6. exercise sheet: Particles and Fields

Exercise 16:

The nonrelativistic version of scalar QED,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \psi^* i\partial_t\psi - \frac{1}{2m}(\mathbf{D}\psi)^* \cdot \mathbf{D}\psi - V(\psi^*\psi), \tag{1}$$

where $\mathbf{D} = \boldsymbol{\nabla} - iq\mathbf{A}$, can be used to describe superconductors. Here, the complex scalar field ψ should be thought of as the wave function of the coherent bosonic state that describes the Cooper pairs; hence, its charge is $q = 2e$. The model has a local gauge invariance.
(a) Local gauge invariance also implies a global phase invariance of the scalar field, $\psi \to e^{-i\alpha}\psi$. Compute the Noether current J^μ of this symmetry. Verify that the spatial components (up to the infinitesimal symmetry parameter α) agree with the Cooper current

$$\mathbf{j} = \frac{i}{2m}\big((\mathbf{D}\psi)^*\psi - \psi^*\mathbf{D}\psi\big). \tag{2}$$

(b) Assume that the potential V inside a superconductor has a minimum at a finite field amplitude at $|\psi| > 0$. Verify that the generic form of the wave function in this ground state then is $\psi = \sqrt{|\rho|}e^{i\varphi}$, where ρ agrees with the Noether current density up to the infinitesimal parameter and $\varphi = \varphi(\mathbf{x}, t)$ is an arbitrary phase.
(c) Compute the Cooper current for this ground state of a constant density $\rho = const.$ and verify that the Maxwell equation $\boldsymbol{\nabla} \times \mathbf{B} = \mathbf{j}$ implies the London equation

$$\nabla^2\mathbf{B} = \frac{1}{\lambda_{\mathrm{L}}^2}\mathbf{B}. \tag{3}$$

Compute λ_{L}^2 and convince yourself that λ_{L} can be interpreted as the penetration depth of a magnetic field into a superconductor, thus explaining the Meißner-Ochsenfeld effect. How is the penetration depth related to the photon mass?

Exercise 17:

Consider an $\mathrm{O}(N)$ invariant scalar model with Langrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^a)(\partial^\mu\phi^a) - V(\rho), \quad \rho = \frac{1}{2}\phi^a\phi^a, \tag{4}$$

where $a = 1, \ldots, N$. Given the potential with some minimum ρ_0, any concrete parametrization of ρ_0 in field space is legitimate and physically equivalent. Thus, the eigenvalues of the mass matrix $m_{ab}^2 = \partial^2 V/\partial\phi^a\partial\phi^b$ can be written in terms of $\mathrm{O}(N)$-invariant quantities. Diagonalize the mass matrix in terms of such invariant expressions. The final expression should also hold for the case $\rho_0 = 0$.

Exercise 18:

Consider the action for a complex two-component scalar field $\phi_i \in \mathbb{C}$, $i = 1, 2$, which may be summarized in a complex vector $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$,

$$\mathcal{L} = (\partial_\mu \phi_i^*)(\partial^\mu \phi_i) + \mu^2 \phi_i^* \phi_i - \frac{\lambda}{3!}\left(|\phi_1|^2 + |\phi_2|^2\right)^2. \tag{5}$$

(a) Identify the global symmetry group of rotations in the complex field space.
(b) For $\mu^2 > 0$, the potential has minima at nonzero field values. Determine this submanifold in field space.
(c) Select a possible vacuum state Φ_0. What is the symmetry group that leaves this ground state invariant?
(d) Now expand the field in terms of real fields denoting excitations on top of the vacuum. Determine the number of Goldstone modes, and compare this number to the number of "broken generators".