9. exercise sheet: Quantum Field Theory

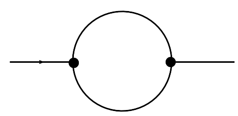
Aufgabe 20:

Consider ϕ^3 theory with Langrange density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2\phi^2 - \frac{g}{3!}\phi^3.$$

in D dimensional spacetime and study the 2-point correlator $G^{(2)} \equiv G$.

(a) Compute the 2-point correlator $G(p^2)$ to order g^2 in momentum space. Hint: let $G(p^2) = G^{[0]}(p^2) + G^{[2]}(p^2) + \ldots$ be the perturbative expansion of G where $G^{[0]}(p^2) = i\Delta_{\mathrm{F}}(p^2)$, and $G^{[2]}(p^2)$ denotes the nontrivial one-loop correction to order g^2. This correction corresponds to the diagram (cf. exercise 18):



(NB: there is another so-called *tadpole* diagram which you have also found in exercise 18; this tadpole only adds an irrelevant momentum independent contribution which can be disregarded in the following).

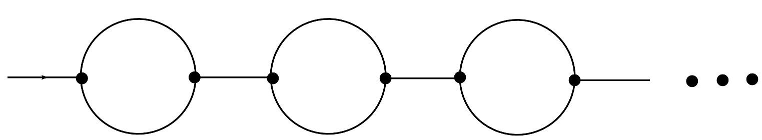
Write this one-loop correction as

$$G^{[2]}(p^2) = \frac{i}{p^2 - m_0^2 + i\epsilon}\big(-i\Sigma(p^2)\big)\frac{i}{p^2 - m_0^2 + i\epsilon},$$

and determine $\Sigma(p^2)$. Further technical hints can be found below.

(b) Show that Σ and accordingly G develop a branch cut for $p^2 > 4m_0^2$ (According to the Lehmann-Källen representation, this corresponds to scattering states with rest energy $> 2m_0$).

(c) In order to investigate the one-particle pole, a resummation of a class of higher loop-corrections is necessary. For this, consider the sum of all contributions to G which consists of chains of diagrams of the type $G^{[2]}$:



Show that a resummation of these diagrams yields the following form of the 2-point correlator:

$$G(p^2) = \frac{i}{p^2 - m_0^2 - \Sigma(p^2)}.$$

(d) Convince yourself that the physical mass m of the one-particle state is no longer given by m_0, but receives a correction which is determined by the (transcendental) equation

$$m^2 = m_0^2 + \Sigma(m^2).$$

Show that the wave function renormalization Z is given by

$$Z = \frac{1}{1 - \frac{\partial \Sigma(p^2=m^2)}{\partial p^2}}.$$

(e) Now consider a $D = 3$ dimensional spacetime and determine m^2 and Z in the limit $g^2/m_0^3 \ll 1$.

(f) What happens in the limit $D \to 4$? For this, consider $D = 4 - \epsilon$ and isolate potential divergencies by expanding the result about $\epsilon = 0$.

Further technical hints:
The technical difficulty consists in the evaluation of a D dimensional momentum space integral of a product of two Feynman-propagators, $\int \frac{d^D q}{(2\pi)^D} \Delta_{\mathrm{F}}(q)\Delta_{\mathrm{F}}(p-q)$. There are several techniques to deal with this. One possibility is to introduce the *propertime* representation for the propagators,

$$\frac{1}{A + i\epsilon} = -i \int_0^\infty ds_1\, e^{i(A+i\epsilon)s_1},$$

e.g., using $A = q^2 - m^2$. Thereby, the q integral turns into a Fresnel integral (a Gauß integral with purely imaginary argument in the exponential). For the computation of the Fresnel integral, perform a rotation of the time direction to Euclidean time, $q^0 \to i q_{\mathrm{E}}^0$, such that $q^2 = q_\mu q^\mu \to -q_{\mathrm{E}}^2 = -q_{\mathrm{E}}^\mu q_{\mathrm{E}}^\mu$.
For the remaining propertime integral with integration variable s_1 and s_2 use the following substitution:

$$s := s_1 + s_2, \quad v := \frac{s_2 - s_1}{s_2 + s_1} \quad \Rightarrow \quad \int_0^\infty ds_1 \int_0^\infty ds_2 \cdots = \frac{1}{2}\int_0^\infty ds\, s \int_{-1}^1 dv \ldots$$

(Convince yourself that this is a correct substitution).
The s integral can be carried out analytically. E.g., assuming that $p^2 < 4m_0^2$, the contour in the complex s plane can be rotated such that $s \to -is$. The resulting integral corresponds to the Euler representation of the Γ function.
There is no need to carry out the remaining v integral in general. A special case is considered in part (e).

Solution of part (a):

$$\Sigma(p^2) = -\frac{g^2}{(3!)^2}\frac{1}{(4\pi)^{D/2}}\,\Gamma(2-(D/2)) \int_0^1 dv \left(m_0^2 - \frac{(1-v^2)}{4}p^2\right)^{\frac{D}{2}-2}$$