3. exercise sheet: Particles and Fields

Exercise 7:

From a pragmatic (physicist's) viewpoint, functional differentiation $\delta/\delta\phi(x)$ can be defined by the conditions that the algebraic rules for standard derivatives apply,

$$\begin{aligned} \frac{\delta}{\delta\phi(x)}\big(F_1[\phi]+F_2[\phi]\big) &= \frac{\delta}{\delta\phi(x)}F_1[\phi]+\frac{\delta}{\delta\phi(x)}F_2[\phi], \quad \text{(linearity)} \\ \frac{\delta}{\delta\phi(x)}\big(F_1[\phi]F_2[\phi]\big) &= F_1[\phi]\frac{\delta}{\delta\phi(x)}F_2[\phi]+F_2[\phi]\frac{\delta}{\delta\phi(x)}F_1[\phi], \quad \text{(Leibniz rule)} \end{aligned} \tag{1}$$

where $F_i[\phi]$ are functionals of ϕ, and that additionally we have:

$$\frac{\delta}{\delta\phi(y)}\phi(x) = \delta^{(D)}(x-y). \tag{2}$$

Verify that

$$\begin{aligned} \frac{\delta}{\delta\phi(y)}\int_x \phi(x)J(x) &= J(y), \\ \frac{\delta}{\delta\phi(y)}\exp\left(\int_x \phi(x)J(x)\right) &= J(y)\,\exp\left(\int_x \phi(x)J(x)\right), \end{aligned} \tag{3}$$

where $\int_x \equiv \int d^Dx$.

Exercise 8:

Given a classical action S for a field $\phi(x)$ in spacetime. We can formulate Hamilton's principle with the aid of the functional derivative:

$$\frac{\delta S[\phi]}{\delta\phi(x)} = 0.$$

Show that for actions of the type $S[\phi] = \int d^Dy\mathcal{L}(\phi, \partial_\mu\phi; y)$, we obtain the Euler-Lagrange equations as discussed in the lecture.

Aufgabe 9:

For a classical field $\phi(\mathbf{x}, t)$ with an associated canonical conjugate momentum density $\pi(\mathbf{x}, t)$, we can define the Poisson brackets analogously to classical mechanics. Let $A[\phi, \pi]$ and $B[\phi, \pi]$ be two general phase space functionals, then the Poisson bracket in $d = D - 1$ space dimensions is given by (we ignore the time argument t in the following for simplicity)

$$\{A, B\} := \int d^d z \left(\frac{\delta A}{\delta \phi(\mathbf{z})} \frac{\delta B}{\delta \pi(\mathbf{z})} - \frac{\delta A}{\delta \pi(\mathbf{z})} \frac{\delta B}{\delta \phi(\mathbf{z})} \right).$$

(a) Verify the fundamental Poisson brackets

$$\{\phi(\mathbf{x}), \phi(\mathbf{y})\} = 0, \quad \{\pi(\mathbf{x}), \pi(\mathbf{y})\} = 0, \quad \{\phi(\mathbf{x}), \pi(\mathbf{y})\} = \delta^{(d)}(\mathbf{x} - \mathbf{y}).$$

The time evolution of the field and the momentum is generated by the Hamilton function H according to the canonical equations of motion

$$\dot{\phi}(\mathbf{x}) = \{\phi(\mathbf{x}), H\}, \quad \dot{\pi}(\mathbf{x}) = \{\pi(\mathbf{x}), H\}.$$

(b) Compute the equations of motion for Klein-Gordon theory with the Hamilton function

$$H \equiv \int d^d y \, \mathcal{H}(\mathbf{y}) = \int d^d y \, \frac{1}{2} \left(\pi^2 + (\boldsymbol{\nabla} \phi)^2 + m^2 \phi^2 \right)$$

where $\mathcal{H}(\mathbf{y})$ is the Hamilton density.