Exercise 1: Ising model in $d = 1$

Consider the Hamiltonian of the one dimensional Ising model

$$\mathcal{H} = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i = -\sum_{i=1}^{N}\left\{J\sigma_i\sigma_{i+1} + \frac{h}{2}(\sigma_i + \sigma_{i+1})\right\}$$

and the partition function expressed as a product of transfer matrices

$$Z = \sum_{\{\sigma\}} \mathrm{e}^{\sum_{i=1}^{N}\left\{\beta J\sigma_i\sigma_{i+1} + \frac{\beta h}{2}(\sigma_i+\sigma_{i+1})\right\}} = \sum_{\{\sigma\}}\prod_{i=1}^{N} T(\sigma_i, \sigma_{i+1})$$

The components of the transfer matrix are

$$T(\sigma_i, \sigma_{i+1}) = \mathrm{e}^{\beta J\sigma_i\sigma_{i+1} + \frac{\beta h}{2}(\sigma_i+\sigma_{i+1})}$$

Part 1. Write down the matrix

$$\mathbf{T} = \begin{bmatrix} T(+1,+1) & T(+1,-1) \\ T(-1,+1) & T(-1,-1) \end{bmatrix}$$

in the basis $\sigma_i \otimes \sigma_{i+1}$.

Part 2. Compute the eigenvalues $\lambda_{1,2}$ of the transfer matrix in this basis. Use them to express the partition function and then take the thermodynamical limit $N \to \infty$.

Optional: Part 3. Compute the spin-spin correlator $\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$.

Hint: Part 3 is much more difficult, so feel free to use Mathematica to compute the eigenvalues (and the eigenvectors if necessary!) It is convenient to modify the Hamiltonian by adding a further space-dependent magnetic field $\mathcal{H} \to \mathcal{H}[b_i] = \mathcal{H} + \sum_{i=1}^{N} b_i\sigma_i$. The spin-spin correlator can then be obtained from the partition function

$$\langle\sigma_i\sigma_j\rangle = \frac{1}{Z}\frac{1}{\beta^2}\frac{\delta^2}{\delta b_i \delta b_j} Z\bigg|_{b_i=0}$$