Theoretisch-Physikalisches Institut
Friedrich-Schiller Universität Jena

Particles and Fields

Winter term 2018/19

Examples sheet 6

Exercise 11

The generators $J_{\rho\sigma}$ of the Lorentz algebra are specified by

$$(J_{\rho\sigma})^{\kappa}{}_{\delta} = i\left(\eta_{\rho\delta}\delta^{\kappa}_{\sigma} - \eta_{\sigma\delta}\delta^{\kappa}_{\rho}\right)\,. \tag{1}$$

Check that the generators $J_{\mu\nu}$ and $J_{\rho\sigma}$ satisfy the following commutation relations

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\left(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}\right)\,. \tag{2}$$

Exercise 12

We consider a N-component field $\phi(x) = (\phi^1(x), \phi^2(x), \ldots, \phi^N(x))$ with components $\phi^a(x)$. Under the Lorentz transformation $x \mapsto \tilde{x} = \Lambda\, x$, the field components $\phi^a(x)$ transform as $\phi^a(x) \mapsto \tilde{\phi}^a(x)$, where

$$\tilde{\phi}^a(x) = D(\Lambda)^a{}_b\, \phi^b(\Lambda^{-1}x)\,. \tag{3}$$

(a) Proof the following identity

$$D(\Lambda)_a{}^b = D(\Lambda^{-1})^b{}_a \tag{4}$$

by using the defining property of the Lorentz group as well as $D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2)$.

(b) Provided that $\phi_a(x)$ transforms as

$$\tilde{\phi}_a(x) = D(\Lambda)_a{}^b\, \phi_b(\Lambda^{-1}x) \tag{5}$$

show that $\tilde{\phi}_a(x)\,\tilde{\phi}^a(x)$ and $\eta^{\mu\nu}\partial_\mu\tilde{\phi}_a(x)\,\partial_\nu\tilde{\phi}^a(x)$ transform as Lorentz scalars. How do possible Lagrangians for free field theories look like?