigma^\mu \Delta^{1/2}{}_\mu = d\Delta^{1/2} \tag{2}$$

Consider also the equation satisfied by the second heat kernel coefficient $a_1(x, x')$

$$a_1 + \sigma^\mu \nabla_\mu a_1 + \Delta^{-1/2} \mathcal{O}_x(\Delta^{1/2} a_0) = a_1 + \sigma^\mu \nabla_\mu a_1 - \Delta^{-1/2} \nabla^2(\Delta^{1/2} a_0) + E a_0 = 0 \tag{3}$$

We denote coincidence limits of bi-tensors as

$$[B] = B(x, x)$$

The first few coincidence limits to begin with are

$$[\sigma] = 0 \qquad [\Delta^{1/2}] = 1 \qquad [a_0] = 1$$

It is convenient to use a notation for which covariant derivatives are denoted as indices

$$\nabla_{\mu_n} \dots \nabla_{\mu_2} \nabla_{\mu_1} \sigma = \sigma_{\mu_1 \mu_2 \dots \mu_n}$$

Notice that the new notation reverses the order of the indices!

Part 1:
Use (1) to show that

$$[\sigma_\mu] = 0$$

Part 2:
Take two covariant derivatives of (1) and use the resulting equation to show that

$$[\sigma_{\mu\nu}] = g_{\mu\nu}$$

Hint: The metric is the only tensor for which $g_{\mu\nu}\, g^\nu{}_\rho = g_{\mu\rho}$.

Part 3:
Take three covariant derivatives of (1) and use the resulting equation to show that

$$[\sigma_{\mu\nu\rho}] = 0$$

Hint: It is useful to arrange all expressions of $[\sigma_{\mu\nu\rho}]$ with the indices in the same order. In doing so a useful manipulation is

$$\sigma_{\mu\rho\nu} = \nabla_\nu \nabla_\rho \, \sigma_\mu = [\nabla_\nu, \nabla_\rho] \, \sigma_\mu + \nabla_\rho \nabla_\nu \, \sigma_\mu = -R_{\nu\rho}{}^\theta{}_\mu \sigma_\theta + \sigma_{\mu\nu\rho}$$

Notice also that curvatures are standard tensors, so in the coincidence limit they factor out

$$\left[R_{\nu\rho}{}^\theta{}_\mu \, \sigma_\theta \right] = R_{\nu\rho}{}^\theta{}_\mu [\sigma_\theta] = 0$$

Part 4:
Take four covariant derivatives of (1) and use the resulting equation to show that

$$[\sigma_{\mu\nu\rho\theta}] = -\frac{1}{3} \left(R_{\mu\rho\nu\theta} + R_{\mu\theta\nu\rho} \right)$$

Hint: Sort all covariant derivatives as in the previous exercise. Furthermore, some standard symmetries of the Riemann tensor could be useful: $R_{\mu\rho\nu\theta} = -R_{\rho\mu\nu\theta}$, $R_{\mu\rho\nu\theta} = R_{\nu\theta\mu\rho}$, and $R_{\mu\rho\nu\theta} + R_{\mu\nu\theta\rho} + R_{\mu\theta\rho\nu} = 0$.

Part 5:
Take one covariant derivative of (2) and use the resulting equation to show that

$$[\Delta^{1/2}{}_\mu] = 0$$

Hint: Consider $\Delta(x, x')^{1/2}$ instead of $\Delta(x, x')$ as the relevant tensor.

Part 6:
Take two covariant derivatives of (2) and use the resulting equation to show that

$$[\Delta^{1/2}{}_{\mu\nu}] = \frac{R_{\mu\nu}}{6}$$

Part 7:
Take the coincidence limit of (3) and show that

$$[a_1] = \frac{R}{6} - E$$
