11. exercise sheet: Particles and Fields

Exercise 31:

In the lectures, we have identified $M_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ with the generator of Lorentz transformations for the Dirac spinors. Verify this explicitly, by showing that $M_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}$ satisfies the Lie algebra of the Lorentz group,

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho}).$$

Hint: First show that $[\gamma_\mu, M_{\rho\sigma}] = i(g_{\mu\rho}\gamma_\sigma - g_{\mu\sigma}\gamma_\rho)$.

Exercise 32:

The Dirac matrices with defining property $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ span an algebra of 4×4 matrices, a basis of which is given by

$$\tilde{\Gamma}^A = \{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma_5, \gamma_5\},$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.
(a) Convince yourself that these are 16 different elements (for $\sigma^{\mu\nu}$, only those with $\mu < \nu$ have to be counted as independent, since $\sigma^{\mu\nu}$ is antisymmetric).
(b) Construct a normalized basis of elements Γ^A, such that they satisfy the normalization condition

$$\mathrm{tr}\,\Gamma^A\Gamma^B = 4\delta^{AB}.$$

Exercise 33:

In theories with fermionic self-interactions, the interaction terms often have the form

$$\bar{\psi}_1\Gamma^A\psi_2\bar{\psi}_3\Gamma^B\psi_4,$$

where ψ_i denotes different spinors (e.g., different flavors, color, momenta ...). Writing the spinor indices explicitly, the interaction involves the Dirac structure $\Gamma^A_{ab}\Gamma^B_{cd}$.
By means of Fierz transformations, the spinor indices can be rearranged,

$$\Gamma^A_{ab}\Gamma^B_{cd} = \sum_{C,D} C^{AB}_{CD}\Gamma^C_{ad}\Gamma^D_{cb},$$

with expansion coefficients C^{AB}_{CD}. Show that these coefficients can be computed in terms of the basis elements Γ^A by

$$C^{AB}_{CD} = \frac{1}{16}\mathrm{tr}\,(\Gamma^A\Gamma^D\Gamma^B\Gamma^C),$$

provided the Γ^A are normalized as in exercise 32.