4. exercise sheet: Particles and Fields

Exercise 10:

Consider a classical field theory which is invariant under translations $x_\mu \to x_\mu - a_\mu$. The corresponding Noether charge is given by the 4-momentum of the field ϕ,

$$P^\mu = \int d^3x (\pi \partial^\mu \phi - g^{\mu 0} \mathcal{L}).$$

Show that the Noether charge is also the generator of spacetime translations, i.e., show that

$$\delta\phi = -a_\mu \{\phi(\mathbf{x}), P^\mu\}.$$

Exercise 11:

Use Noether's theorem to construct the (canonical) energy-momentum tensor for the classical electromagnetic field from the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1}$$

Convince yourself that this does not result in a symmetric tensor, i.e., $T^{\mu\nu} \neq T^{\nu\mu}$; also the result is not gauge invariant. A symmetric gauge-invariant tensor can nevertheless be constructed by adding a term of the form

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}, \quad \text{where } K^{\lambda\mu\nu} = -K^{\mu\lambda\nu}. \tag{2}$$

The anti-symmetry of $K^{\lambda\mu\nu}$ with respect to its first two indices guarantees that Noether's conservation law still applies, $\partial_\mu T^{\mu\nu} = 0$. Show that this construction with $K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu$ leads to the desired (symmetric, gauge-invariant) result. Show also the components are related to standard quantities, such as the energy density $\varepsilon = T^{00} = \frac{1}{2}(E^2 + B^2)$ and the Poynting vector (momentum density) $\mathbf{S} = \mathbf{E} \times \mathbf{B}$, where $S^i = \hat{T}^{0i}$. (Hint: for the last step, use $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.) Also show that $\hat{T}^{\mu\nu}$ is traceless.

Exercise 12:

Consider the action,

$$S = \int d^4x \left(\frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\bar{\lambda}}{4!} \phi^p \right), \tag{3}$$

with some power p for the nonlinear term. Study the behavior of the action under scale transformations (*dilatations*)

$$x \to \lambda x, \quad \phi(x) \to \lambda^{-D} \phi(\lambda^{-1} x), \quad \lambda > 0, \tag{4}$$

where D is a scaling exponent (dilatation weight) for the field ϕ. Under which conditions is the action scale invariant? Determine the corresponding conserved current using Noether's theorem.