Exercise 10: Heat kernel methods

Some theory first.
Consider the representation of the one loop effective action

$$\Gamma[\varphi] = \frac{1}{2}\,\mathrm{Tr}\log\left(\mathcal{O} + m^2\right)$$

with the operator $\mathcal{O} = -\partial^2 + \frac{g}{2}\varphi^2$. We also define the flat space Laplacian as $\Delta = -\partial^2$. We have already seen that the action can be represented as

$$\Gamma[\varphi] = -\frac{1}{2}\,\mathrm{Tr}\int \frac{\mathrm{d}s}{s}\,\mathrm{e}^{-sm^2}\,\mathcal{H}(s)$$

with $\mathcal{H}(s) \equiv \mathrm{e}^{-s\mathcal{O}}$ known as the *heat kernel* of the operator $\mathcal{O}$. The parameter s is often called *proper time*. (The mass is assumed here to be a number that commutes with everything.)

The trace can be performed by giving a representation to $\mathcal{H}(s)$. In coordinate space it is a matrix which is formally written as $\mathcal{H}(s;x,x') \equiv \langle x|\mathrm{e}^{-s\mathcal{O}}|x'\rangle$. In practice $\mathcal{H}(s;x,x')$ is the solution of the differential equation

$$\partial_s\mathcal{H}(s;x,x') + \mathcal{O}_x\mathcal{H}(s;x,x') = 0 \qquad\qquad \mathcal{H}(0;x,x') = \delta(x,x')$$

This function can be interpreted as the propagation to the location x and in the time s of some heat from a point source located at the point $x = x'$ and time $s = 0$ according to the operator $\mathcal{O}$. The heat kernel forms an Abelian one dimensional semigroup

$$\int \mathrm{d}^d x'\,\mathcal{H}(s_1;x,x')\mathcal{H}(s_2;x',x'') = \mathcal{H}(s_1+s_2;x,x'')$$

We define also the heat kernel $\mathcal{H}_0(s)$ of the simpler operator $\Delta = -\partial^2$ analogously

$$\partial_s\mathcal{H}_0(s;x,x') + \Delta_x\mathcal{H}_0(s;x,x') = 0 \qquad\qquad \mathcal{H}_0(0;x,x') = \delta(x,x')$$

This latter heat kernel is simpler because it can be computed easily in momentum space and becomes a Gaussian distribution

$$\tilde{\mathcal{H}}_0(s;p)\delta_{p,p'} = \langle p|\mathcal{H}_0(s)|p'\rangle = \mathrm{e}^{-sp^2}\delta_{p,p'}$$

in which we used momentum conservation to factor out a delta function in the definition of $\tilde{\mathcal{H}}_0(s;p)$. In coordinate space it is also a Gaussian after Fourier transforming

$$\mathcal{H}_0(s;x,x') = \frac{1}{(4\pi s)^{d/2}}\exp\left(-\frac{|x-x'|^2}{4s}\right)$$

You can see that Δ is covariant under Euclidean transformations (translations and rotations) because $\mathcal{H}_0(s;x,x')$ is just a function of $|x-x'|$.

Let $E = \frac{g}{2}\varphi^2$ be the *endomorphism* that distinguishes the two operators $\mathcal{O} = \Delta + E$. The endomorphism is a local function $E = E_x = \frac{g}{2}\varphi(x)^2$, so it is not a differential operator like Δ, and we indicate by $\tilde{E}_p$ its Fourier transform. We can write the heat kernel of $\mathcal{O}$ using the simpler $\mathcal{H}_0(s)$ thanks to the (proper time) ordered expansion in E

$$
\begin{aligned}
&\mathcal{H}(s;x,x') = \mathcal{H}_0(s;x,x') - s\int_y\int_0^1 \mathrm{d}t_1\, \mathcal{H}_0(s(1-t_1);x,y)E_y\mathcal{H}_0(st_1;y,x') \\
&+s^2\int_{y_1,y_2}\int_0^1 \mathrm{d}t_1 \int_0^{t_1}\mathrm{d}t_2\, \mathcal{H}_0(s(1-t_1);x,y_1)E_{y_1}\mathcal{H}_0(s(t_1-t_2);y_1,y_2)E_{y_2}\mathcal{H}_0(st_2;y_2,x') \\
&+\dots
\end{aligned}
$$

in which spacetime integrations are condensed $\int_y \equiv \int \mathrm{d}^d y$. (We are going to show how to derive this expansion in class.) Inserting the above expression in the one of the effective action

$$
\Gamma[\varphi] = -\frac{1}{2}\int\frac{\mathrm{d}s}{s}\int \mathrm{d}^d x\, \mathrm{e}^{-sm^2}\,\mathcal{H}(s;x,x)
$$

and considering only the part quadratic in E we get

$$
\begin{aligned}
\Gamma|_{E^2} \quad = \quad & -\frac{1}{2}\int \mathrm{d}s\, s\, \mathrm{e}^{-sm^2}\int_{y_1,y_2}\int_0^1\mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\,\mathcal{H}_0(s(1-t_1+t_2);y_2,y_1)E_{y_1}\times \\
& \times\mathcal{H}_0(s(t_1-t_2);y_1,y_2)E_{y_2}
\end{aligned}
$$

after using the semigroup property (and the cyclicity of the trace).

Part 1:
Compute the Fourier transform of

$$
\int_{y_1,y_2}\mathcal{H}_0(s(1-t_1+t_2);y_2,y_1)E_{y_1}\mathcal{H}_0(s(t_1-t_2);y_1,y_2)E_{y_2} = \int_{p,q}\tilde{E}_p\,\mathcal{A}(s;t_1,t_2;p,q)\,\tilde{E}_{-p}
$$

Hint:
Recall that $\tilde{E}_p$ is the Fourier transform of the endomorphism.

Part 2:
Define the variable $\xi = t_1 - t_2$ and show that

$$
(1-t_1+t_2)q^2 + (t_1-t_2)(q+p)^2 = (q+\xi p)^2 + \xi(1-\xi)p^2
$$

then insert this result in the amplitude $\mathcal{A}$ and perform the integration over q.

Hint:
After the manipulation of the exponent recall that the integration measure $\mathrm{d}^d q$ is invariant under translations.

Part 3:
Use the result of the previous point (keeping the dependence on the momentum p) to compute the integration over the proper time s inside $\Gamma|_{E^2}$.

Hint:
You can rewrite the integration over the two parameters t_1 and t_2 using the fact that:

$$\int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \, g(t_1 - t_2) = \frac{1}{2} \int_0^1 \mathrm{d}\xi \, g(\xi) \,,$$

for g a function such that $g(\xi) = g(1 - \xi)$.

Part 4:
Expand $\Gamma|_{E^2}$ in $\epsilon = 4 - d$ (including the finite part at ϵ^0) and compute the integration over ξ.

Hint:
You can make use of the following primitive function

$$\begin{aligned} \int \mathrm{d}\xi \, \log\left(m^2 + \xi(1-\xi)p^2\right) \quad &= \quad -\sqrt{\frac{p^2 + 4m^2}{p^2}} \operatorname{arctanh}\left(\frac{2\xi - 1}{\sqrt{\frac{p^2+4m^2}{p^2}}}\right) \\ &\quad -\xi + \left(\xi - \frac{1}{2}\right) \log\left(m^2 + \xi(1-\xi)p^2\right) \end{aligned}$$

and recall that $\operatorname{arctanh}(-x) = -\operatorname{arctanh}(x)$.
