3. exercise sheet: Quantum Field Theory

Aufgabe 8:

Quantize the complex free scalar field.

(a) Start from the Lagrangian density for the classical complex scalar field,

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2|\phi|^2,$$

and then construct the Hamiltonian density: $\mathcal{H} = \pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2|\phi|^2$. Then replace the canonical variables by corresponding operators, in particular $\phi^*, \pi^* \to \phi^\dagger, \pi^\dagger$.

(b) Now quantize the field operators by introducing ladder operators a_1, a_2 and $a_1^\dagger, a_2^\dagger$ for the real field components $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. Next, define

$$a(\mathbf{p}) = \frac{1}{\sqrt{2}}(a_1(\mathbf{p}) + ia_2(\mathbf{p})), \quad b(\mathbf{p}) = \frac{1}{\sqrt{2}}(a_1(\mathbf{p}) - ia_2(\mathbf{p})),$$

and show that these “complex” ladder operators satisfy two independent ladder operator algebras.

(c) Express the complex field and momentum density operators in a suitable way in terms of $a, a^\dagger, b$, and $b^\dagger$.

(d) Show that the Hamilton operator can be written as

$$H = \int \frac{d^dp}{(2\pi)^d} \, \omega_{\mathbf{p}}(a^\dagger(\mathbf{p})a(\mathbf{p}) + b^\dagger(\mathbf{p})b(\mathbf{p})) + \text{zero-point energies}$$

(which implies that both sets of states generated by $a^\dagger$ and $b^\dagger$ contribute positively to the energy spectrum).

(e) Consider the Noether charge $Q = i\int dx(\phi^\dagger\partial^0\phi - \phi\partial^0\phi^\dagger)$, and show that the ladder operators $a^\dagger$ and $b^\dagger$ generate field excitations with opposite charges.