Exercise 11: Form-factors and decoupling

In the last exercise we computed some local and non-local contributions to the effective action for an interacting scalar field theory. Using the notation for which $[\varphi^2]_p$ is the Fourier transform of the square of the field φ, we established that at order φ^4 the one loop effective action is

$$\Gamma|_{\varphi^4} = -\frac{g^2}{2 \cdot 4(4\pi)^2 \epsilon} \int_x \varphi^4 + \int_p [\varphi^2]_{-p} F\left(p, m; g\right) [\varphi^2]_p + \mathcal{O}(\epsilon)$$

in which we defined the **non-local form-factor** F as

$$F\left(p, m; g\right) = \frac{2g^2}{(4\pi)^2} \frac{m}{p} \sqrt{4 + \frac{p^2}{m^2}} \operatorname{arctanh}\left(\frac{p}{m\sqrt{4 + \frac{p^2}{m^2}}}\right) + \frac{g^2}{(4\pi)^2}\left[\gamma - 2 + \log\left(\frac{m^2}{4\pi}\right)\right]$$

The $\frac{1}{\epsilon}$ pole is the divergence that requires first regularization and then renormalization. The process of renormalization forces us to introduce a reference scale μ and results in a renormalization group beta function for the coupling

$$\beta_{\rm MS} = \frac{3g^2}{(4\pi)^2}$$

after minimal subtraction (MS) of the divergence as we have seen during the lectures. After subtraction the renormalized effective will contain only the finite part of $\Gamma|_{\varphi^4}$.

Through this exercise we want to understand if we can give a more physical intuition to the renormalization group in particle physics. First recall that the finite effective action is now

$$\Gamma|_{\varphi^4} = \int_p [\varphi^2]_{-p} F\left(p, m; g\right) [\varphi^2]_p$$

which can be related to the scattering of four scalar field's states by taking the appropriate number of functional derivatives. Qualitatively, we can imagine that the form-factor $F\left(p, m; g\right)$ is a **momentum-dependent coupling constant** $g(p) \equiv F\left(p, m; g\right)$ in which the relevant momentum scale $p = |p_\mu|$ is related to some scattering energy (and in particular to the variables s, t and u of the notes, even if we do not work out this relation explicitly here).

Having made this definition, we are naturally lead to interpret p as a renormalization group scale and $\beta = p\frac{\mathrm{d}}{\mathrm{d}p} g(p)$ as the (new) renormalization group running according to this scale. This running is certainly richer than the one of $\beta_{\rm MS}$ because of the explicit presence of the mass, below we want to see how much richer it is and what can we learn from it.

Part 1:
Define the variable $x = \frac{p^2}{m^2}$ and rewrite the form-factor of $\Gamma|_{\varphi^4}$ in terms of x. Rewrite the renormalization group operator $p\frac{\mathrm{d}}{\mathrm{d}p}$ in terms of the variable x as well.

Part 2:
Compute the beta function

$$\beta = 4!\, p \frac{\mathrm{d}}{\mathrm{d}p} F\left(p, m; g\right)$$

up to order g^2. Make sure that the final result is expressed in terms of x.

Hint: You better use the results of the previous point. Also recall that $\beta \sim g^2$ and $\beta_m \sim g$!

Part 3:
Expand the beta function to find the leading contributions in the asymptotic regimes $x \sim \infty$ and $x \sim 0$.

Hint: Use the following limits

$$\operatorname{arctanh}\left(\sqrt{\frac{x}{4+x}}\right) = \begin{cases} \dfrac{1}{x}, & \text{for} \quad x \sim \infty \\ \dfrac{\sqrt{x}}{2}, & \text{for} \quad x \sim 0 \end{cases}$$

Part 4:
Verify that

$$\begin{aligned} \beta(x) &= \beta_{\mathrm{MS}} && \text{for} \quad x \sim \infty \\ \beta(x) &\to 0 && \text{for} \quad x \to 0 \end{aligned}$$

It should be clear that the limits $x \sim \infty$ ($p^2 \gg m^2$) and $x \sim 0$ ($p^2 \ll m^2$), represent the UV and IR behaviors of the momentum-dependent coupling respectively. Once this connection is done then we understand that

- In the ultraviolet the new beta function β coincides with β_{MS}. It happens because the scale μ of dimensional regularization is a very high energy scale that is bigger, in particular, of any physical mass $\mu^2 \gg m^2$. In UV all the beta functions coincide because they are **universal** (here used with a slightly different meaning than in statistical mechanics).
- In the infrared the new beta function goes to zero. This is known as the **decoupling** of the field's fluctuations: below the field's mass quantum fluctuations stop propagating and cannot contribute to the quantum effects! The decoupling is predicted by the **Appelquist-Carazzone theorem** of QFT. It is very important to consider decoupling effects when studying the standard model of particle physics, because in different energy ranges there are different matter fields that contribute to the running!