Exercise 1: Ising model in $d = 1$

Consider the Hamiltonian of the one dimensional Ising model

$$\mathcal{H} \;=\; -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i \;=\; -\sum_{i=1}^{N}\left\{J\sigma_i\sigma_{i+1} + \frac{h}{2}\left(\sigma_i + \sigma_{i+1}\right)\right\}$$

and the partition function expressed as a product of transfer matrices

$$Z \;=\; \sum_{\{\sigma\}} \mathrm{e}^{\sum_{i=1}^{N}\left\{\beta J\sigma_i\sigma_{i+1} + \frac{\beta h}{2}(\sigma_i+\sigma_{i+1})\right\}} = \sum_{\{\sigma\}}\prod_{i=1}^{N} T(\sigma_i, \sigma_{i+1})$$

The components of the transfer matrix are

$$T(\sigma_i, \sigma_{i+1}) = \mathrm{e}^{\beta J\sigma_i\sigma_{i+1} + \frac{\beta h}{2}(\sigma_i+\sigma_{i+1})}$$

Part 1. Write down the matrix

$$\mathbf{T} = \begin{bmatrix} T(+1,+1) & T(+1,-1) \\ T(-1,+1) & T(-1,-1) \end{bmatrix}$$

in the basis $\sigma_i \otimes \sigma_{i+1}$.

Part 2. Compute the eigenvalues $\lambda_{1,2}$ of the transfer matrix in this basis. Use them to express the partition function and then take the thermodynamical limit $N \to \infty$.

Optional: Part 3. Compute the spin-spin correlator $\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$.

Hint: Part 3 is much more difficult, so feel free to use Mathematica to compute the eigenvalues (and the eigenvectors if necessary!) It is convenient to modify the Hamiltonian by adding a further space-dependent magnetic field $\mathcal{H} \to \mathcal{H}[b_i] = \mathcal{H} + \sum_{i=1}^{N} b_i\sigma_i$. The spin-spin correlator can then be obtained from the partition function

$$\langle\sigma_i\sigma_j\rangle = \frac{1}{Z}\frac{1}{\beta^2}\frac{\delta^2}{\delta b_i \delta b_j} Z\bigg|_{b_i=0}$$

Exercise 2: Scaling relations and universal properties

Assume the generalized homogeneous scaling form of the free energy near criticality

$$F_s(\lambda^{a_t}t, \lambda^{a_h}h) = \lambda F_s(t,h)$$

The number λ is a dimensionless constant of arbitrary value. All thermodynamical exponents can be computed from the above expression.

Let us compute one as example: First take a derivative with respect to h on both sides

$$\frac{\partial}{\partial h}F_s(\lambda^{a_t}t, \lambda^{a_h}h) = \lambda\frac{\partial}{\partial h}F_s(t,h)$$

We use the fact that the derivative of F_s with respect to h is the (negative of the) magnetization

$$\lambda^{a_h}M_s(\lambda^{a_t}t, \lambda^{a_h}h) = \lambda M_s(t,h)$$

and solve for $M_s(t,h)$ as follows

$$M_s(t,h) = \lambda^{a_h-1}M_s(\lambda^{a_t}t, \lambda^{a_h}h)$$

Finally eliminate from the argument on the right hand side by taking $\lambda = t^{-\frac{1}{a_t}}$. We get the scaling form

$$M_s(t,h) = t^{\frac{1-a_h}{a_t}}M_s(1, t^{-\frac{a_h}{a_t}}h) = t^{\frac{1-a_h}{a_t}}M_s(1, t^{-\Delta}h) \equiv t^{\frac{1-a_h}{a_t}}g_M\left(\frac{h}{t^{\Delta}}\right)$$

The exponent β is defined from the scaling at zero magnetic field $M_s(t,0) \sim t^{\beta}$, and the function g_M is regular by definition. This implies

$$\beta = \frac{1-a_h}{a_t}$$

Part 1. Use the same procedure to determine the exponents α, γ and δ.

Part 2. Use the determined exponents to *check* the scaling relations (Rushbrooke's, Griffiths' and Widom's identities). Alternatively feel free to derive them as described in class.

In class we have seen that following hyperscaling arguments it is possible to introduce two new critical exponents ν and η of field-theoretical nature and relate them to α and γ (Josephson's and Fisher's identities).

Part 3. Use all identies to show

$$\begin{aligned} \alpha &= 2-\nu d & \qquad \gamma &= \nu(2-\eta) \\ \beta &= \tfrac{1}{2}(d-2+\eta)\nu & \qquad \delta &= \tfrac{d+2-\eta}{d-2+\eta} \end{aligned}$$

(You do not have to derive hyperscaling identities here, feel free to bring them from the notes.)

The critical exponents are *universal* features of the phase transition. However, they are *not* the only universal features that one can compute. In general, a quantity such as the correlation length ξ behaves as

$$\xi = \begin{cases} \xi_0^+ t^{-\nu} & \text{for } t>0 \text{ and } h=0 \\ \xi_0^-(-t)^{-\nu} & \text{for } t<0 \text{ and } h=0 \end{cases}$$

We have introduced two overall coefficients $\xi_0^{\pm}$ known as **amplitudes** which normalize the powerlaw behavior and are related to the scaling form. Similar amplitudes can be introduced for the other quantities, but their value is not universal. However the ratio $\frac{\xi_0^+}{\xi_0^-}$, which is often called **amplitude ratio**, is a universal quantity that can be estimated (for example) with mean-field analysis.

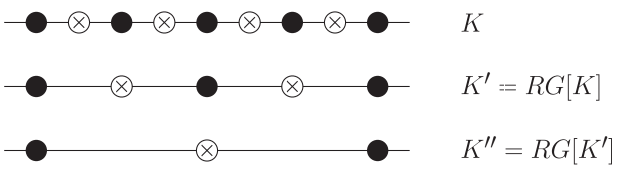
Optional: Part 4. Using the Ginzburg-Landau approach of the notes with $r = r_0\,|t|$, and the analysis of the two-point function show

$$\left(\frac{\xi_0^+}{\xi_0^-}\right)^2 = 2$$

Exercise 3: Real space renormalization of the two-dimensional Ising model

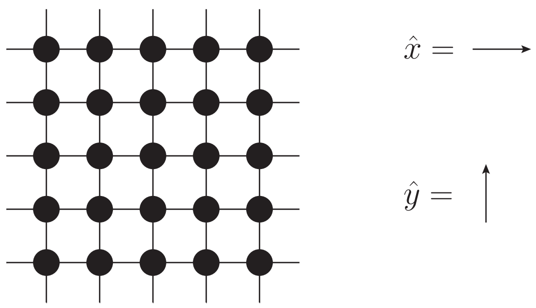
The objective of this exercise is to construct a real-space renormalization of the two dimensional Ising model using only the analysis that we have made for the one dimensional case. *Keep in mind that you should be able to infer the renormalization without doing any actual computation* besides the last numerical estimate that you can do with your favorite programming language.

First recall how we perform a renormalization step of the one dimensional model: we explicitly sum over half of the spins and we conclude that the Ising's coupling $K = \beta J$ of the original lattice is mapped to the coupling $K' = RG[K] = \frac{1}{2}\ln\cosh(2K)$ in the resulting lattice. The renormalization group flow is obtained by nesting this operation and it is easy to see that after few steps the flow leads the coupling to the (high temperature) fixed point $K^* = 0$. The figure below summarizes the procedure: circles correspond to spins, and lines to the bond interactions. Full black circles are spins that are not summed over, while circles with a cross are spins that we sum. The figure shows the first two steps of the blocking procedure.



The explicit RG transformation derived in class is repeated here $K' = RG[K] = \frac{1}{2}\ln\cosh(2K)$ and will be needed later.

Now we move to a two dimensional lattice. In class we discused how to sum an alternating sublattice, but for this exercise we are going to do something different. We are going to contruct the RG procedure in three steps. First let us introduce the unit vectors $\hat{x}$ and $\hat{y}$ as shown in the figure below.



A spin σ_i located in position $i = \{x, y\}$ couples with the four neigboring spins as follows

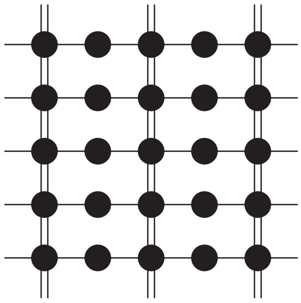
$$-\beta\mathcal{H} \sim K\Big\{\sigma_i\sigma_{i+\hat{x}} + \sigma_i\sigma_{i-\hat{x}} + \sigma_i\sigma_{i+\hat{y}} + \sigma_i\sigma_{i+\hat{y}}\Big\}$$

in which we are showing only the terms of the Hamiltonian which include σ_i. **The first procedure's step** is to allow the interaction K to be anisotropic as follows

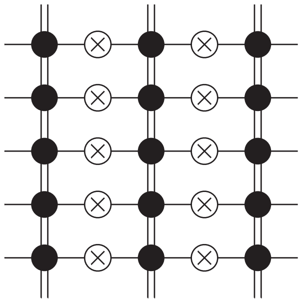
$$-\beta\mathcal{H} \sim K_x\Big\{\sigma_i\sigma_{i+\hat{x}} + \sigma_i\sigma_{i-\hat{x}}\Big\} + K_y\Big\{\sigma_i\sigma_{i+\hat{y}} + \sigma_i\sigma_{i+\hat{y}}\Big\}$$

Spins are now interacting with different strengths for different cardinal directions, but clearly requiring $K_x = K_y = K$ we can get to the original Hamiltonian!

The second procedure's step is to approximate the Hamiltonian by "sliding" half of the bond interactions in the $\hat{y}$ direction to the other half as shown in the figure blow. In practice: we neglect half of the vertical bond interactions, while the other half becomes *twice as strong*.



The third procedure's step is to sum over the spins which are interacting *only in the horizontal direction* as shown in the following figure.



Together the three procedure's steps form our initial RG step.

Question 1: Write down the effect of an RG step to the couplings: $\{K'_x, K'_y\} = RG_1[\{K_x, K_y\}]$.

From the answer to the first question it should be clear that it doesn't make sense to iterate this single RG step.

Question 2: Can you explain qualitatively why?

Now consider another RG step which we call $\{K'_x, K'_y\} = RG_2[\{K_x, K_y\}]$ which is of the same form but has x and y switched (so it is rotated by $\pi/2$).

Question 3: Consider the nested steps $\{K'_x, K'_y\} = RG_2[RG_1[\{K_x, K_y\}]]$. What are the fixed points for K_x and K_y? (Do it numerically.) Using either fixed point estimate the exponent α as done in class with scaling methods. Using Josephson's hyperscaling identity estimate the exponent ν too.

Optional question 4: Are these estimates better or worse than those obtained with the method shown in class? Can you give some qualitative argument why?

Exercise 4: The effective action

Consider a free massive scalar field in d dimensions with action

$$S[\varphi] = \frac{1}{2}\int \mathrm{d}^d x \left\{(\partial\varphi)^2 + m^2\varphi^2\right\} = \frac{1}{2}\int \mathrm{d}^d x\, \varphi\left(-\partial^2 + m^2\right)\varphi$$

and the path integral

$$\mathrm{e}^{-\Gamma[\varphi]} = \int D\chi\, \mathrm{e}^{-S[\varphi+\chi]}$$

The physical interpretation of the path integral above goes as follows: the field χ are **fluctuations** over a **background field** φ. By integrating the fuctuations we are obtaining an effective action for the field configuration φ which is valid at $\langle\chi\rangle = 0$.

Part 1:
Use the path-integral to show that the effective action is

$$\Gamma[\varphi] = S[\varphi] + \frac{1}{2}\,\mathrm{Tr}\log\mathcal{O}$$

in which $\mathcal{O} = -\partial^2 + m^2$.

Hints for part 1:
The action is free and therefore the dependences on χ and φ are separable (exactly like when we did momentum shell RG in class). Use the Gaussian integral formula for the functional determinant.

Part 2:
Apply the Laplace transform method to show

$$\frac{1}{2}\,\mathrm{Tr}\log\mathcal{O} = \frac{1}{2}\int_0^\infty \mathrm{d}s\,\mathcal{L}^{-1}[f](s)\,\mathrm{Tr}\,\mathrm{e}^{-s\mathcal{O}}$$

in which $\mathcal{L}^{-1}[f](s)$ is the inverse Laplace transform of the function $f(x) = \log x$.

Hints for part 2:
Given a function $f(x)$, the relation with its inverse Laplace transform is

$$f(x) = \int_0^\infty \mathrm{d}s\,\mathcal{L}^{-1}[f](s)\mathrm{e}^{-sx}$$

Part 3:
Show that the inverse Laplace transform of the logarithm is

$$\mathcal{L}^{-1}[\log](s) = -\frac{1}{s}$$

and therefore

$$\frac{1}{2}\,\mathrm{Tr}\log\mathcal{O} = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}s}{s}\;\mathrm{Tr}\,\mathrm{e}^{-s\mathcal{O}}$$

Hints for part 3:
The relation of the inverse Laplace transform with the original function is

$$\mathcal{L}^{-1}[f](s) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{d}z\, f(z)\mathrm{e}^{sz}$$

with $\gamma \in \mathbb{R}$ such that all the poles z_i of $f(z)$ lie on the left part of the complex plane with respect to γ: $\gamma > Re(z_i)\ \forall\, i$. You might want to use the following property of the inverse Laplace transform:

$$\mathcal{L}^{-1}[f'](s) = -s\mathcal{L}^{-1}[f](s)\,,$$

with f' the derivative of f.

Part 4:
Use the momentum space representation of the operator

$$\mathrm{e}^{-s\mathcal{O}}\,|p\rangle = \mathrm{e}^{-s(p^2+m^2)}\,|p\rangle$$

for a normalized state $|p\rangle$, and of the trace

$$\mathrm{Tr}(...) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\langle p|\,(...)\,|p\rangle$$

to find an explicit formula for

$$\frac{1}{2}\,\mathrm{Tr}\log\mathcal{O}$$

(integrate first in p and then in s). What are the assumptions that you have to make on d and m^2 for the integral to be convergent? When do you have to make these assumptions? What happens if $m^2 = 0$?

Hints for part 4:
Recall the integral form of the Euler Gamma function:

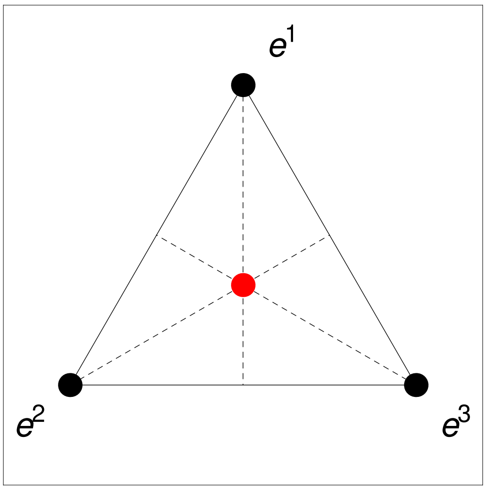
$$\Gamma(z+1) = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t}\,t^{z-1}\,,$$

which is convergent for $z > 0$.

Exercise 5: Three states Potts model

Consider the vertices e_a^α for $a \in \{1,2\}$ and $\alpha \in \{1,2,3\}$ of the regular 2-simplex often known as "equilateral triangle"

$$e^1 = \{0, 1\} \qquad e^2 = \{-\sqrt{3}/2, -1/2\} \qquad e^3 = \{\sqrt{3}/2, -1/2\} \tag{1}$$



(Notice that we are using a normalization which is different from the notes.)

Introduce two scalar fields $\phi_a(x)$ and combine them in the field $\psi^\alpha = \sum_{a=1,2} e_a^\alpha \phi_a$. We define an action for the 3-states Potts model

$$S[\phi] = \int \mathrm{d}^d x \sum_{\alpha=1}^{3} \Big\{ \frac{1}{3} \partial_\mu \psi^\alpha \partial^\mu \psi^\alpha + g (\psi^\alpha)^3 \Big\} \tag{2}$$

The model is invariant under the group S_3 of the permutations of three objects: a permutation $p \in S_3$ acts on the vertices of the triangle as

$$p : \{e^1, e^2, e^3\} \to \{e^{p(1)}, e^{p(2)}, e^{p(3)}\}$$

Geometrically, the permutations of three elements correspond to the transformations that leave the triangle invariant. They are

- the identity transformation: $\{1,2,3\} \to \{1,2,3\}$,
- two rotational symmetries: $\{1,2,3\} \to \{2,3,1\}$ and $\{1,2,3\} \to \{3,1,2\}$ (for 120° and 240°),
- and three reflections: $\{1,2,3\} \to \{1,3,2\}$, $\{1,2,3\} \to \{3,2,1\}$ and $\{1,2,3\} \to \{2,1,3\}$.

Part 1:
Give an explicit expression of $S[\phi]$ in terms of ϕ_1 and ϕ_2. In particular show that the potential becomes

$$V(\phi_1, \phi_2) = \frac{3}{4}(\phi_2)^3 - \frac{9}{4}(\phi_1)^2\phi_2 \tag{3}$$

Part 2:
Consider the rotations with angles $\omega = \pm\frac{2\pi}{3}$ in the (ϕ_1, ϕ_2)-plane given by the matrices

$$R(2\pi/3) = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} \qquad R(-2\pi/3) = R(2\pi/3)^T = R(2\pi/3)^{-1} \tag{4}$$

The rotations act on the fields as

$$\phi_a \to \phi'_a = R(\omega)_a{}^b\,\phi_b \tag{5}$$

Show that the potential $V(\phi_1, \phi_2)$ is invariant under those two rotations, $V(\phi_1, \phi_2) = V(\phi'_1, \phi'_2)$.

Part 3:
Why is the potential invariant under two rotations? How many more nontrivial symmetries do you expect the potential to have?

(Optional but very easy!) Part 4:
Consider a generic rotation $R(\alpha)$ of an angle α, and the reflection along the ϕ_2 axis

$$\mathcal{I} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

Can you write all transformations that leave the potential invariant as products of one specific rotation and the reflection $\mathcal{I}$? If yes, which angle α did you find convenient to use and why? *There are multiple answers and you should be able to solve this graphically.*

A bit more theory.
If instead we consider the action with quartic interaction there are in general two couplings

$$S[\phi] = \int \mathrm{d}^d x \Big\{ \frac{1}{3}\sum_{\alpha=1}^{3} \partial_\mu\psi^\alpha\partial^\mu\psi^\alpha + \lambda_1 \sum_{\alpha,\beta=1}^{3} (\psi^\alpha)^2(\psi^\beta)^2 + \lambda_2 \sum_{\alpha=1}^{3} (\psi^\alpha)^4 \Big\} \tag{7}$$

but in the case $q = 3$ the potential is a function of only the combination $\lambda = \frac{9}{4}\lambda_1 + \frac{9}{8}\lambda_2$. The potential becomes

$$V(\phi_1, \phi_2) = \lambda\left((\phi_1)^2 + (\phi_2)^2\right) \tag{8}$$

In general the quartic interaction will have additional parity $\mathbb{Z}_2$ symmetries $\phi_i \to -\phi_i$ as compared to S_q, but, after direct inspection, in the case $q = 2$ the potential becomes a function of the norm of $\{\phi_1, \phi_2\}$ and therefore has symmetry enhanced to the full $O(2)$ group.

Exercise 6: Dimensionless variables and critical properties

Consider a scale k dependent potential $V_k(\phi)$ with renormalization group flow:

$$k\partial_k V_k(\phi) = k^d F\left(\frac{V''(\phi)}{Z_k k^2}\right) \tag{1}$$

Assume that the function F is nowhere singular. The above notation means that the potential is a function of ϕ that depends also on k: the derivative ∂_k only acts on the explicit dependence of V_k and not on the argument ϕ. In other words, the renormalization group flow of the potential $V_k(\phi)$ is *the logarithmic derivative with respect to k at fixed field ϕ.*

The scale dependent constant Z_k has renormalization group flow $k\partial_k Z_k = -\eta Z_k$ and it normalizes the kinetic term $\frac{Z_k}{2}(\partial\phi)^2$. Define the *dimensionless renormalized field*

$$\varphi \equiv k^{-d/2+1} Z_k^{1/2} \phi \tag{2}$$

and the dimensionless potential

$$v_k(\varphi) \equiv k^{-d} V_k(\phi) = k^{-d} V_k(k^{d/2-1} Z_k^{-1/2} \varphi) \tag{3}$$

The dimensionless renormalized field has canonically normalized kinetic term in units of the scale k.

Part 1:
Show that the renormalization group flow of the dimensionless potential (that is the logarithmic scale derivative at fixed φ) is

$$k\partial_k v_k(\varphi) = -dv(\varphi) + \frac{1}{2}(d-2+\eta)\varphi v_k'(\varphi) + F(v_k''(\varphi)) \tag{4}$$

Some definitions: the first few terms of the scale derivative depend only on the rescaling

$$-dv(\varphi) + \frac{1}{2}(d-2+\eta)\varphi v_k'(\varphi)$$

and generally are referred to as **scaling part** to distinguish it from $F(v_k''(\varphi))$ which carries the quantum or statistical effects.

Part 2:
Approximate the potential to two couplings $v_k(\varphi) = \frac{1}{2} g_2 \varphi^2 + \frac{1}{4!} g_4 \varphi^4$ and project the flow as

$$k\partial_k v_k(\varphi) = \frac{1}{2}\beta_{g_2}\varphi^2 + \frac{1}{4!}\beta_{g_4}\varphi^4 \tag{5}$$

Give explicit expressions for the beta functions β_{g_2} and β_{g_4} and identitfy their scaling parts.

(Optional) Part 3:
Take $\eta = B g_4^2$ and $F(x) = Ax^2$ for A and B two positive constants. Expand in $d = 4 - \epsilon$ and find the fixed points of $\beta_{g_2} = 0 = \beta_{g_4}$ and the eigenvalues of the stability matrix *at the leading order in* ϵ.

Does the result depend on A and B? Why?

Some comments that will be discussed during the standard or the exercise class:

Assuming in general that $\eta \sim g_4^2$, it is possible to construct the leading orders in the ϵ expansion of both fixed points and eigenvalues for a general function F, making the above result universal.

From $\beta_{g_4} = 0$ you can find $g_4 = \epsilon/(3F''(g_2))$, and substituting it in $\beta_{g_2} = 0$ you find $g_2 = \epsilon F'(g_2)/(6F''(g_2))$. Using the fact that both g_4 and g_2 are proportional to ϵ and the regularity of $F(x)$ in zero we get that to $\mathcal{O}(\epsilon)$ the nontrivial fixed point is

$$g_2^* = \frac{\epsilon}{6}\frac{F'(0)}{F''(0)} \qquad\qquad g_4^* = \frac{\epsilon}{3F''(0)}$$

The stability matrix at this fixed point becomes

$$\begin{bmatrix} -2 + \frac{\epsilon}{3} & \left(1 + \frac{\epsilon}{6}\right) F'(0) \\ 0 & \epsilon \end{bmatrix}$$

and the eigenvalues can be found trivially because it is a triangular matrix. The negative of these eigenvalues are (related to) the critical exponents.

Consider the eigenvector of the critical exponent $\theta = 2 - \frac{\epsilon}{3}$. In the above basis it is $\{1, 0\}$ which corresponds to the operator φ^2. This means that close to the fixed point we can deform

$$v(\varphi) = v^*(\varphi) + \left(\frac{k}{k_0}\right)^{\theta} \varphi^2$$

with k_0 an arbitrary reference mass scale.

Now identify $v^*(\varphi)$ with the critical point of a system: the reduced temperature is thus $\frac{T - T_c}{T_c} \sim (k/k_0)^{\theta}$. (We do not use the symbol t for the reduced temperature to not confuse it with the logarithm of the scale.)

Finally perform a rescaling of the system. The momentum scale transforms as $k \to \lambda \cdot k$, which implies that $\frac{T - T_c}{T_c} \sim \lambda^{\theta}$. Recalling that the exponent ν is related to the scaling of $\frac{T - T_c}{T_c}$ as $\frac{T - T_c}{T_c} \sim \lambda^{1/\nu}$ under the hyperscaling hypothesis, we deduce that $\nu = 1/\theta = 1/2 + \epsilon/12$.

Exercise 7: Vertex expansion

Consider the Wetterich equation

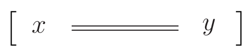
$$k\partial_k \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\,\mathcal{G}_k k\partial_k \mathcal{R}_k$$

in which we denoted the **(modified) propagator** as

$$\mathcal{G}_k = \left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}$$

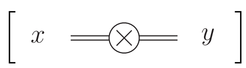
If you graphically represent the propagator as

$$\mathcal{G}_k(x,y) = \left(\frac{\delta^2\Gamma_k}{\delta\phi(x)\delta\phi(y)} + \mathcal{R}_k(x,y)\right)^{-1} =$$



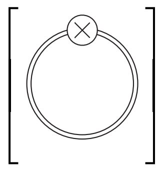
and the derivative of the cutoff as

$$k\partial_k \mathcal{R}_k(x,y) =$$



then the RG flow equation has the following representation

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2}$$



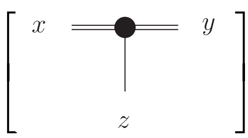
in which a closed loop means that we are taking the trace.

We want to construct the **vertex expansion** by acting with functional derivatives on the RG flow of the effective average action. We need to know how a derivative acts on the propagator

$$\frac{\delta}{\delta\phi(z)}\mathcal{G}_k(x,y) = -\int \mathrm{d}^d z_1 \mathrm{d}^d z_2\, \mathcal{G}_k(x,z_1)\,\frac{\delta^3\Gamma_k}{\delta\phi(z_1)\delta\phi(z_2)\delta\phi(z)}\,\mathcal{G}_k(z_2,y)$$

Graphically the action becomes very simple

$$\frac{\delta}{\delta\phi(z)}\mathcal{G}_k(x,y) = -$$



in which we denoted with a black dot the vertex coming from the derivatives of $\Gamma_k[\phi]$. We distinguish internal from external lines by denoting the former with a double line and the latter with a standard line, so we remember to not attach propagators to the external ones.

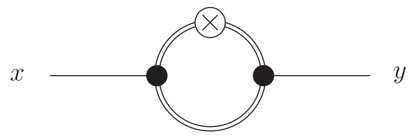
Part 1:
Give the graphical representation of the first derivative of the flow

$$\frac{\delta}{\delta\phi(x)} k\partial_k \Gamma_k[\phi]$$

This is the flow of the one-point function (which is the first vertex of the theory).

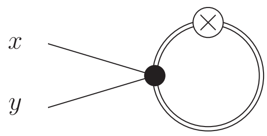
Part 2:
Show (graphically) that the following representation is correct

$$\frac{\delta^2}{\delta\phi(x)\delta\phi(y)} k\partial_k \Gamma_k[\phi] = \frac{1}{2}\left[\vphantom{\frac{1}{2}}\right.$$



$$\left.\vphantom{\frac{1}{2}}\right] + (x \leftrightarrow y)$$

$$-\frac{1}{2}\left[\vphantom{\frac{1}{2}}\right.$$



$$\left.\vphantom{\frac{1}{2}}\right]$$

in which $(x \leftrightarrow y)$ repeats the preceeding term with an exchange of x and y. Assuming full symmetry in the exchange of the two coordinates $\{x, y\}$ how many "topologically" different diagrams do you have?

(Optional) Part 3:
Assuming full symmetry under the exchange of the three coordinates $\{x, y, z\}$, give a graphical representation of the third derivative of the flow

$$\frac{\delta^3}{\delta\phi(x)\delta\phi(y)\delta\phi(z)} k\partial_k \Gamma_k[\phi]$$

(Optional) Part 4:
Assuming full symmetry under the exchange of the four coordinates $\{x, y, z, w\}$, give a graphical representation of the fourth derivative of the flow

$$\frac{\delta^4}{\delta\phi(x)\delta\phi(y)\delta\phi(z)\delta\phi(w)} k\partial_k \Gamma_k[\phi]$$

Exercise 8: Projection of the vertex expansion

In this exercise we want to evaluate the vertex expansion of Exercise 7 for an action truncated to the local potential approximation

$$\Gamma_k[\phi] = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial\phi)^2 + V_k(\phi) \right\}$$

We want to evaluate the renormalization group flow of each vertex at constant field ϕ in momentum space. You can begin to familiarize with some steps of the procedure:

- The propagator is a function of one momentum conjugate to the difference of the coordinates
$$\mathcal{G}_k(x,y) = \int_q G_k\left(q^2\right) \, \mathrm{e}^{iq\cdot(x-y)}$$
We choose the Fourier transform to be a function of q^2 using rotational and translational invariance.
- The cutoff and its derivative are also functions of q^2 for the same reasons
$$\mathcal{R}_k(x,y) = \int_q R_k\left(q^2\right) \, \mathrm{e}^{iq\cdot(x-y)}\,, \qquad k\partial_k\mathcal{R}_k(x,y) = \int_q k\partial_k R_k\left(q^2\right) \, \mathrm{e}^{iq\cdot(x-y)}$$
The functions $G_k(q^2)$ and $R_k(q^2)$ are obviously related
$$G_k(q^2) = \left(q^2 + V_k''(\phi) + R_k(q^2)\right)^{-1}$$
- The vertices with three or more legs become local because the potential is local. For example
$$\frac{\delta^3\Gamma_k[\phi]}{\delta\phi_{p_1}\delta\phi_{p_2}\delta\phi_{p_3}} = V_k^{(3)}(\phi)\,(2\pi)^d\,\delta(p_1+p_2+p_3)$$

Part 1:
The evaluation of the zero point function gives

$$k\partial_k\Gamma[\phi]|_{\phi=\text{const.}} = \frac{1}{2}\int \mathrm{d}^d x \int_q G_k\left(q^2\right) k\partial_k R_k\left(q^2\right)$$

Evaluate the right hand side of the above formula for the cutoff $R_k\left(q^2\right) = \left(k^2-q^2\right)\theta\left(k^2-q^2\right)$ with $\theta(x)$ the Heaviside theta function. Deduce the flow of $V_k(\phi)$.

Hints:
The momentum integral is $\int_q \equiv \int \frac{\mathrm{d}^d q}{(2\pi)^d}$ and can immediately be changed to polar coordinates. Furthermore, you might find the following property useful

$$f\left(A\theta(x-x_0) + B\theta(x_0-x)\right) = f(A)\theta(x-x_0) + f(B)\theta(x_0-x)$$

(Heaviside thetas are projectors over the space of functions.)

Part 2:
The evaluation of the two point function gives

$$k\partial_k \frac{\delta^2}{\delta\phi_p\delta\phi_{-p}}\Gamma[\phi]\bigg|_{\phi=\text{const.}} = V^{(3)}(\phi)^2 \int_q G_k\left((q+p)^2\right) G_k\left(q^2\right)^2 k\partial_k R_k\left(q^2\right) \\ -\frac{1}{2}V^{(4)}(\phi)\int_q G_k\left(q^2\right)^2 k\partial_k R_k\left(q^2\right)$$

Show that upon Taylor expanding the above formula in p_μ, the coefficient of p^2 of the right hand side of the above formula is

$$V^{(3)}(\phi)^2 \int_q \left(G_k'\left(q^2\right) + q^2\frac{2}{d}G_k''\left(q^2\right)\right) G_k\left(q^2\right)^2 k\partial_k R_k\left(q^2\right)$$

Hints:
Use the following properties that can be proved through invariance under rotational symmetry

$$\int_q q_\mu f(q^2) = 0\,, \qquad \int_q q_\mu q_\nu f(q^2) = \frac{1}{d} g_{\mu\nu}\int_q q^2 f(q^2)$$

(Optional) Part 3:
Evaluate

$$V^{(3)}(\phi)^2 \int_q \left(G_k'\left(q^2\right) + q^2\frac{2}{d}G_k''\left(q^2\right)\right) G_k\left(q^2\right)^2 k\partial_k R_k\left(q^2\right)$$

for the cutoff $R_k\left(q^2\right) = \left(k^2-q^2\right)\theta\left(k^2-q^2\right)$.

Hints:
The derivative of the Heaviside function is the Dirac delta and they satisfy the following properties

$$\int \mathrm{d}x\,\theta(x-x_0)\theta(x_0-x)f(x) = 0\,, \qquad \int \mathrm{d}x\,\theta(x-x_0)\delta(x-x_0)f(x) = \frac{1}{2}f(x_0)$$

(Delta and Heaviside functions are distributions: a distribution is defined as the limit of a sequences of functions in the space of all functions. The formula on the right assumes that the elements of limiting sequence for the Dirac delta are the derivatives of the elements of the limiting sequence of the Heviside theta.)

(Optional) Part 4:
We have not introduced a wavefunction normalization Z_k in the truncation to the local potential of $\Gamma_k[\phi]$, but it should be clear that it is generated by the flow. Assuming that we start from $Z_k = 1$, after one infinitesimal RG step it becomes

$$Z_{k+\delta k} = Z_k + \delta Z_k = Z_k + \frac{\partial Z_k}{\partial k}\delta k = Z_k\left(1-\eta\,\delta k\right) = 1-\eta\,\frac{\delta k}{k}$$

in which we used $\eta = -\partial \log Z_k/\partial \log k$. Can you estimate the anomalous dimension η from the result of Part 3? What value does it have for $\phi = 0$ if the potential is symmetric?

Exercise 9: Perturbative RG and the renormalized potential

In this exercise we want to give a functional representation based on a renormalized potential to our lecture's results based on perturbation theory for small coupling.

Recall that during the lecture we promoted the standard ϕ^4 Lagrangian in $d = 4$ dimensions

$$\mathcal{L}_4 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}g_2\phi^2 + \frac{1}{4!}g_4\phi^4$$

to a d-dimensional action by introducing a scale μ as

$$\mathcal{L}_d = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}g_2\phi^2 + \frac{1}{4!}\mu^{4-d}g_4\phi^4$$

The "promotion" was done in such a way that the field ϕ has always canonical dimension, but the couplings g_2 and g_4 have maintained the same dimension when $4 \to d$. The potential is for now restricted to be a polynomial of the fourth order.

Part 1:
Consider the generalization of $\mathcal{L}_4$ to a full functional potential

$$\mathcal{L}_4 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi)$$

The original Lagrangian can be recovered by expanding $V(\phi) = \frac{1}{2}g_2\phi^2 + \frac{1}{4!}g_4\phi^4$ and the couplings are the Taylor coefficients of this expansion. Find the generalization of the potential that is needed to promote the Lagrangian to $\mathcal{L}_d$ and leaves the dimension of the couplings invariant.

Hint:
You may try the ansatz

$$\mathcal{L}_d = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \mu^A V(\mu^B\phi)$$

and determine the unknown constants A and B by comparing with the coupling's Lagrangian.

After 1) dimensional regularization, 2) minimal subtraction of the $\frac{1}{\epsilon}$ poles and 3) an unimportant rescaling of factors of 4π, the beta function of the coupling g_4 at two loops in $d = 4$ is

$$\beta_{g_4} = 3g_4^2 - \frac{17}{3}g_4^3$$

and the anomalous dimension is

$$\eta = \frac{1}{6}g_4^2$$

Part 2:
Assume that in the functional representation the potential has beta function

$$\beta_V = C_1 \left(V''(\phi)\right)^2 + C_2 V''(\phi) \left(V'''(\phi)\right)^2 + \frac{1}{2}\eta\,\phi\,V'(\phi)$$

and the anomalous dimension η has the form

$$\eta = C_3 \left(V^{(4)}(0)\right)^2$$

Determine the constants C_1, C_2 and C_3.

Hint:
There is no need to compute them from diagrams if you substitute $V(\phi) = \frac{1}{4!}g_4\phi^4 + \ldots$ and interpret β_V as the generator of the beta functions $\beta_V = \frac{1}{4!}\beta_{g_4}\phi^4 + \ldots$. It is probably convenient to determine the constant C_3 of the anomalous dimension first.

Now we are interested in computing the critical exponents *below* four dimensions using the beta functions and the perturbative expansion of $d = 4$.

One trick to do this is to "trade" the expansion in g_4 for a $\varepsilon = 4 - d$ expansion. We use the fact that $V(\phi)$ and ϕ are canonically normalized in d dimensions and switch to a dimensionless potential $v(\varphi) = \mu^{-d}V(\mu^{d/2-1}\varphi)$. (Notice that the field ϕ is already renormalized and the contribution of η is already inside β_V above so there is no need to rescale by a further wavefunction.)

We are purposedly distinguishing between $d = 4 - \varepsilon$ (which parametrizes any dimension below the upper critical one for statistical field theory) and $d = 4 - \epsilon$ (which analytically continued the theory to make it finite). *After the new additional rescaling all couplings are dimensionless* and the beta function is

$$\begin{aligned}
\beta_v &= -dv(\varphi) + \frac{d-2}{2}\varphi v'(\varphi) + \beta_V|_{V\to v} \\
&= -dv(\varphi) + \frac{d-2+\eta}{2}\varphi v'(\varphi) + \frac{1}{2}\left(v''(\varphi)\right)^2 - \frac{1}{2}v''(\phi)\left(v'''(\phi)\right)^2 \\
\eta &= = \frac{1}{6}(v^{(4)}(0))^2
\end{aligned}$$

(This expression has a familiar scaling term that should remind you of another exercise. It should also suggest you the correct values of C_1, C_2 and C_3 as a check of part 2!) The fixed point solutions of $\beta_v = 0$ are of the form $v^*(\varphi) = \frac{1}{4!}\lambda_4^*(\varepsilon)\varphi^4$.

(Optional) Part 3:

- Find all the fixed points $\lambda_4^*(\varepsilon)$ up to order ε^2. How many are they and why?
- Which fixed point is the one that we need for the ε expansion and why?
- Find a simple way to determine the critical exponent η and ν to order ε^2.
(The results should be $\eta = \frac{\varepsilon^2}{54}$ and $\nu = \frac{1}{2} + \frac{\varepsilon}{12} + \frac{7\varepsilon^2}{162}$.)

Exercise 10: Heat kernel methods

Some theory first.
Consider the representation of the one loop effective action

$$\Gamma[\varphi] = \frac{1}{2}\,\mathrm{Tr}\log\left(\mathcal{O} + m^2\right)$$

with the operator $\mathcal{O} = -\partial^2 + \frac{g}{2}\varphi^2$. We also define the flat space Laplacian as $\Delta = -\partial^2$. We have already seen that the action can be represented as

$$\Gamma[\varphi] = -\frac{1}{2}\,\mathrm{Tr}\int \frac{\mathrm{d}s}{s}\,\mathrm{e}^{-sm^2}\,\mathcal{H}(s)$$

with $\mathcal{H}(s) \equiv \mathrm{e}^{-s\mathcal{O}}$ known as the *heat kernel* of the operator $\mathcal{O}$. The parameter s is often called *proper time*. (The mass is assumed here to be a number that commutes with everything.)

The trace can be performed by giving a representation to $\mathcal{H}(s)$. In coordinate space it is a matrix which is formally written as $\mathcal{H}(s;x,x') \equiv \langle x|\mathrm{e}^{-s\mathcal{O}}|x'\rangle$. In practice $\mathcal{H}(s;x,x')$ is the solution of the differential equation

$$\partial_s\mathcal{H}(s;x,x') + \mathcal{O}_x\mathcal{H}(s;x,x') = 0 \qquad\qquad \mathcal{H}(0;x,x') = \delta(x,x')$$

This function can be interpreted as the propagation to the location x and in the time s of some heat from a point source located at the point $x = x'$ and time $s = 0$ according to the operator $\mathcal{O}$. The heat kernel forms an Abelian one dimensional semigroup

$$\int \mathrm{d}^dx'\,\mathcal{H}(s_1;x,x')\mathcal{H}(s_2;x',x'') = \mathcal{H}(s_1+s_2;x,x'')$$

We define also the heat kernel $\mathcal{H}_0(s)$ of the simpler operator $\Delta = -\partial^2$ analogously

$$\partial_s\mathcal{H}_0(s;x,x') + \Delta_x\mathcal{H}_0(s;x,x') = 0 \qquad\qquad \mathcal{H}_0(0;x,x') = \delta(x,x')$$

This latter heat kernel is simpler because it can be computed easily in momentum space and becomes a Gaussian distribution

$$\tilde{\mathcal{H}}_0(s;p)\delta_{p,p'} = \langle p|\mathcal{H}_0(s)|p'\rangle = \mathrm{e}^{-sp^2}\delta_{p,p'}$$

in which we used momentum conservation to factor out a delta function in the definition of $\tilde{\mathcal{H}}_0(s;p)$. In coordinate space it is also a Gaussian after Fourier transforming

$$\mathcal{H}_0(s;x,x') = \frac{1}{(4\pi s)^{d/2}}\exp\left(-\frac{|x-x'|^2}{4s}\right)$$

You can see that Δ is covariant under Euclidean transformations (translations and rotations) because $\mathcal{H}_0(s;x,x')$ is just a function of $|x-x'|$.

Let $E = \frac{g}{2}\varphi^2$ be the *endomorphism* that distinguishes the two operators $\mathcal{O} = \Delta + E$. The endomorphism is a local function $E = E_x = \frac{g}{2}\varphi(x)^2$, so it is not a differential operator like Δ, and we indicate by $\tilde{E}_p$ its Fourier transform. We can write the heat kernel of $\mathcal{O}$ using the simpler $\mathcal{H}_0(s)$ thanks to the (proper time) ordered expansion in E

$$
\begin{aligned}
&\mathcal{H}(s;x,x') = \mathcal{H}_0(s;x,x') - s\int_y \int_0^1 \mathrm{d}t_1\, \mathcal{H}_0(s(1-t_1);x,y)E_y\mathcal{H}_0(st_1;y,x') \\
&\quad + s^2 \int_{y_1,y_2}\int_0^1 \mathrm{d}t_1 \int_0^{t_1}\mathrm{d}t_2\, \mathcal{H}_0(s(1-t_1);x,y_1)E_{y_1}\mathcal{H}_0(s(t_1-t_2);y_1,y_2)E_{y_2}\mathcal{H}_0(st_2;y_2,x') \\
&\quad + \dots
\end{aligned}
$$

in which spacetime integrations are condensed $\int_y \equiv \int \mathrm{d}^d y$. (We are going to show how to derive this expansion in class.) Inserting the above expression in the one of the effective action

$$
\Gamma[\varphi] = -\frac{1}{2}\int \frac{\mathrm{d}s}{s}\int \mathrm{d}^d x\, \mathrm{e}^{-sm^2}\,\mathcal{H}(s;x,x)
$$

and considering only the part quadratic in E we get

$$
\begin{aligned}
\Gamma|_{E^2} \quad = \quad & -\frac{1}{2}\int \mathrm{d}s\, s\, \mathrm{e}^{-sm^2}\int_{y_1,y_2}\int_0^1 \mathrm{d}t_1 \int_0^{t_1}\mathrm{d}t_2\, \mathcal{H}_0(s(1-t_1+t_2);y_2,y_1)E_{y_1}\times \\
& \times \mathcal{H}_0(s(t_1-t_2);y_1,y_2)E_{y_2}
\end{aligned}
$$

after using the semigroup property (and the cyclicity of the trace).

Part 1:
Compute the Fourier transform of

$$
\int_{y_1,y_2}\mathcal{H}_0(s(1-t_1+t_2);y_2,y_1)E_{y_1}\mathcal{H}_0(s(t_1-t_2);y_1,y_2)E_{y_2} = \int_{p,q}\tilde{E}_p\,\mathcal{A}(s;t_1,t_2;p,q)\,\tilde{E}_{-p}
$$

Hint:
Recall that $\tilde{E}_p$ is the Fourier transform of the endomorphism.

Part 2:
Define the variable $\xi = t_1 - t_2$ and show that

$$
(1-t_1+t_2)q^2 + (t_1-t_2)(q+p)^2 = (q+\xi p)^2 + \xi(1-\xi)p^2
$$

then insert this result in the amplitude $\mathcal{A}$ and perform the integration over q.

Hint:
After the manipulation of the exponent recall that the integration measure $\mathrm{d}^d q$ is invariant under translations.

Part 3:
Use the result of the previous point (keeping the dependence on the momentum p) to compute the integration over the proper time s inside $\Gamma|_{E^2}$.

Hint:
You can rewrite the integration over the two parameters t_1 and t_2 using the fact that:

$$\int_0^1 \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \; g(t_1 - t_2) = \frac{1}{2} \int_0^1 \mathrm{d}\xi \; g(\xi)\,,$$

for g a function such that $g(\xi) = g(1-\xi)$.

Part 4:
Expand $\Gamma|_{E^2}$ in $\epsilon = 4 - d$ (including the finite part at ϵ^0) and compute the integration over ξ.

Hint:
You can make use of the following primitive function

$$\begin{aligned} \int \mathrm{d}\xi \; \log\left(m^2 + \xi(1-\xi)p^2\right) \quad &= \quad -\sqrt{\frac{p^2 + 4m^2}{p^2}}\operatorname{arctanh}\left(\frac{2\xi - 1}{\sqrt{\frac{p^2+4m^2}{p^2}}}\right) \\ &\quad -\xi + \left(\xi - \frac{1}{2}\right)\log\left(m^2 + \xi(1-\xi)p^2\right) \end{aligned}$$

and recall that $\operatorname{arctanh}(-x) = -\operatorname{arctanh}(x)$.

Exercise 11: Form-factors and decoupling

In the last exercise we computed some local and non-local contributions to the effective action for an interacting scalar field theory. Using the notation for which $[\varphi^2]_p$ is the Fourier transform of the square of the field φ, we established that at order φ^4 the one loop effective action is

$$\Gamma|_{\varphi^4} = -\frac{g^2}{2\cdot 4(4\pi)^2\epsilon}\int_x \varphi^4 + \int_p [\varphi^2]_{-p} F\left(p,m;g\right)[\varphi^2]_p + \mathcal{O}(\epsilon)$$

in which we defined the **non-local form-factor** F as

$$F\left(p,m;g\right) = \frac{2g^2}{(4\pi)^2}\frac{m}{p}\sqrt{4+\frac{p^2}{m^2}}\,\mathrm{arctanh}\left(\frac{p}{m\sqrt{4+\frac{p^2}{m^2}}}\right) + \frac{g^2}{(4\pi)^2}\left[\gamma - 2 + \log\left(\frac{m^2}{4\pi}\right)\right]$$

The $\frac{1}{\epsilon}$ pole is the divergence that requires first regularization and then renormalization. The process of renormalization forces us to introduce a reference scale μ and results in a renormalization group beta function for the coupling

$$\beta_{\mathrm{MS}} = \frac{3g^2}{(4\pi)^2}$$

after minimal subtraction (MS) of the divergence as we have seen during the lectures. After subtraction the renormalized effective will contain only the finite part of $\Gamma|_{\varphi^4}$.

Through this exercise we want to understand if we can give a more physical intuition to the renormalization group in particle physics. First recall that the finite effective action is now

$$\Gamma|_{\varphi^4} = \int_p [\varphi^2]_{-p} F\left(p,m;g\right)[\varphi^2]_p$$

which can be related to the scattering of four scalar field's states by taking the appropriate number of functional derivatives. Qualitatively, we can imagine that the form-factor $F\left(p,m;g\right)$ is a **momentum-dependent coupling constant** $g(p) \equiv F\left(p,m;g\right)$ in which the relevant momentum scale $p = |p_\mu|$ is related to some scattering energy (and in particular to the variables s, t and u of the notes, even if we do not work out this relation explicitly here).

Having made this definition, we are naturally lead to interpret p as a renormalization group scale and $\beta = p\frac{\mathrm{d}}{\mathrm{d}p}g(p)$ as the (new) renormalization group running according to this scale. This running is certainly richer than the one of β_{MS} because of the explicit presence of the mass, below we want to see how much richer it is and what can we learn from it.

Part 1:
Define the variable $x = \frac{p^2}{m^2}$ and rewrite the form-factor of $\Gamma|_{\varphi^4}$ in terms of x. Rewrite the renormalization group operator $p\frac{\mathrm{d}}{\mathrm{d}p}$ in terms of the variable x as well.

Part 2:
Compute the beta function

$$\beta = 4!\, p \frac{\mathrm{d}}{\mathrm{d}p} F\,(p, m; g)$$

up to order g^2. Make sure that the final result is expressed in terms of x.

Hint: You better use the results of the previous point. Also recall that $\beta \sim g^2$ and $\beta_m \sim g$!

Part 3:
Expand the beta function to find the leading contributions in the asymptotic regimes $x \sim \infty$ and $x \sim 0$.

Hint: Use the following limits

$$\operatorname{arctanh}\left(\sqrt{\frac{x}{4+x}}\right) = \begin{cases} \dfrac{1}{x}, & \text{for} \quad x \sim \infty \\ \dfrac{\sqrt{x}}{2}, & \text{for} \quad x \sim 0 \end{cases}$$

Part 4:
Verify that

$$\begin{aligned} &\beta(x) = \beta_{\mathrm{MS}} && \text{for} \quad x \sim \infty \\ &\beta(x) \to 0 && \text{for} \quad x \to 0 \end{aligned}$$

It should be clear that the limits $x \sim \infty$ ($p^2 \gg m^2$) and $x \sim 0$ ($p^2 \ll m^2$), represent the UV and IR behaviors of the momentum-dependent coupling respectively. Once this connection is done then we understand that

- In the ultraviolet the new beta function β coincides with β_{MS}. It happens because the scale μ of dimensional regularization is a very high energy scale that is bigger, in particular, of any physical mass $\mu^2 \gg m^2$. In UV all the beta functions coincide because they are **universal** (here used with a slightly different meaning than in statistical mechanics).
- In the infrared the new beta function goes to zero. This is known as the **decoupling** of the field's fluctuations: below the field's mass quantum fluctuations stop propagating and cannot contribute to the quantum effects! The decoupling is predicted by the **Appelquist-Carazzone theorem** of QFT. It is very important to consider decoupling effects when studying the standard model of particle physics, because in different energy ranges there are different matter fields that contribute to the running!

Exercise 12: Coincidence limits

This is going to be the most tedious exercise so far. Do not get intimidated: it took me less than four A4 pages and I am very bad at computations!

Consider the crucial relations of the Synge's function $\sigma(x, x')$

$$\sigma_\mu \sigma^\mu = 2\sigma \tag{1}$$

and the (root of the) van Vleck's determinant $\Delta(x, x')^{1/2}$

$$\Delta^{1/2} {\sigma_\mu}^\mu + 2\sigma^\mu {\Delta^{1/2}}_\mu = d\Delta^{1/2} \tag{2}$$

Consider also the equation satisfied by the second heat kernel coefficient $a_1(x, x')$

$$a_1 + \sigma^\mu \nabla_\mu a_1 + \Delta^{-1/2} \mathcal{O}_x(\Delta^{1/2} a_0) = a_1 + \sigma^\mu \nabla_\mu a_1 - \Delta^{-1/2} \nabla^2 (\Delta^{1/2} a_0) + E a_0 = 0 \tag{3}$$

We denote coincidence limits of bi-tensors as

$$[B] = B(x, x)$$

The first few coincidence limits to begin with are

$$[\sigma] = 0 \qquad\qquad [\Delta^{1/2}] = 1 \qquad\qquad [a_0] = 1$$

It is convenient to use a notation for which covariant derivatives are denoted as indices

$$\nabla_{\mu_n} \dots \nabla_{\mu_2} \nabla_{\mu_1} \sigma = \sigma_{\mu_1 \mu_2 \dots \mu_n}$$

Notice that the new notation reverses the order of the indices!

Part 1:
Use (1) to show that

$$[\sigma_\mu] = 0$$

Part 2:
Take two covariant derivatives of (1) and use the resulting equation to show that

$$[\sigma_{\mu\nu}] = g_{\mu\nu}$$

Hint: The metric is the only tensor for which $g_{\mu\nu}\, {g^\nu}_\rho = g_{\mu\rho}$.

Part 3:
Take three covariant derivatives of (1) and use the resulting equation to show that

$$[\sigma_{\mu\nu\rho}] = 0$$

Hint: It is useful to arrange all expressions of $[\sigma_{\mu\nu\rho}]$ with the indices in the same order. In doing so a useful manipulation is

$$\sigma_{\mu\rho\nu} = \nabla_\nu \nabla_\rho \, \sigma_\mu = [\nabla_\nu, \nabla_\rho] \, \sigma_\mu + \nabla_\rho \nabla_\nu \, \sigma_\mu = -R_{\nu\rho}{}^{\theta}{}_{\mu} \sigma_\theta + \sigma_{\mu\nu\rho}$$

Notice also that curvatures are standard tensors, so in the coincidence limit they factor out

$$\left[R_{\nu\rho}{}^{\theta}{}_{\mu} \, \sigma_\theta\right] = R_{\nu\rho}{}^{\theta}{}_{\mu}[\sigma_\theta] = 0$$

Part 4:
Take four covariant derivatives of (1) and use the resulting equation to show that

$$[\sigma_{\mu\nu\rho\theta}] = -\frac{1}{3} \left(R_{\mu\rho\nu\theta} + R_{\mu\theta\nu\rho}\right)$$

Hint: Sort all covariant derivatives as in the previous exercise. Furthermore, some standard symmetries of the Riemann tensor could be useful: $R_{\mu\rho\nu\theta} = -R_{\rho\mu\nu\theta}$, $R_{\mu\rho\nu\theta} = R_{\nu\theta\mu\rho}$, and $R_{\mu\rho\nu\theta} + R_{\mu\nu\theta\rho} + R_{\mu\theta\rho\nu} = 0$.

Part 5:
Take one covariant derivative of (2) and use the resulting equation to show that

$$[\Delta^{1/2}{}_{\mu}] = 0$$

Hint: Consider $\Delta(x, x')^{1/2}$ instead of $\Delta(x, x')$ as the relevant tensor.

Part 6:
Take two covariant derivatives of (2) and use the resulting equation to show that

$$[\Delta^{1/2}{}_{\mu\nu}] = \frac{R_{\mu\nu}}{6}$$

Part 7:
Take the coincidence limit of (3) and show that

$$[a_1] = \frac{R}{6} - E$$
