10. exercise sheet: Particles and Fields

Exercise 28:

In the chiral basis, there are four independent solutions of the free Dirac equation of the form $\psi(x) = u(p)e^{-ipx}$, and $\psi(x) = v(p)e^{ipx}$ where

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\,\xi^s \\ \sqrt{p\cdot\sigma}\,\xi^s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\,\eta^s \\ -\sqrt{p\cdot\sigma}\,\eta^s \end{pmatrix}, \quad s = 1, 2,$$

Here ξ^s and η^s denote 2-component base spinors.
Provided the base spinors are orthonormalized

$$\sum_{s=1,2} \xi^s \xi^{s\dagger} = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

show that the spin sums satisfy:

$$\sum_s u^s(p)\bar{u}^s(p) = \gamma\cdot p + m, \quad \sum_s v^s(p)\bar{v}^s(p) = \gamma\cdot p - m.$$

Exercise 29:

In the lectures, we have worked with Weyl as well as Dirac fermions and have written down the Majorana mass term in terms of the Weyl spinors. The *Majorana spinor* is defined as a Dirac spinor with the property of being its own charge conjugate,

$$\psi^c_{\mathrm{M}} = \psi_{\mathrm{M}}, \quad \text{where } \psi^c = -i\gamma^2\psi^* \tag{1}$$

defines the charge conjugate of a Dirac spinor (i.e. the transformation that turns particles into antiparticles and vice versa).

(a) Start from an ansatz $\psi_{\mathrm{M}} = \begin{pmatrix} \eta \\ \xi \end{pmatrix}$, and use the defining property to show that the Majorana spinor can equivalently be written as

$$\psi_{\mathrm{M}} = \begin{pmatrix} -i\sigma^2\xi^* \\ \xi \end{pmatrix} = \begin{pmatrix} \eta \\ i\sigma^2\eta^* \end{pmatrix} \tag{2}$$

(b) Compute explicitly the Lagrangian for the Majorana spinor in terms of its chiral component η. For this, plug ψ_{M} into the Dirac Lagrangian $\mathcal{L}_{\mathrm{D}} = \bar{\psi}i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$. Convince yourself in this way that the final result is equivalent to the Lagrangian of the Weyl fermion η including a Majorana mass term up to an irrelevant global factor of 2.
Hint: you may find the relation $\sigma^2\bar{\sigma}^\mu\sigma^2 = (\sigma^\mu)^T$ useful. Also remember that the component of a spinor is a Grassmann variable.
Conclusion: The Majorana particle is a particular kind of Dirac fermion that has the property of being its own charge conjugate. This requirement reduces the number of degrees of freedom of the particle from 4 for a Dirac fermion to 2 for a Majorana fermion.

Exercise 30:

Motivation: In the lecture, we found the important identity

$$\bar{A}\gamma^{\mu}A = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}, \tag{3}$$

where A is connected with the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation of the Lorentz group. There is an alternative way to interpret this equation: it connects the Lorentz transformation of the 4-vector γ_{μ} (RHS) with a "rotation" in spinor space (LHS), precisely such that the Dirac γ matrices look the same in any Lorentz frame. In fact, this alternative viewpoint is more general (and also allows for a straightforward generalization to curved space), and hence deserves to by studied in the following:

Exercise:

(a) Verify that the Dirac algebra

$$\{\gamma_{\mu},\gamma_{\nu}\} = 2g_{\mu\nu} \tag{4}$$

is invariant under generalized "rotations" of spinor space, so called spin-base transformations, $\gamma^{\mu} \to \mathcal{S}\gamma^{\mu}\mathcal{S}^{-1}$, where $\mathcal{S}$ is allowed to be an element of the general linear group of 4×4 matrices with complex components GL(4,$\mathbb{C}$).

(b) Verify that the Dirac equation

$$(i\gamma_{\mu}\partial^{\mu} - m)\psi(x) = 0, \tag{5}$$

is invariant under spin base transformations, provided that the Dirac spinor transforms as $\psi \to \mathcal{S}\psi$.

(c) Now, we define the Lorentz-transformed Dirac matrices: $\gamma'_{\mu} = \Lambda_{\mu}{}^{\nu}\gamma_{\nu}$, i.e., somewhat contrary to the philospophy of Eq. (3), we accept that the Dirac matrices look differently in a different Lorentz frame. Show, that also the γ'_{μ} satisfy the Dirac algebra (4).

(d) Use this to show that the Dirac equation (5) is also satisfied in the primed Lorentz system, provided the Dirac spinors now transform component-wise as scalars, i.e., $\psi'(x') = \psi(x)$ under Lorentz transformations.

Conclusion: (a)–(d) demonstrate that the Dirac equation is separately and independently invariant under spin-base transformations $\mathcal{S} \in \mathrm{GL}(4,\mathbb{C})$ and Lorentz transformations $\Lambda_{\mu}^{\nu} \in \mathrm{SO}(3,1)$.

In this light, Equation (3) can be interpreted as the statement that it is always possible to perform simultaneously a Lorentz and a spin-base transformation such that the Dirac matrices γ_{μ} have the same representation in any Lorentz frame. These spin-base transformations $\mathcal{S} = A$ form a subgroup of GL(4,$\mathbb{C}$) corresponding to two representations of SL(2,$\mathbb{C}$).