5. exercise sheet: Quantum Field Theory

Aufgabe 11:

Charge conjugation, i.e., the transformation of replacing charges by anti-charges and vice versa, can be a symmetry for many theories. The symmetry transformation of charge conjugation for a complex scalar field can be realized by a unitary operator:

$$U = \exp\left[-i\frac{\pi}{2}\int \frac{d^dp}{(2\pi)^d}\,\left(a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + b^\dagger_{\mathbf{p}}b_{\mathbf{p}} - a^\dagger_{\mathbf{p}}b_{\mathbf{p}} - b^\dagger_{\mathbf{p}}a_{\mathbf{p}}\right)\right].$$

(a) Verify that $U^\dagger = U^{-1}$.

(b) Show that U implements charge conjugation on the level of the field operators,

$$\phi^\dagger(\mathbf{x}) = U\phi(\mathbf{x})U^\dagger.$$

Hint: First show that $Ua_{\mathbf{p}}U^\dagger = b_{\mathbf{p}}$ for which you can use a variant of the BCH formula:

$$e^{-A}Be^{A} = \sum_{k=0}^{\infty}\frac{1}{k!}\underbrace{\Big[\dots\Big[[B,A],A\Big],\dots,A\Big]}_{k \text{ times}}.$$

Aufgabe 12:

(a) Show that the Feynman propagator can be decomposed into particle and anti-particle contributions of the form

$$\Delta_{\mathrm{F}}(x) = \theta(x^0)\Delta^+(x) + \theta(-x^0)\Delta^-(x),$$

such that it obtains a time-ordered structure.

Definitions: using $D = d+1$ and $\bar{p}^\mu = (E_{\mathbf{p}}, \mathbf{p})$, we have

$$\Delta_{\mathrm{F}}(x) = \int \frac{d^Dp}{(2\pi)^D}\frac{1}{p^2 - m^2 + i\epsilon}\,e^{-ipx}, \quad \Delta^\pm(x) = \frac{1}{i}\int\frac{d^dp}{(2\pi)^d}\frac{1}{2E_{\mathbf{p}}}\,e^{\mp i\bar{p}x}.$$

(b) The $i\epsilon$ prescription for the Feynman propagator clarifies how the contour of the p^0 integral goes around the two poles of the integrand at $p^0 = \pm E_{\mathbf{p}}$ in the complex p^0 plane.

Now, consider alternatively a contour that goes around both poles in the upper half-plane. Show that this contour leads to the *retarded* propagator,

$$\Delta_{\mathrm{R}}(x) = \theta(x^0)\Delta(x), \qquad \text{mit} \quad \Delta(x) = \Delta^+(x) - \Delta^-(x).$$

(c) Show that the contour that goes around both poles in the lower half-plane leads to the *advanced* propagator,

$$\Delta_{\mathrm{A}}(x) = -\theta(-x^0)\Delta(x).$$