# 2. EXERCISE SHEET: QUANTUM FIELD THEORY

**Aufgabe 4:**

Proof the Noether theorem for classical field theory.

(a) Assume that $\phi \to \phi + \delta\phi$ is an infinitesimal symmetry transformation such that the Lagrangian density changes at most by a total derivative $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$, where $\delta\mathcal{L} = \partial_\mu K^\mu$. Now express the variation of $\mathcal{L} = \mathcal{L}(\phi, \partial\phi)$ in terms of the variation of the field $\delta\phi$.

(b) Use the equations of motion to show that the 4-current $J^\mu$ satisfies a continuity equation:

$$\partial_\mu J^\mu = 0, \quad J^\mu = \pi^\mu \delta\phi - K^\mu, \quad \pi^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}.$$

(c) Show that this implies (under suitable conditions) the conservation of the Noether charge $Q := \int d^dx J^0$, where $d$ is the number of space dimensions.

**Aufgabe 5:**

Consider a scalar field theory defined in terms of some Lagrangian $\mathcal{L}(\phi, \partial\phi)$ which is invariant under spacetime translations by a constant 4-vector $a^\mu$, i.e., $\phi(x) \to \phi(x-a)$ leaves the equations of motion invariant. Show that the Noether theorem implies that the energy-momentum tensor is conserved:

$$0 = \partial_\mu T^{\mu\nu}, \quad T^{\mu\nu} = \pi^\mu \partial^\nu\phi - g^{\mu\nu}\mathcal{L}, \quad \pi^\mu := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}.$$

(a) Derive the infinitesimal field transformation $\delta\phi$ in terms of a first order Taylor expansion assuming that $a^\mu$ is chosen infinitesimally.

(b) Analogously, derive the infinitesimal transformation of the Lagrangian $\delta\mathcal{L}$ and determine the form of $K^\mu$.

(c) Derive the desired result from the Noether theorem. Discuss also the corresponding Noether charge.

**Aufgabe 6:**

For the field theory of a complex scalar field with Lagrangian $\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - U(\phi^*\phi)$, show that the invariance under phase transformations $\phi \to \exp(i\alpha)\phi$ implies the existence of a Noether current of the form $J^\mu = -2\mathrm{Im}(\phi^*\partial^\mu\phi)$ (up to irrelevant constant factors).

**Aufgabe 7:**

Consider the Lagrangian of an interacting real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4.$$

Derive the equation of motion in both ways, using (a) the Euler-Lagrange equation and (b) the canonical equations of motion of the Hamilton formalism.