5. exercise sheet: Particles and Fields

Exercise 13:

Consider the Lagrangian for a triplet of real scalar fields ϕ^a, $(a = 1, 2, 3)$, defining a classical field theory with rotational O(3) invariance in field space,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - V(\phi^a \phi^a), \tag{1}$$

(a) Verify that the action is indeed invariant under infinitesimal rotations in field space, which can be written analogously to rotations in coordinate space as $\phi^a \to \phi^a + \theta \epsilon^{abc} \hat{n}^b \phi^c$, where $\hat{n}^b$ is a unit vector defining the rotation axis, and $\theta \ll 1$ is an infinitesimal rotation angle.
(b) Compute the Noether current and the Noether charge.
(c) Verify the conservation of the Noether charge explicitly by using the equation of motion.

Exercise 14:

Consider an almost O(N) invariant scalar model with Langrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - \left(-\frac{1}{2}\mu^2 \phi^a \phi^a + \frac{\lambda}{4!}(\phi^a \phi^a)^2\right) - \delta V, \tag{2}$$

where $a = 1, \ldots, N$ and δV is a potential term that breaks O(N) symmetry explicitly. Upon spanning ϕ^a by a parametrization $\phi = (\pi^i, \Sigma)$, where $i = 1, \ldots, N-1$, δV takes the form $\delta V = -h\Sigma$ with a positive constant parameter $h > 0$.
(a) Determine the position of the global minimum of the potential to first order in h.
(b) Verify that the would-be Goldstone bosons acquire a mass. Compute the mass to first order in h.

Exercise 15:

Consider the action for a free O(N) symmetric field theory,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a). \tag{3}$$

(a) Convince yourself that the theory becomes interacting by simply imposing the constraint $\phi^a \phi^a = 1$. For this, use the parametrization of exercise 14, eliminate the Σ field by the constraint, and compute the leading interactions for small π^i fluctuations.
(b) This model is called a nonlinear σ model. Construct a suitable limit procedure such that the nonlinear model arises from the linear σ model (of exercise 14 with $\delta V = 0$).