12. exercise sheet: Particles and Fields

Exercise 34:

In exercise 26, we showed that a theory defined by the action

$$S = \int d^4x \left(\bar{\psi} i \partial_\mu \gamma^\mu \psi + \frac{\lambda}{2} \left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 \right] \right)$$

is invariant under vector transformations $\mathrm{U}(1)_\mathrm{V}$ $[\psi \to e^{i\alpha}\psi,\ \bar{\psi} \to e^{-i\alpha}\bar{\psi}]$, as well as axial transformations $\mathrm{U}(1)_\mathrm{V}$ $[\psi \to e^{i\alpha\gamma_5}\psi,\ \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_5}]$, by studying the transformations infinitesimally.
Proof this invariance also directly for finite transformations by writing the interaction in terms of $\psi_{\mathrm{L,R}}$ and $\bar{\psi}_{\mathrm{L,R}}$ using the projectors $P_{\mathrm{L,R}}$ of exercise 25 and studying the invariance of the resulting interaction under chiral transformations $\mathrm{U}(1)_\mathrm{R}$ and $\mathrm{U}(1)_\mathrm{L}$.

Exercise 35:

(a) The symmetry of the action of exercise 34 can also be verified by first showing that the Fierz transformations of exercise 33 imply the identity (Hint: take into account that the spinors are Grassmann-valued.)

$$\left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 \right] = -\frac{1}{2} \left[(\bar{\psi}\gamma_\mu\psi)^2 - (\bar{\psi}\gamma_\mu\gamma_5\psi)^2 \right].$$

Following exercise 27, verify the chiral/axial symmetry of the model.
(b) Generalize this identity to the case of N_f flavors ψ_a, $\bar{\psi}_a$, $a = 1, \dots, N_\mathrm{f}$.

Exercise 36:

In the theory of strong interactions (QCD), gluon exchange can give rise to chirally invariant effective quark interactions of the type $(\bar{\psi}_a\gamma_\mu\psi_a)^2$ and $(\bar{\psi}_a\gamma_\mu\gamma_5\psi_a)^2$. According to exercise 35 (b), the anti-symmetric combination can be rewritten as

$$\left[(\bar{\psi}_a\gamma_\mu\psi_a)^2 - (\bar{\psi}_a\gamma_\mu\gamma_5\psi_a)^2 \right] = -2 \left[(\bar{\psi}_a\psi_b)(\bar{\psi}_b\psi_a) - (\bar{\psi}_a\gamma_5\psi_b)(\bar{\psi}_b\gamma_5\psi_a) \right].$$

(a) Start with a theory of the form

$$S = \int d^4x \left(\bar{\psi} i \partial_\mu \gamma^\mu \psi - \frac{\lambda}{4} \left[(\bar{\psi}_a\gamma_\mu\psi_a)^2 - (\bar{\psi}_a\gamma_\mu\gamma_5\psi_a)^2 \right] \right),$$

use the Fierz rearrangement, and rewrite the purely fermionic model in terms of a fermion-scalar model by means of the Hubbard-Stratonovich transformation. Discuss the symmetry transformation of the scalar field. Hint: a matrix-valued scalar field is needed.
(b) Construct the general Yukawa theory exhibiting the same symmetries as in (a). For this, the representation of the interaction in terms of right- and left-handed spinors may be useful. What is the most-general potential to fourth order in the scalar field?