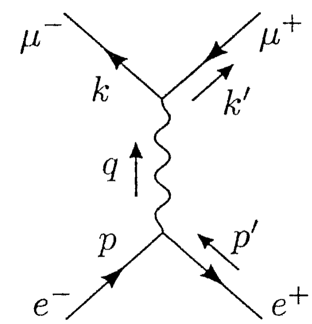
12. exercise sheet: Quantum Field Theory

Aufgabe 26:

Compute the differential cross section for $\mu^+\mu^-$ production in an e^+e^- collider to lowest order. (This can occur if the center of mass energy is sufficiently large. It is one of the most important cross sections as it serves as the "normalizer" for all other production cross sections at an e^+e^- collider).

(a) Use the QED Feynman rules (in Feynman gauge) to compute the diagram. Show that the squared scattering amplitude is given by



$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\Big(\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')\Big)\Big(\bar{u}(k)\gamma^\mu v(k')\bar{v}(k')\gamma^\nu u(k)\Big).$$

(b) Use the spin sums as well as the γ trace technology from the last sheets to show that the squared amplitude (neglecting the electron mass $m_e \simeq 0$) can be written as

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^2}{q^4}\Big[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')\Big],$$

where m_μ denotes the muon mass.

(c) For an e^+e^- collider, the lab frame is a center-of-mass (CM) frame. Choosing, e.g., $p = (E, E\hat{\mathbf{e}}_z), p = (E, -E\hat{\mathbf{e}}_z)$, and $k = (E, \mathbf{k})$, we must have $k' = (E, -\mathbf{k})$. For this situation, use the fact that the phase space integral of the general formula for the cross section simplifies to

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_p 2E_{p'}|v_p - v_{p'}|}\frac{|\mathbf{k}|}{(4\pi)^2 E_{\text{CM}}}\frac{1}{4}|\mathcal{M}|^2$$

to show that the cross section results in

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\text{CM}}}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right],$$

where $\alpha = \frac{e^2}{4\pi}$ denotes the fine-structure constant, θ is the scattering angle, i.e., $\mathbf{k}\cdot\hat{\mathbf{e}}_z = |\mathbf{k}|\cos\theta$, and $E_{\text{CM}} = 2E$.

(d) Determine, how the total cross section behaves for large energies $E, E_{\text{CM}} \gg m_\mu$.

Aufgabe 27:

Consider a "discretized free quantum field theory" with discrete field variables ϕ_i (the index $i = 1 \ldots n$ shall collectively represent further indices such as Lorentz or inner-symmetry indices as well as spacetime or momentum variables). Show that the following generating functional $Z[J]$ can be computed in closed form

$$\begin{aligned} Z[J] &= \int \prod_i d\phi_i \, \exp\left(-\frac{1}{2}\phi_i \, M^i{}_j \, \phi^j + J_i \phi^i\right) \\ &= (2\pi)^{n/2} (\det \mathbf{M})^{-1/2} \exp\left(\frac{1}{2} J_i \, (\mathbf{M}^{-1})^i{}_j \, J^j\right), \end{aligned} \tag{1}$$

where $(\mathbf{M})^i{}_j = M^i{}_j$ denotes a symmetric matrix. (Einstein's sum convention is used here and in the following.)

Hint:

Use the Gaußian integral

$$\int_{-\infty}^{\infty} dx \, \exp\left(-\frac{1}{2} a \, x^2\right) = \sqrt{\frac{2\pi}{a}} \tag{2}$$

as well as a formal diagonalization of the matrix $\mathbf{M}$ with eigenvalues λ_I and normalized eigenvectors v_I^i, where $I = 1 \ldots n$ labels the different eigenvalues and -vectors, to show that

$$\int \prod_i d\phi_i \, \exp\left(-\frac{1}{2}\phi_i \, M^i{}_j \, \phi^j\right) = (2\pi)^{n/2} (\det \mathbf{M})^{-1/2}. \tag{3}$$

Then study the full generating functional and use a completion of the squares.