4. Exercise sheet: Quantum Field Theory

Aufgabe 9:
A Lorentz transformation is a linear transformation of a 4-vector

$$v^\mu \to v'^\mu = \Lambda^\mu{}_\nu v^\nu,$$

that preserves the norm

$$v^2 = g_{\mu\nu} v^\mu v^\nu \equiv v^\mu v_\mu, \quad g = (+,-,-,-).$$

(a) Show that Λ satisfies the identity:

$$g_{\kappa\lambda} = g_{\mu\nu} \Lambda^\mu{}_\kappa \Lambda^\nu{}_\lambda.$$

(b) Consider infinitesimal Lorentz transformations $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu$, $\epsilon^\mu{}_\nu \ll 1$. Show that

$$\epsilon_{\kappa\lambda} + \epsilon_{\lambda\kappa} = 0$$

holds such that Lorentz transformations are specified in terms of 6 independent parameters (in 4 spacetime dimensions).

(c) Now consider a scalar field $\phi(x)$ and the corresponding Klein-Gordon Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \partial\phi)$. Use the transformation property of a scalar field, $\phi \to \phi' = \phi(\Lambda x)$, under Lorentz transformations to derive the associated Noether current J^μ. Show that this current can be written as:

$$J^\mu = \frac{1}{2}\epsilon_{\kappa\lambda}\mathcal{J}^{\mu\kappa\lambda}, \quad \mathcal{J}^{\mu\kappa\lambda} = T^{\mu\kappa}x^\nu - T^{\mu\nu}x^\kappa,$$

where $T^{\mu\nu}$ is the canonical energy-momentum tensor of the Klein-Gordon field.

(d) Convince yourself that the quantity

$$L^k := -\frac{1}{2}\epsilon^{kij}\int d^dx J^{0ij}$$

can be interpreted as an angular momentum of the field, in analogy to the field momentum $\mathbf{P}$ (Hint: for this, show that the corresponding momentum densities satisfies a relation that is reminiscent to $\mathbf{L} = \mathbf{x} \times \mathbf{P}$.)

Aufgabe 10:
Show that the normalization of the 1-particle states chosen in the lecture

$$|\mathbf{p}\rangle = \sqrt{2E_\mathbf{p}}\, a^\dagger(\mathbf{p})|0\rangle$$

implies that the inner product is Lorentz-invariant, i.e,

$$\langle \mathbf{q}|\mathbf{p}\rangle = \langle \mathbf{q}'|\mathbf{p}'\rangle,$$

where $\mathbf{q}', \mathbf{p}'$ are the momentum coordinates with respect to a Lorentz-transformed frame.

Hint: it is sufficient to show the invariance with respect to a Lorentz boost along a specific direction, say the 3-direction, such that $p'^3 = \gamma(p^3 - \beta E_\mathbf{p})$ und $E'_\mathbf{p} = \gamma(E_\mathbf{p} - \beta p^3)$.