Theoretisch-Physikalisches Institut
Friedrich-Schiller Universität Jena

Particles and Fields

Winter term 2018/19

Examples sheet 7

Exercise 13

Show that

$$\mathcal{J}^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$$

is a representation of the Lorentz algebra

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = i\left(\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\rho}J^{\mu\sigma} + \eta^{\nu\sigma}J^{\mu\rho} - \eta^{\mu\sigma}J^{\nu\rho}\right)$$

if the Dirac matrices γ^μ satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathbb{1}.$$

Exercise 14

We consider two different sets of Dirac matrices for a four-dimensional spacetime in the following. The chiral basis is given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (-\mathbb{1}, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu = (-\mathbb{1}, -\boldsymbol{\sigma})$ and $\boldsymbol{\sigma}$ is a vector consisting of the Pauli matrices. The Majorana basis is given by

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}.$$

(i) Show that both sets of Dirac matrices, the Majorana and the chiral basis satisfy the defining relation of the Clifford algebra.

(ii) Compute the matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in both basis. In particular, for the chiral basis find the projection operators $\mathcal{P}_\pm = \frac{1}{2}\left(\mathbb{1} \pm \gamma_5\right)$ and determine the set of Ψ_L and Ψ_R satisfying $\mathcal{P}_+\Psi_L = \Psi_L$ and $\mathcal{P}_-\Psi_R = \Psi_R$, respectively. Ψ_L and Ψ_R are called left- and righthanded Weyl spinors. What do they look like in the Majorana basis?

(iii) The charge conjugation matrix $\mathcal{C}$ is defined as

$$\mathcal{C}^\dagger\gamma^\mu\mathcal{C} = -\left(\gamma^\mu\right)^\star, \qquad \mathcal{C}^\dagger\mathcal{C} = \mathbb{1}.$$

Show that in the chiral basis $\mathcal{C} = i\gamma^2$ satisfies the conditions stated above, while in the Majorana basis the charge conjugation matrix is given by $\mathcal{C} = \mathbb{1}$.

(iv) Majorana spinors[1] are defined by the condition $\Psi^{(c)} = \Psi$, where $\Psi^{(c)}$ is the Charge conjugated spinor $\Psi^{(c)} = \mathcal{C}\Psi^\star$. Show that the Majorana spinors are real within the Majorana basis. What do Majorana spinors look like in the chiral basis?

[1] Do not mix up the technical terms Majorana spinors and Majorana basis. Majorana spinors may be defined for any set of Dirac matrices in four dimensional Minkowski spacetime. However, within the Majorana basis, the spinors take a very simple form.