7. exercise sheet: Particles and Fields

Exercise 19:

The generators $M_{\mu\nu} \equiv -M_{\nu\mu}$ of the Lorentz group SO(3,1) satisfy the Lie algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho}).$$

Show that the components

$$J_i \equiv \frac{1}{2}\epsilon_{ijk}M^{jk}, \quad K_i \equiv M_{i0} = -M_{0i}, \quad (i, j, k = 1, 2, 3)$$

satisfy the algebraic relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k$$

such that $\mathbf{J}$ generates an angular momentum algebra and $\mathbf{K}$ generates the Lorentz *boosts*.

Exercise 20:

Start from the Lie algebra for the generators $\mathbf{J}$ and $\mathbf{K}$ of the preceding exercise and introduce the combinations

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}), \quad \mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}).$$

Show that these generators satisfy the following Lie algebra

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad [B_i, B_j] = i\epsilon_{ijk}B_k, \quad [A_i, B_j] = 0,$$

such that the Lorentz algebra can actually be decomposed into two mutually commuting angular momentum algebras.

Exercise 21:

Show that the scalar product of two SL(2,$\mathbb{C}$) spinors

$$\xi\zeta \equiv \xi^\alpha\zeta_\alpha := \epsilon^{\alpha\beta}\xi_\beta\zeta_\alpha, \quad \text{where } \epsilon^{\alpha\beta} \equiv i(\sigma_2)^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

is invariant under Lorentz transformations $\xi'_\alpha = a_\alpha{}^\beta\xi_\beta$, $\zeta'_\alpha = a_\alpha{}^\beta\zeta_\beta$. Here a is an element of SL(2,$\mathbb{C}$), i.e. a complex 2×2 matrix with $\det a = 1$.
Hint: First proof and then use the following formula for general 2×2 matrices M:

$$\epsilon M^{\mathrm{T}}\epsilon^{\mathrm{T}} = (\det M)M^{-1},$$

where the superscript T denotes matrix transposition.