THEORETISCH-PHYSIKALISCHES INSTITUT
FRIEDRICH-SCHILLER UNIVERSITÄT JENA

Particles and Fields

Winter term 2018/19

Examples sheet 9

Exercise 17

In the following we construct the plane wave solution for a free massive Dirac fermion in the chiral basis, which is given by the following choice of Dirac matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (-\mathbb{1}, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu = (-\mathbb{1}, -\boldsymbol{\sigma})$ and $\boldsymbol{\sigma}$ is a vector consisting of the Pauli matrices.

(i) Show that the ansatz

$$\Psi = u(\boldsymbol{p})\, e^{ip\cdot x} + v(\boldsymbol{p})\, e^{-ip\cdot x}, \tag{1}$$

with $p \cdot x = p_\mu x^\mu$ solves the Dirac equation provided that

$$u(\boldsymbol{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}, \qquad v(\boldsymbol{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix} \tag{2}$$

and $p_0 = \omega_{\boldsymbol{p}} \equiv \sqrt{\boldsymbol{p}^2 + m^2}$. ξ and η are two normalized, linealry independent two-component spinors, which may be choosen to be orthogonal, e.g. $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(ii) Specialize your solution to the case $p^\mu = (E, 0, 0, p)$, i.e. a Dirac fermion propagating in x_3 direction, and consider the massless limit $m \to 0$.

Exercise 18

In the following we consider two Lagrangian densities for massive spin-1 fields.

(a) Determine the energy density

$$\mathcal{H} = \frac{\partial \mathcal{L}_1}{\partial \dot{A}_\mu}\dot{A}_\mu - \mathcal{L}_1$$

using the Lagrangian density

$$\mathcal{L}_1 = -\frac{1}{2}\partial_\mu A_\nu\, \partial^\mu A^\nu - \frac{1}{2}m^2 A_\mu A^\mu. \tag{3}$$

Also compute the energy $E = \int d^{d-1}\boldsymbol{x}\mathcal{H}$. In particular show that the energy density and the energy are not bounded from below.

(b) In the following we consider

$$\mathcal{L}_2 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu, \qquad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4}$$

Compute the equations of motion and determine the energy density. While the energy density may still be negative, show that the total energy is always positive (Hint: rewrite negative terms in the energy density as boudary terms plus equations of motion).