9. Exercise sheet: Particles and Fields

Exercise 25:

(a) Use the fundamental defining property of the Dirac algebra,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu},$$

to show that $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ anti-commutes with all γ^μ, i.e., $\{\gamma_5, \gamma^\mu\} = 0$, independently of the representation.
(b) Also show that $\gamma_5^2 = \mathbb{1}$ holds independently of the representation.
(c) Verify that the right- and left-handed projectors

$$P_\mathrm{R} := \frac{1}{2}(1+\gamma_5), \quad P_\mathrm{L} := \frac{1}{2}(1-\gamma_5),$$

satisfy the projector algebra

$$P_{\mathrm{R,L}}^2 = P_{\mathrm{R,L}}, \quad P_\mathrm{R}P_\mathrm{L} = 0, \quad P_\mathrm{R} + P_\mathrm{L} = 1.$$

(d) Use this to show that the kinetic term of Dirac's theory can be decomposed into separate kinetic terms for ψ_L and ψ_R, where $\psi_{\mathrm{R,L}} = P_{\mathrm{R,L}}\psi$. Show in addition that this does not hold for the mass term.
Hint: Note that $\bar\psi_{\mathrm{L,R}} = \bar\psi P_{\mathrm{R,L}}$ (why?).

Exercise 26:

Show that a theory defined by the action

$$S = \int d^4x \left(\bar\psi i\partial_\mu\gamma^\mu\psi - \frac{\lambda}{2}\left[(\bar\psi\psi)^2 - (\bar\psi\gamma_5\psi)^2\right]\right)$$

is invariant under vector transformations $\mathrm{U}(1)_\mathrm{V}$ as well as axial transformations $\mathrm{U}(1)_\mathrm{V}$,

$$\begin{aligned} \mathrm{U}(1)_\mathrm{V}: &\quad \psi \to e^{i\alpha}\psi, \quad \bar\psi \to e^{-i\alpha}\bar\psi, \\ \mathrm{U}(1)_\mathrm{A}: &\quad \psi \to e^{i\alpha\gamma_5}\psi, \quad \bar\psi \to \bar\psi e^{i\alpha\gamma_5}, \end{aligned}$$

(Hint: it is sufficient to consider infinitesimal transformations), and compute the corresponding Noether currents.

Exercise 27:

Consider the action of a fermion theory

$$S = \int d^4x \left(\bar\psi^a i\partial_\mu\gamma^\mu\psi^a - \frac{\lambda}{2}(\bar\psi^a\gamma_\mu\psi^a)^2\right), \quad a = 1, \ldots, N_\mathrm{f},$$

where N_f denotes the number of fermion *flavors*. What are the continuous symmetries of the theory?