Theoretisch-Physikalisches Institut
Friedrich-Schiller Universität Jena

Particles and Fields

Winter term 2018/19

Examples sheet 2

Exercise 3: Complex scalar fields

Consider two non-interacting real scalar fields ϕ_1 and ϕ_2 with common mass m and interacting via the potential $\mathcal{V}_{int}$ which depends only on $1/2(\phi_1^2+\phi_2^2)$. Hence, the Lagrangian of that theory reads

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_1(x)\partial_\nu\phi_1(x) - \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi_2(x)\partial_\nu\phi_2(x) - \frac{1}{2}m^2(\phi_1(x)^2+\phi_2(x)^2) - \mathcal{V}_{\rm int}\left[\frac{1}{2}(\phi_1(x)^2+\phi_2(x)^2)\right]. \qquad (1)$$

(i) Show that the Lagrangian can be written in terms of the complex scalar field ϕ

$$\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x)+i\phi_2(x)\right)$$

and its complex conjugate, $\phi^\star$.

(ii) Derive the equations of motion for ϕ and $\phi^\star$ assuming that ϕ and $\phi^\star$ are independent fields. Are the equations of motion consistent with the ones for ϕ_1 and ϕ_2?

(iii) Use the Lagrangian derived in part (i) to determine the Hamiltonian density for the complex scalar field ϕ! Does it agree with the Hamiltonian density derived from Lagrangian (1)?

Exercise 4

(i) Given a functional $A[\phi_a,\Pi_a,t]$ of the form

$$A[\phi_a,\Pi_a,t] = \int d^{d-1}\boldsymbol{x}\mathcal{A}(\phi_a(x),\partial_i\phi_a(x),\Pi_a(x),x)\,, \qquad (2)$$

show that

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial A}{\partial t} + \{A,H\}_{P.B.} \qquad (3)$$

holds.

Note: Assume that the field vanishes at the boundary of the spatial manifold. In particular, neglect the boundary terms when integrating by parts.

(ii) Show that the continuity equation

$$\frac{\partial\mathcal{H}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{S} = 0$$

holds if the Hamiltonian density $\mathcal{H}$ (or equivalently, the Lagrangian densitiy) is explicitly time independent. Here, $\boldsymbol{S}$ is the energy flux whose components are given by

$$S_i = \frac{\partial\mathcal{L}}{\partial(\partial_i\phi_a)}\frac{\partial\phi_a}{\partial t}\,.$$