String geometry and spin geometry on loop spaces

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Conference “Classical and quantum symmetries in mathematics and physics”, July 2016, Jena

Parallel section “Mathematical aspects of string theory and string geometry”
String geometry and spin geometry on loop spaces
— two approaches to anomaly cancellation in supersymmetric sigma models

1.) Anomalies in supersymmetric sigma models

2.) String geometry

3.) Spin structures on loop spaces

4.) Transgression – from string geometry to spin geometry
The supersymmetric sigma model:

- target space: Riemannian manifold $M$.
- world sheet: Riemann surface $\Sigma$ with a spin structure $\mathcal{S}$.
- world sheet embeddings:
  \[ \phi \in C^\infty(\Sigma, M) \]
- spinors:
  \[ \psi \in L^2(\Sigma, \mathcal{S} \otimes \phi^* TM) \]
Origin of the anomaly: give sense to the fermionic path integral

\[ A(\phi) = \int D\psi \exp \left( \int_\Sigma \langle \psi, \mathcal{D}_\phi \psi \rangle d\text{vol}_\Sigma \right). \]

Well-known solution: \( A(\phi) \) is a well-defined element in a Pfaffian line bundle:

\[ A(\phi) \in \text{Pfaff}(\mathcal{D}) \]

\[ \phi \in C^\infty(\Sigma, M) \]

Integrand of the bosonic path integral is not a function, but a section in a complex line bundle – anomaly!
Anomalies of this kind are treated by the Green-Schwarz anomaly mechanism:

1.) Make sure that the line bundle is trivializable.
2.) Provide, for all worldsheets $\Sigma$, a trivialization.

By the formula

"Section – trivialization = smooth function"

the integrand of the path integral becomes a smooth function.
Step 1 – make sure that $\text{Pfaff}(\mathcal{D})$ is trivializable.

**Theorem (Freed ’86)**

*If $M$ is a spin manifold, then*

$$c_1(\text{Pfaff}(\mathcal{D})) = \int_{\Sigma} \text{ev}^*\left(\frac{1}{2}p_1(M)\right)$$

*where*

- $\text{ev} : C^\infty(\Sigma, M) \times \Sigma \to M$ is the evaluation map
- $\frac{1}{2}p_1(M) \in H^4(M, \mathbb{Z})$ is the first fractional Pontryagin class

**Sufficient condition:** $\frac{1}{2}p_1(M) = 0$.

Spin manifold that satisfy this condition are called **string manifolds**.
Step 2 of the Green-Schwarz mechanism:

– provide a trivialization of $Pfaff(\mathcal{D})$.

In order to provide such a trivialization consistently for all worldsheets $\Sigma$, two interesting geometric theories have been developed:

- Spin geometry on the loop space $LM := C^\infty(\Sigma, M)$
  Witten ’86, Killingback ’87, Alvarez et al. ’87,...

- String geometry on $M$
  Stolz-Teichner ’03, Sati-Schreiber-Stasheff ’09,...
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Main idea of string geometry:

\[ \frac{1}{2} p_1(M) \in H^4(M, \mathbb{Z}) \]

is the “level” of a Chern-Simons theory over \( M \): we need a notion of a “trivialization” of this Chern-Simons theory that

- exists if and only if \( M \) is a string manifold
- it induces a trivialization of \( Pfaff(D) \).

Stolz-Teichner proposed to use certain extended field theories, where a trivialization is a 2-dimensional twisted field theory. A precise definition of these notions is in process to be developed. Here we describe fields theories in terms of the gauge fields: connections on (higher) gerbes.
A short (and informal) reminder on \( n \)-gerbes and connections:

0-gerbes = \( S^1 \)-bundles

- open cover \( U_\alpha \) and transition functions \( g_{\alpha\beta} \)
- connections: local 1-forms \( A_\alpha \)
- classified by Chern class in \( H^2(X, \mathbb{Z}) \)

(1-)gerbes, a.k.a. “B-fields”

- open cover and transition line bundles + higher structure
- connections: local 2-forms + connections on transition bundles
- classified by “Dixmier-Douady class” in \( H^3(X, \mathbb{Z}) \)

2-gerbes

- open cover and transition gerbes + higher structure
- connections: local 3-forms, connections on transition gerbes,…
- classified by a nameless characteristic class in \( H^4(X, \mathbb{Z}) \)
Example 1 – Basic gerbe $G_{bas}$ over a compact, simple, connected simply-connected Lie group $G$.

(Meinrenken ’02, Gawędzki-Reis ’02)

- conjugation-invariant open sets $U_\alpha$ corresponding to open subsets of the Weyl alcove, with $\alpha$ the vertices of the alcove.

- $U_\alpha \cap U_\beta$ deformation retracts to the coadjoint orbit $\mathcal{O} \subseteq \mathfrak{g}^*$ through $\alpha - \beta$. This orbit is integrable: we pull back the prequantum line bundle with its Kostant connection.

- the Dixmier-Douady class is a generator of $H^3(G, \mathbb{Z}) = \mathbb{Z}$.

- the curvature is the bi-invariant closed 3-form $H$ corresponding to the trilinear form $\langle X, [Y, Z] \rangle$.

In terms of field theories, the basic gerbe $G_{bas}$ corresponds to the level $k = 1$ Wess-Zumino-Witten model on $G$. 
Example 2 – Chern-Simons 2-gerbe $\mathcal{CS}_M$ over a spin manifold
(Carey et al. ’05, KW ’07)

- Let $FM$ be the frame bundle of $M$, with its structure group reduced to $\text{Spin}(n)$. Use the projection $FM \rightarrow M$ as the “open cover”.
- The transition gerbe is the pullback of the basic gerbe $\mathcal{G}_{bas}$ along the “difference map” $\delta: FM \times_M FM \rightarrow \text{Spin}(n)$.
- The local 3-form of the connection is the Chern-Simons 3-form of the Levi-Civita-connection $A$,

\[ \langle A \wedge dA \rangle + \frac{2}{3} \langle A \wedge [A \wedge A] \rangle \in \Omega^3(FM). \]

- Its characteristic class is $\frac{1}{2}p_1(M) \in H^4(M, \mathbb{Z})$.

In terms of field theories, the Chern-Simons 2-gerbe $\mathcal{CS}_M$ corresponds to the Chern-Simons theory over $M$ with level $\frac{1}{2}p_1(M)$. 
Two more facts about \( n \)-gerbes:

- For every \( n \)-gerbe, there is a notion of a trivialization, such that trivializations exist if and only if the characteristic class vanishes.
- Moreover, if the \( n \)-gerbe is equipped with a connection, then there is a notion of connections on the trivialization (additional structure for \( n > 0 \)).

**Definition**

- a **string structure** on \( M \) is a trivialization of the Chern-Simons 2-gerbe \( CS_M \).
- a **string connection** is a connection on the string structure.
- a **geometric string structure** is the pair of a string structure and a string connection.
Result 1 – **Existence** of string connections

**Theorem (KW ’09)**

*Every string structure admits a string connection. Moreover, the set of string connections on a fixed string structure is affine.*

As a consequence, we obtain the following equivalences:

\[ \frac{1}{2} p_1(M) = 0 \iff M \text{ admits a string structure} \iff M \text{ admits a geometric string structure.} \]

Thus, geometric string structures complete Step 1 in the Green-Schwarz mechanism.
Result 2 – Anomaly cancellation

Theorem (Bunke ’10)

Every geometric string structure determines a trivialization of the line bundle Pfaff(\mathcal{D}).

This result is proved by performing a detailed analysis of the index theory of the Pfaffian line bundle. It is a remarkable line between higher-categorical geometry and classical analysis.

By the theorem, geometric string structures complete Step 2 in the anomaly cancellation mechanism.

In other words, the supersymmetric sigma model requires to fix a geometric string structure on its target space $M$. 
Result 3 – Classification of string structures

- Equivalence classes of string structures are parameterized by

\[ H^3(M, \mathbb{Z}) \cong \{ \text{Isomorphism classes of gerbes over } M \} \]

- Equivalence classes of geometric string structures are parameterized by the differential cohomology group

\[ \hat{H}^3(M, \mathbb{Z}) \cong \{ \text{Isomorphism classes of gerbes with connection over } M \} \]

In particular, 2-forms \( B \in \Omega^2(M) \) (connections on the trivial gerbe) act on the string connections. Under this action, the trivialization of \( Pfaff(\mathcal{D}) \) changes by

\[ \exp 2\pi i \int_\Sigma B. \]

In particular, the trivialization depends on the string connection.
Result 4 – The covariant derivative of a string connection

Every geometric string structure on $M$ determines a 3-form $K \in \Omega^3(M)$ with $dK = \frac{1}{2} \langle F_A \wedge F_A \rangle$.

The Pfaffian line bundle $Pfaff(\mathcal{D})$ comes equipped with the Bismut-Freed connection. The trivialization of $Pfaff(\mathcal{D})$ has covariant derivative

$$\int_{\Sigma} ev^* K \in \Omega^1(\mathcal{C}^\infty(\Sigma, M)).$$

In particular, the trivialization is not parallel.

Höhn-Stolz conjecture: if $\text{Ric}_g > 0$ and $K = 0$, then the Witten genus of $M$ vanishes in $tmf^{-n}(pt)$. 
Result 5 – The string 2-group

String structures can also be understood in terms of a (higher) reduction problem in non-abelian gerbes.

There is a central extension

\[ B \text{U}(1) \rightarrow \text{String}(n) \rightarrow \text{Spin}(n) \]

of Lie 2-groups, and one can try to “reduce” the frame bundle $FM$ to a non-abelian gerbe with structure 2-group $\text{String}(n)$.

**Theorem (KW-Nikolaus ’12)**

*The Chern-Simons 2-gerbe is the (higher) lifting gerbe of this reduction problem, i.e. there is a 1:1 correspondence between string structures and reductions of $FM$ to $\text{String}(n)$.*

An analogous understanding of string connections has not been developed so far.
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We come to the second approach to Step 2 of the Green-Schwarz mechanism

— spin geometry on the loop space $LM = C^\infty(\Sigma, M)$.

Motivation: a string in $M$ is a point in $LM$, and supersymmetric point-particles are well-understood and treated with spin geometry.

Why still interesting? Many aspects of string geometry are open:

- representation theory of the string 2-group
- the analog of the spinor bundle ("stringor bundle")
- the analog of the Dirac operator and its index ($tmf$-valued?)

Hope: “higher-categorical geometry” can benefit from the well-developed “classical geometry” of the loop space.
Main idea: the class

\[ \lambda := \int_{S^1} \text{ev}^* \left( \frac{1}{2} p_1(M) \right) \in H^3(LM, \mathbb{Z}) \]

is the analog of the 3rd Stiefel-Whitney class for the loop space, and can be treated like an obstruction against Spin\(^c\)(\(n\))-structures.

The frame bundle of \(LM\) is \(LFM\), which is a principal \(L\text{Spin}(n)\)-bundle over \(LM\).

**Theorem (Killingback '87; McLaughlin '92)**

\(\lambda\) vanishes if and only if the structure group of \(LFM\) lifts to the universal loop group extension

\[ 1 \to U(1) \to \widehat{L\text{Spin}}(n) \to L\text{Spin}(n) \to 1. \]
Definition

A spin structure on $LM$ is a lift of the structure group of $LFM$ from $L\text{Spin}(n)$ to $L\hat{\text{Spin}}(n)$.

The Levi-Civita connection $A$ on $M$ defines a “looped” connection on $LFM$. A corresponding lift of this connection is called a spin connection, and the pair of a spin structure and a spin connection is called geometric spin structure.

One can show (Manoharan ’02) that every spin structure admits a spin connection. Hence, we have an equivalence

$$\lambda = 0 \iff LM \text{ admits a geometric spin structure}$$

Two problems:

- $\frac{1}{2}p_1(M) = 0 \implies \lambda = 0$ but no equivalence (Pilch-Warner ’88)
- It is not clear how geometric spin structures provide trivializations of $\text{Pfaff}(\mathcal{D})$. 
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Theorem (Murray ’95)

For every lifting problem there exists a gerbe ("lifting gerbe") whose trivializations are precisely the possible lifts.

The lifting gerbe $S_{LM}$ for spin structures over $LM$ ("spin lifting gerbe") is the following:

- its "open cover" is $LFM \rightarrow LM$.
- its transition bundle is the pullback of $L\delta : LFM \times_{LM} LFM \rightarrow LSpin(n)$ along $L\delta : LFM \times_{LM} LFM \rightarrow LSpin(n)$
- A local connection 2-form can be constructed using the Levi-Civita connection $A$ and a twisted Higgs field (Gomi ’03).
Theorem

The spin lifting gerbe $S_{LM}$ is the transgression of the Chern-Simons 2-gerbe $CS_M$.

Main ingredients of the proof:

- transgression of gerbes to the loop space (Brylinski ’93) takes the basic gerbe $G_{bas}$ to the universal central extension; this gives coincidence of the transition bundles.
- transgression of the Chern-Simons 3-form is the local connection 2-form of the spin lifting gerbe (Coquereaux-Pilch ’98).

There is an induced functor on categories of trivializations:

\[
\begin{align*}
\{ \text{Geometric string structures on } M \} & \longrightarrow \{ \text{Geometric spin structures on } LM \} \\
\end{align*}
\]

Thus: string geometry transgresses to spin geometry on $LM$. 
The fact that the vanishing of $\frac{1}{2}p_1(M)$ is not equivalent to the vanishing of $\lambda$ corresponds to the fact that the functor

$$\left\{ \text{Geometric string structures on } M \right\} \longrightarrow \left\{ \text{Geometric spin structures on } LM \right\}$$

is neither injective nor surjective.

The problem can be traced back to the fact that Brylinski's transgression functor

$$Grb^\nabla(M) \longrightarrow Bun^\nabla(LM)$$

is neither injective nor surjective.

Solution: add additional structure on the loop space side in such a way that the transgression functor becomes an equivalence of categories.
We consider the following additional structures on a line bundle $P$ over $LM$:

- **loop fusion** – an associative rule

\[
P_{\gamma_1 \cup \gamma_2} \otimes P_{\gamma_2 \cup \gamma_3} \to P_{\gamma_1 \cup \gamma_3}
\]

relating the fibres of $P$ over the three loops obtained from the figure.

- **thin homotopy equivariance** – if two loops $\tau_1$ and $\tau_2$ are thin homotopic (homotopic via a rank-one-homotopy), then there are coherent maps $P_{\tau_1} \to P_{\tau_2}$ between the fibres of $P$.

- **superficial connections** – connections on $P$ whose parallel transport along a thin homotopy gives the thin homotopy equivariant structure.
These structure lead to new categories of line bundles over $LM$:

- $\mathcal{FusBun}^{th}(LM)$ – line bundles equipped with a fusion product and a thin homotopy equivariant structure.
- $\mathcal{FusBun}^{\nabla_s}(LM)$ – line bundles equipped with a fusion product and a superficial connection.

**Theorem (KW '10)**

There is a commutative diagram of categories and functors

\[
\begin{array}{ccc}
\mathcal{G}rb^\nabla(M) & \longleftrightarrow & \mathcal{FusBun}^{\nabla_s}(LM) \\
\downarrow & & \downarrow \\
\mathcal{G}rb(M) & \longleftrightarrow & \mathcal{FusBun}^{th}(LM)
\end{array}
\]

whose horizontal arrows are equivalences, and whose vertical arrows forget the connections (and only keep the induced thin homotopy equivariant structure).
A corresponding modification can be performed with spin structures and spin connections over loop spaces.

**Theorem (KW '14)**

There is a commutative diagram of categories and functors:

\[
\begin{array}{ccc}
\{ \text{Geometric string structures over } M \} & \xleftarrow{\sim} & \{ \text{Fusion spin structures with superficial and fusive spin connections} \} \\
\downarrow & & \downarrow \\
\{ \text{String structures over } M \} & \xleftarrow{\sim} & \{ \text{Thin fusion spin structures} \}
\end{array}
\]

whose horizontal arrows are equivalences.
Conclusions:

- **String geometry** provides new geometric structures suitable for the anomaly cancellation in supersymmetric sigma models.

- **Spin geometry** on loop spaces is a similar attempt using classical geometry on the loop space; however, it fails to correctly perform the cancellation mechanism.

- If spin geometry is coupled to loop fusion and thin homotopies, the two geometries become **equivalent**.

Thank you very much!
References

O. Alvarez, T. P. Killingback, M. Mangano, and P. Windey, “The Dirac-Ramond operator in string theory and loop space index theorems”.


U. Bunke, “String Structures and Trivialisations of a Pfaffian Line Bundle”.
[arxiv:0909.0846]

[arxiv:math/0410013]

D. S. Freed and G. W. Moore, “Setting the quantum integrand of M-theory”.
[arxiv:hep-th/0409135]

D. S. Freed, “Determinants, torsion, and strings”.


T. Nikolaus and K. Waldorf, “Lifting problems and transgression for non-abelian gerbes”.
[arxiv:1112.4702]

[arxiv:0801.3480]

S. Stolz and P. Teichner, “What is an elliptic object?”

K. Waldorf, “Multiplicative bundle gerbes with connection”.
[arxiv:0804.4835v4]

K. Waldorf, “Transgression to loop spaces and its inverse, III: Gerbes and thin fusion bundles”.
[arxiv:1109.0480]

K. Waldorf, “String connections and Chern-Simons theory”.
[arxiv:0906.0117]
K. Waldorf, “String geometry vs. spin geometry on loop spaces”.  
[arxiv:1403.5656]

E. Witten, “The index of the Dirac operator on loop space”.  