8. exercise sheet: Particles and Fields

Exercise 22:

Consider the relation between Lorentz 4-vectors and spinors. This relation is constructed with the help of

$$(\sigma_\mu)_{\alpha\dot{\beta}} = (\mathbb{1}, \boldsymbol{\sigma}), \quad (\bar{\sigma}_\mu)^{\dot{\alpha}\beta} = (\mathbb{1}, -\boldsymbol{\sigma}),$$

where $\boldsymbol{\sigma}$ are the Pauli matrices.
Proof the following identities:

$$(1) \qquad \frac{1}{2}\mathrm{tr}\,(\bar{\sigma}^\mu \sigma_\nu) = \delta^\mu_\nu$$

$$(2) \qquad (\sigma^\mu)_{\alpha\dot{\beta}} (\bar{\sigma}_\mu)^{\dot{\gamma}\delta} = 2\delta^\delta_\alpha \delta^{\dot{\gamma}}_{\dot{\beta}}.$$

$$(3) \qquad \sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = \bar{\sigma}_\mu \sigma_\nu + \bar{\sigma}_\nu \sigma_\mu = 2\, g_{\mu\nu}.$$

Exercise 23:

(a) Deduce with the aid of the representation of the Lorentz transformation matrix $\Lambda^\mu{}_\nu$ in terms of spin and boost generators **J** und **K** the relation between Λ and the Lorentz transformation matrix for spinors a:

$$\sigma_\mu \Lambda^\mu{}_\nu = a \sigma_\nu a^\dagger.$$

(b) In turn, show that a 4-vector constructed from two independent SL(2,$\mathbb{C}$) spinors $\xi^\alpha, \eta^{\dot{\beta}}$

$$V_\mu = \xi^\alpha (\sigma_\mu)_{\alpha\dot{\beta}} \eta^{\dot{\beta}}$$

has the correct transformation properties under Lorentz transformations.
Hint: the relation between the transposed of a 2×2 matrix and its inverse from the preceding exercise sheet may also be helpful.

Exercise 24:

(a) Consider a Grassmann algebra consisting of only two different numbers θ_1 und θ_2 with the properties

$$\theta_1^2 = 0, \quad \theta_2^2 = 0, \quad \{\theta_1, \theta_2\} = 0.$$

Show that a representation of this algebra can be constructed with the aid of the Pauli matrices σ_i and the combination

$$\sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2)$$

such that the Grassmann algebra can be realized by 4×4 matrices, e.g., $\theta_1 \to \sigma_- \otimes \mathbb{1}$ and $\theta_2 \to \sigma_3 \otimes \sigma_-$.

(b) Convince yourself that the following identities hold independently of the representation

$$\exp(\theta_1\theta_2) = \frac{1}{1 - \theta_1\theta_2} = 1 + \ln(1 + \theta_1\theta_2).$$

(c) Determine the set of solutions x of the equation

$$x^2 = 1 + \theta_1\theta_2.$$

(d)* Given a Grassmann algebra with n different numbers

$$\theta_i^2 = 0, \quad \{\theta_i, \theta_j\} = 0, \quad i, j = 1, \ldots, n.$$

How many linearly independent elements does the algebra have?

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* Extra exercise for the willing.