

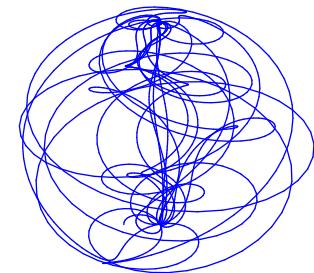
# Nonperturbative worldline dynamics

## – a feasibility study –

Holger Gies, Heidelberg U.



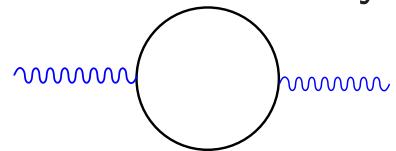
- all-loop orders on the worldline
- nonperturbative worldline numerics
- effective action for a scalar model



Collaborators: J. Sánchez-Guillén, R.A. Vázquez, [\(HEP-TH/0505275\)](#)

# QFT techniques

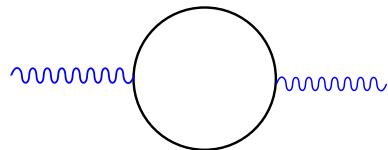
perturbation theory



small  $g$ , small  $A$

# QFT techniques

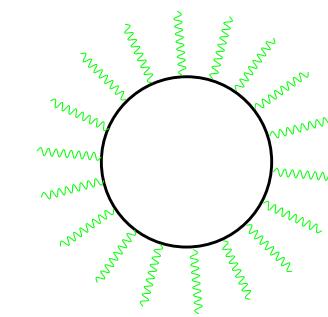
perturbation theory



small  $g$ , small  $A$



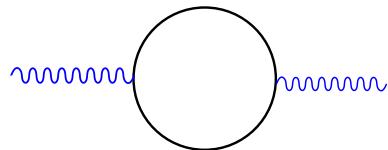
effective action techniques



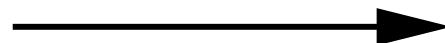
small  $g$ , arbitrary  $A$

# QFT techniques

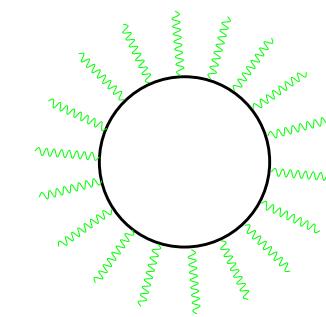
perturbation theory



small  $g$ , small  $A$



effective action techniques



small  $g$ , arbitrary  $A$



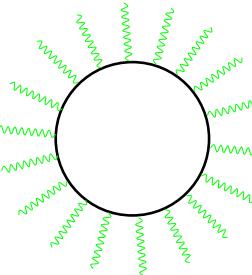
“quenched approximation”

small- $N_f$  expansion

arbitrary  $g$ , small  $A$

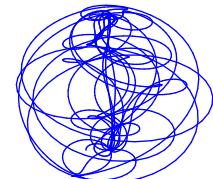
# Worldline techniques so far

- ▷ Simple example: scalar QED at one loop

$$\Gamma^1 = -\ln \int \mathcal{D}\phi e^{-\int -|D(\mathbf{A})\phi|^2 + m^2|\phi|^2} =$$


$$= \ln \det [-(\partial + i\mathbf{A})^2 + m^2]$$

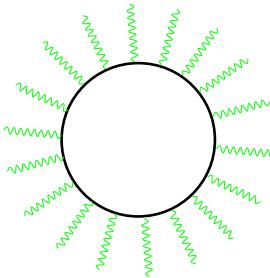
$$= - \int_{1/\Lambda^2}^{\infty} \frac{d\mathbf{T}}{\mathbf{T}} e^{-m^2 \mathbf{T}} \mathcal{N} \int \mathcal{D}\mathbf{x}(\tau) e^{-\int_0^{\mathbf{T}} d\tau \left( \frac{\dot{\mathbf{x}}^2}{4} + i \dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}(\tau)) \right)}, \quad \mathbf{x}(\mathbf{T}) =$$



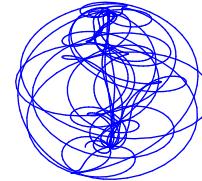
- effective action (1PI)  $\sim \int$  closed worldlines  $\mathbf{x}(\tau)$
- worldline  $\sim$  spacetime trajectory of  $\phi$  fluctuations

# Worldline techniques so far

- ▷ Simple example: scalar QED at one loop

$$\Gamma^1 = -\ln \int \mathcal{D}\phi e^{-\int -|D(\textcolor{green}{A})\phi|^2 + m^2|\phi|^2} =$$


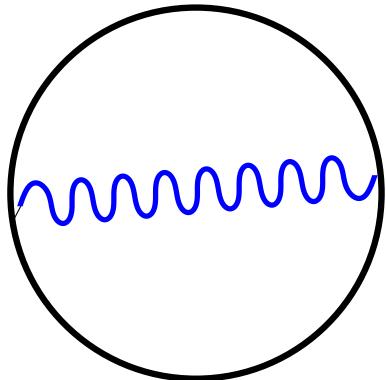
$$= \ln \det [-(\partial + i\textcolor{green}{A})^2 + m^2]$$

$$= - \int_{\textcolor{red}{T}} \left\langle e^{-i \oint d\textcolor{blue}{x} \cdot \textcolor{green}{A}(\textcolor{blue}{x})} \right\rangle_{\textcolor{blue}{x}}, \quad x(\textcolor{red}{T}) =$$


- **gauge-field** interaction  $\sim$  “Wegner-Wilson loop”
- all 1-loop diagrams in one expression,  $p$  integrals already done

## Higher loops *per pedes*

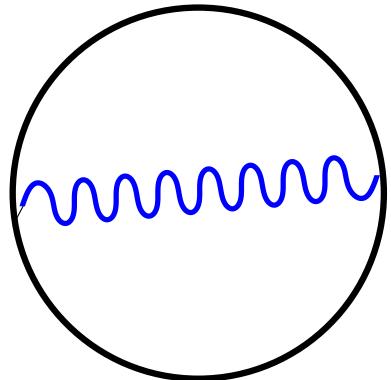
- ▷ Feynman diagrammar:



$$\sim \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \prod_i \Delta_i(q_i)$$

## Higher loops *per pedes*

▷ Worldline:



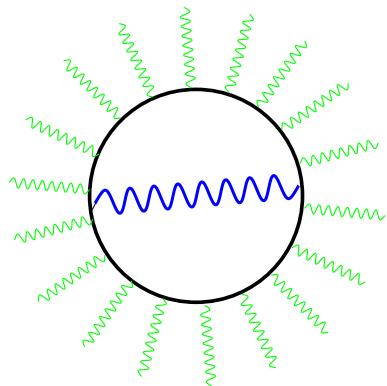
$$\sim \int_{\textcolor{red}{T}} \left\langle \int d\tau_1 d\tau_2 \Delta(\textcolor{blue}{x}(\tau_1), \textcolor{blue}{x}(\tau_2)) \right\rangle_{\textcolor{blue}{x}}$$

▷ photon propagator in coordinate space

$$\Delta(\textcolor{blue}{x}_1, \textcolor{blue}{x}_2) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} e^{ip(\textcolor{blue}{x}_1 - \textcolor{blue}{x}_2)} = \frac{\Gamma\left(\frac{D-2}{2}\right)}{4\pi^{D/2}} \frac{1}{|\textcolor{blue}{x}_1 - \textcolor{blue}{x}_2|^{D-2}}$$

# Higher loops *per pedes*

▷ Worldline:



$$\sim \int_{\textcolor{red}{T}} \left\langle e^{-i \oint d\mathbf{x} \cdot \textcolor{green}{A}(\mathbf{x})} \int d\tau_1 d\tau_2 \Delta(\mathbf{x}(\tau_1), \mathbf{x}(\tau_2)) \right\rangle_{\mathbf{x}}$$

## Higher loops *per pedes*

- ▷ Feynman diagrammar:

$$\begin{array}{c} \text{Diagram: } + \\ \text{Two circles with blue wavy lines inside.} \end{array} \sim \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \int \frac{d^D p_3}{(2\pi)^D} \prod_i \Delta_i(q_i) \\ + \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \int \frac{d^D p_3}{(2\pi)^D} \prod_i \Delta_i(q_i) \end{array}$$

## Higher loops *per pedes*

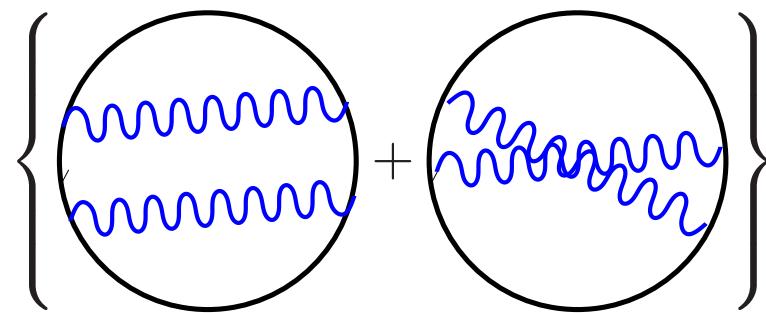
▷ Worldline:

$$\left\{ \begin{array}{c} \text{Diagram 1: A circle with a blue wavy line inside.} \\ + \\ \text{Diagram 2: A circle with a more complex blue wavy line inside.} \end{array} \right\} \sim \int_T \left\langle \int d\tau_1 d\tau_2 d\tau_3 d\tau_4 \Delta(\mathbf{x}(\tau_1), \mathbf{x}(\tau_2)) \Delta(\mathbf{x}(\tau_3), \mathbf{x}(\tau_4)) \right\rangle_x$$

▷ both diagrams in one expression

## Higher loops *per pedes*

▷ Worldline:

$$\left\{ \text{Diagram A} + \text{Diagram B} \right\} \sim \int_{\textcolor{red}{T}} \left\langle \left( \int d\tau_1 d\tau_2 \Delta(\textcolor{blue}{x}(\tau_1), \textcolor{blue}{x}(\tau_2)) \right)^2 \right\rangle_{\textcolor{blue}{x}}$$


## Higher loops *per pedes*

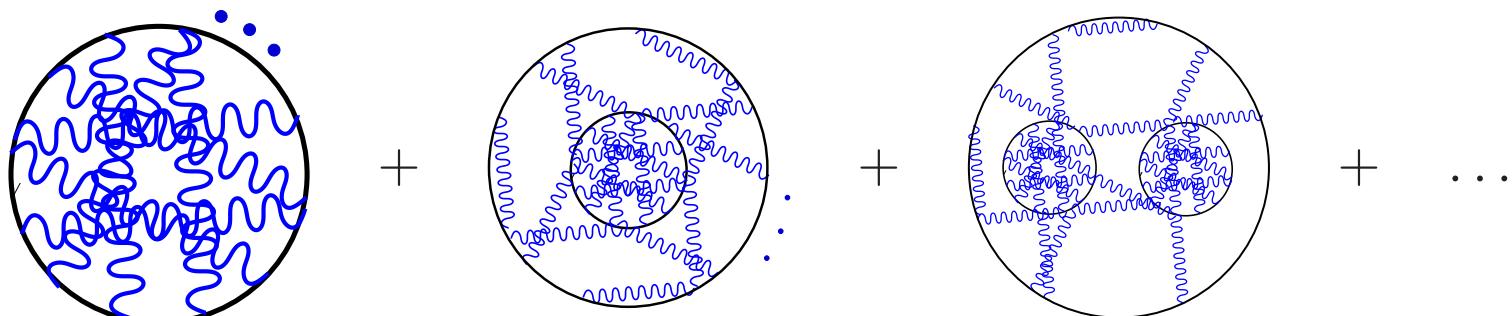
- ▷ Worldline, all possible photon insertions:

$$\sum \text{ (diagram)} \sim \int_{\textcolor{red}{T}} \left\langle \exp \left( -\frac{\textcolor{violet}{g}^2}{2} \int d\tau_1 d\tau_2 \Delta(\textcolor{blue}{x}(\tau_1), \textcolor{blue}{x}(\tau_2)) \right) \right\rangle_{\textcolor{blue}{x}}$$

⇒ “quenched approximation” (further charged loops neglected)

(FREYMAN'50)

# Systematics: small- $N_f$ expansion

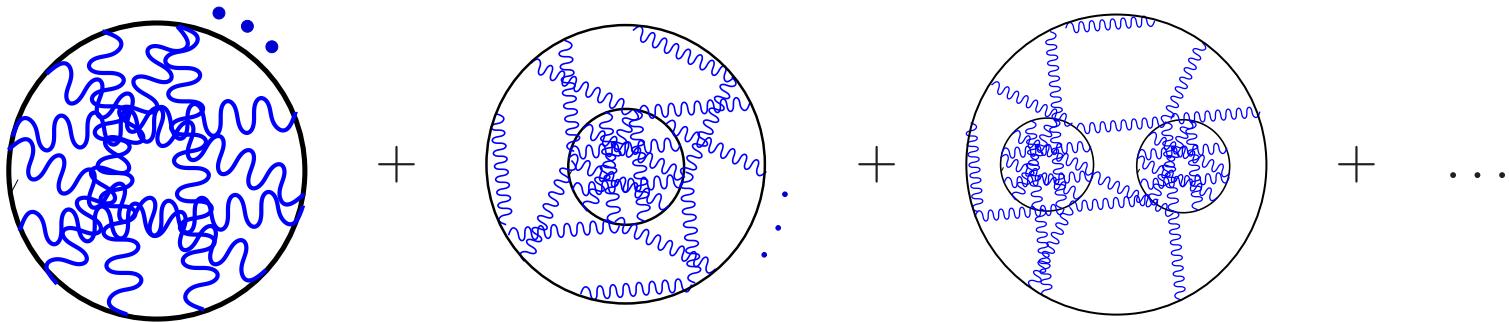


$$\sim N_f \int_T \left\langle e^{-\frac{g^2}{2} \iint \Delta} \right\rangle_x + N_f^2 \int_{T_1, T_2} \left\langle F_2\{\mathbf{x}_1, \mathbf{x}_2\} \right\rangle_{x_1, x_2} + N_f^3 \int_{T_1, T_2, T_3} \left\langle F_3\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \right\rangle_{x_1, x_2, x_3} + \dots$$

⇒ “particle- $\hbar$  expansion”

(HALPERN&SIEGEL'77)

# Systematics: small- $N_f$ expansion



- ▶ higher-order  $N_f$  negligible, if charged fluctuation trajectories are uncorrelated in spacetime

$$\text{---} \quad \text{---} \quad \text{---} : \quad \left\langle F\{\textcolor{blue}{x}_1, \textcolor{blue}{x}_2, \dots\} \right\rangle_{\textcolor{blue}{x}_1, \textcolor{blue}{x}_2, \dots} \rightarrow 0$$

$$\text{---} \quad \text{---} \quad \text{---} : \quad \left\langle F\{\textcolor{blue}{x}_1, \textcolor{blue}{x}_2, \dots\} \right\rangle_{\textcolor{blue}{x}_1, \textcolor{blue}{x}_2, \dots} = \mathcal{O}(1)$$

⇒ arbitrary  $g$ , “small”  $A$  ( $\dots$  but not perturbative in  $A$ )

# A scalar model in quenched approximation

$$\mathcal{L}(\phi, \textcolor{blue}{A}) = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}(\partial_\mu \textcolor{blue}{A})^2 - \frac{i}{2} \textcolor{violet}{h} \textcolor{blue}{A} \phi^2.$$

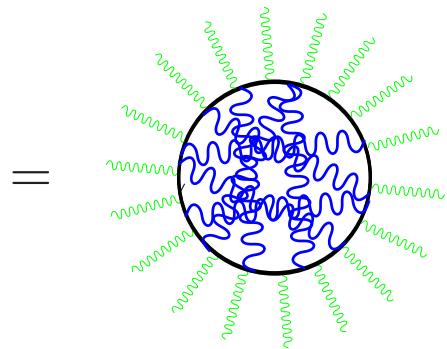
$\phi$ : “charged” matter field,     $A$ : “scalar” photon

- well-defined perturbative expansion
  - well-defined small- $N_f$  expansion
  - $\sim h A \phi^2$  superrenormalizable,  $[h] = 1$ , in  $D = 4$
  - imaginary interaction  $\sim$  QED (imaginary Wick-Cutkosky model)

# Photon effective action

- ▷ quenched approximation

$$\Gamma_{\text{QA}}[\mathcal{A}] = \int_x \frac{1}{2} (\partial_\mu \mathcal{A})^2 - \frac{1}{2(4\pi)^{D/2}} \int_0^\infty \frac{dT}{T^{1+D/2}} e^{-m^2 T} \left\langle e^{i\mathbf{h} \int d\tau \mathcal{A}} e^{-g V[\mathbf{x}]} \right\rangle_{\mathbf{x}}$$

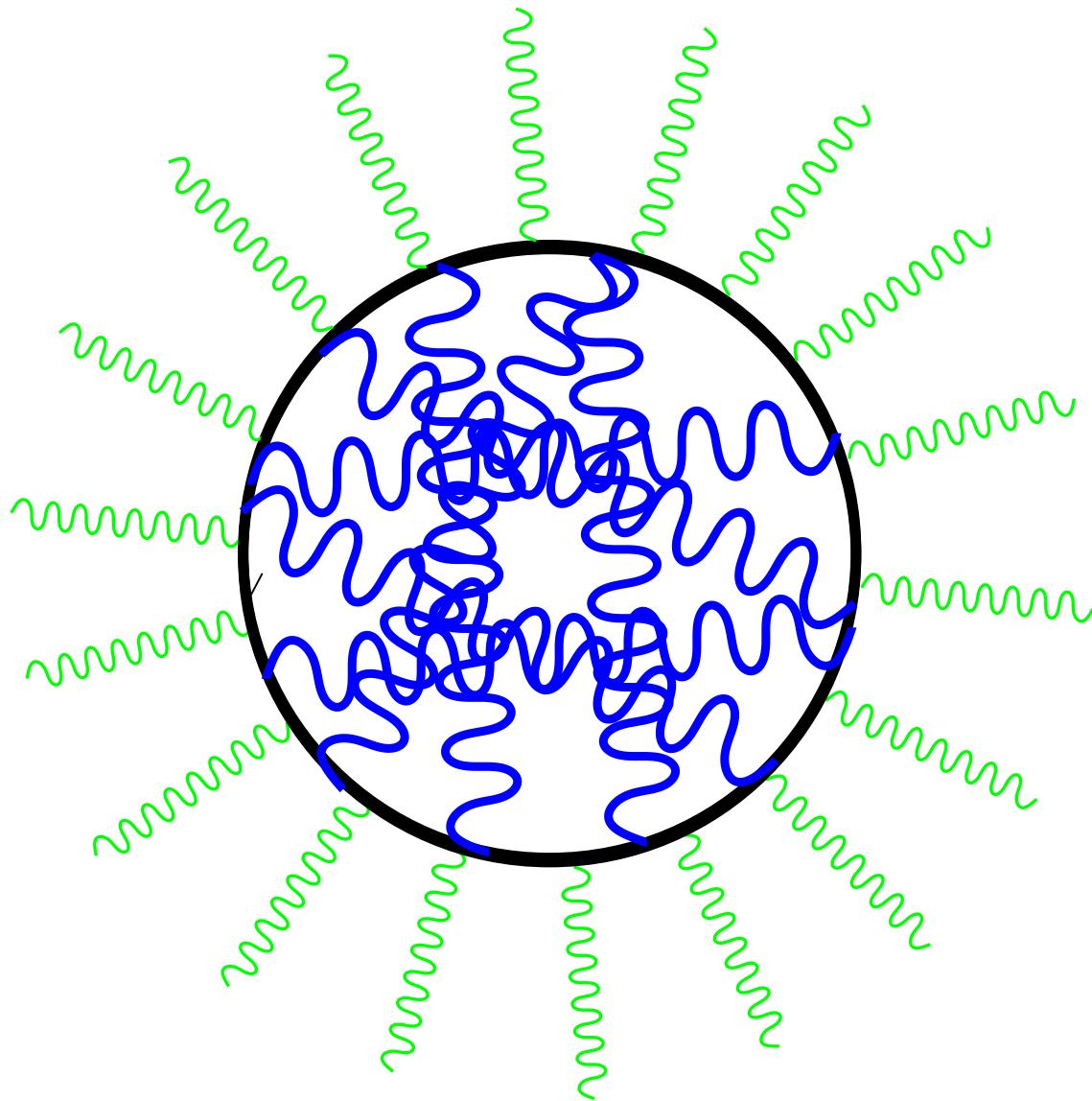


- ▷ Worldline self-interaction potential

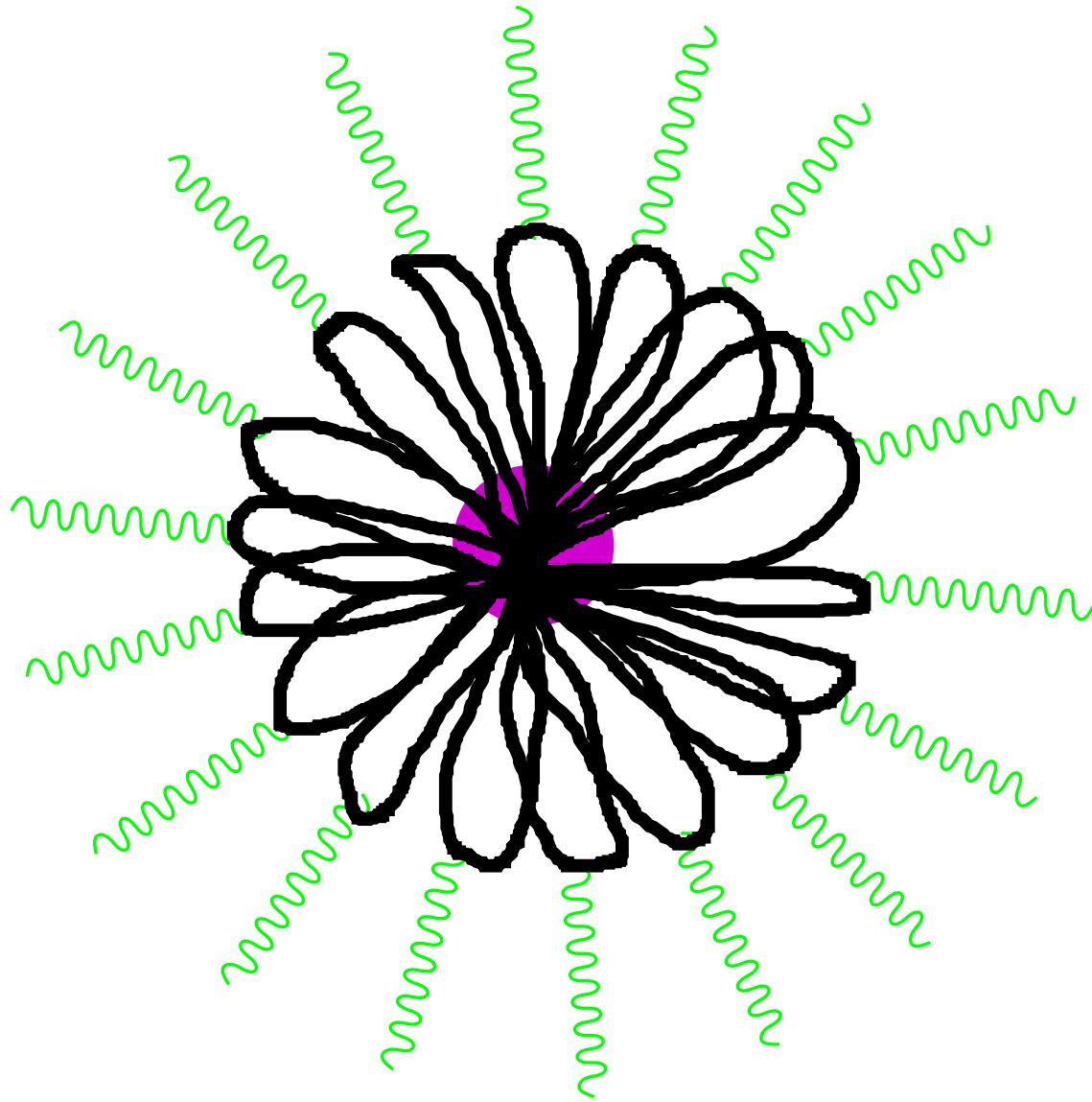
$$g := \frac{h^2}{8\pi^{D/2}} \Gamma\left(\frac{D-2}{2}\right)$$

$$g V[\mathbf{x}] := \frac{h^2}{2} \int d\tau_1 d\tau_2 \Delta(\mathbf{x}(\tau_1), \mathbf{x}(\tau_2)) \equiv \frac{h^2}{8\pi^{D/2}} \Gamma\left(\frac{D-2}{2}\right) \int_0^T d\tau_1 d\tau_2 \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|^{D-2}}$$

# Self-interaction potential



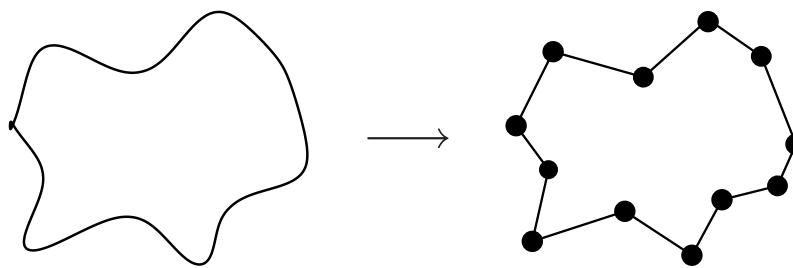
# Self-interaction potential



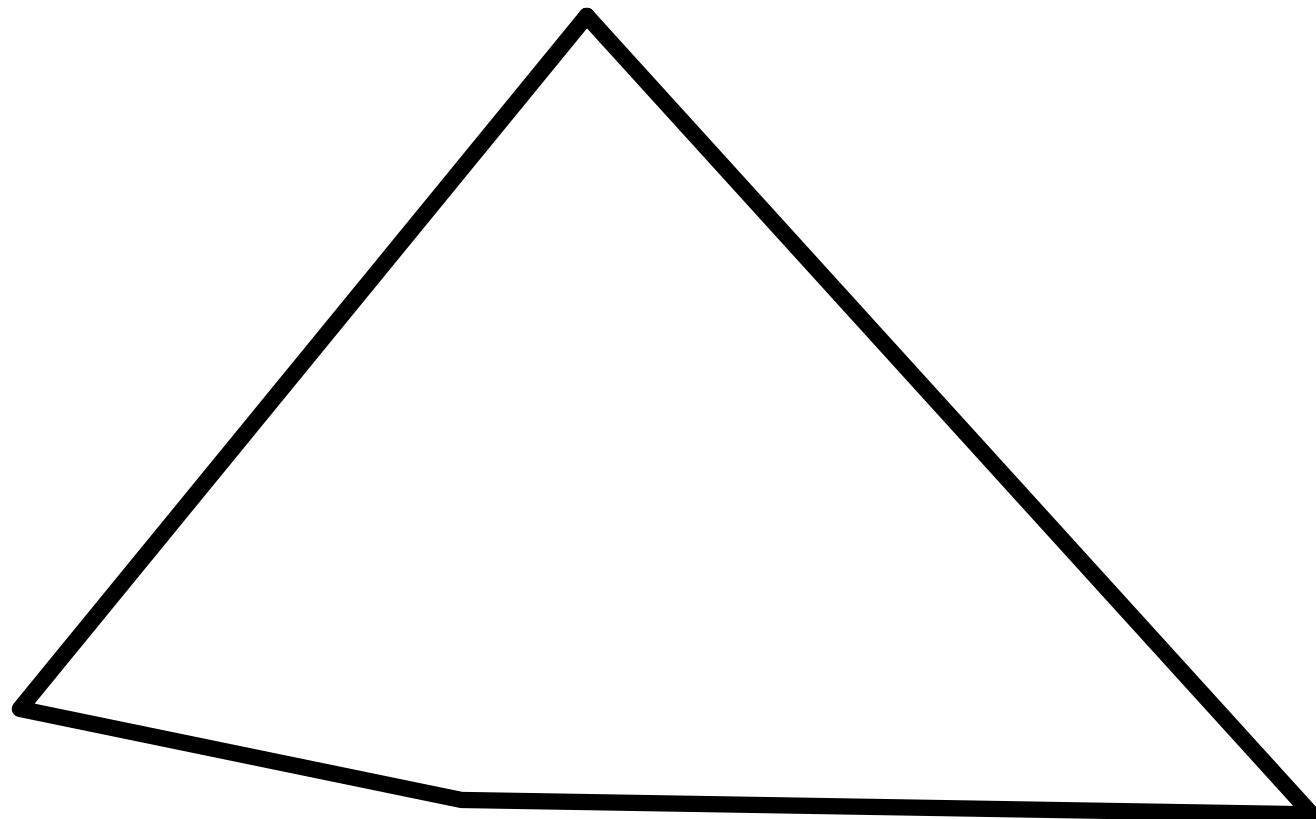
# Worldline numerics

$$\int_{\mathbf{x}(1)=\mathbf{x}(0)} \mathcal{D}\mathbf{x}(t) \longrightarrow \sum_{l=1}^{n_L}, \quad n_L = \# \text{ of worldlines}$$

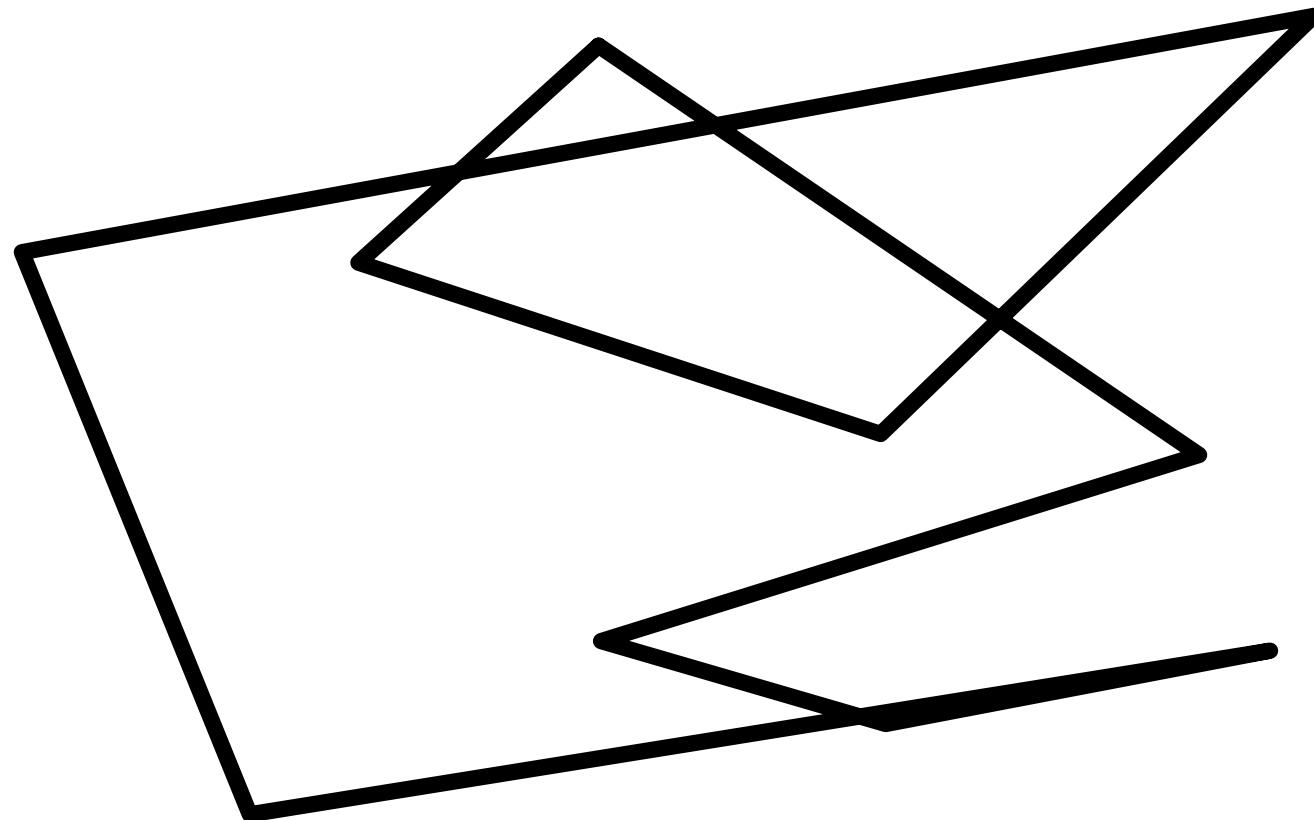
$$\mathbf{x}(t) \longrightarrow \mathbf{x}_i, \quad i = 1, \dots, N \text{ (ppl)}$$



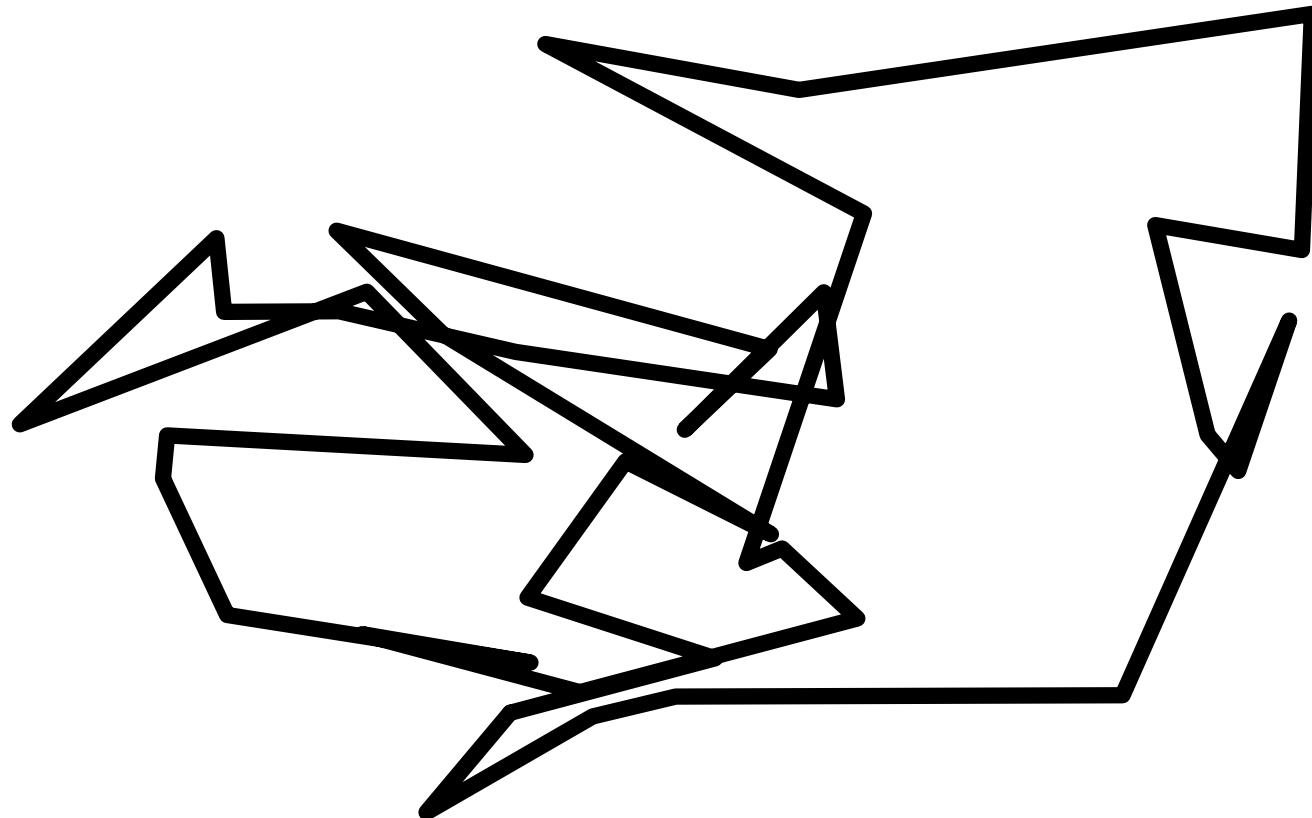
For instance,  $N = 4$  ppl



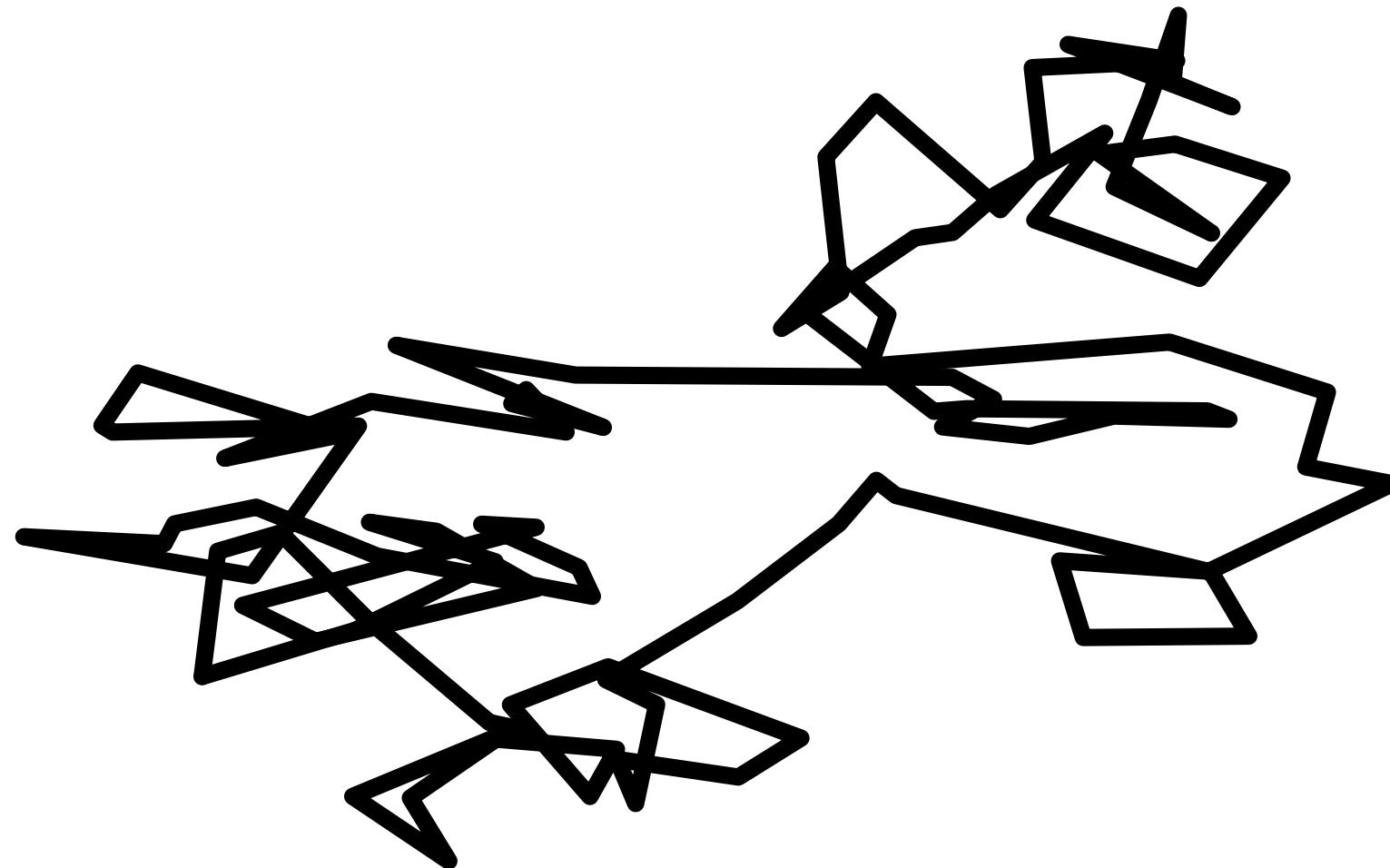
For instance,  $N = 10$  ppl



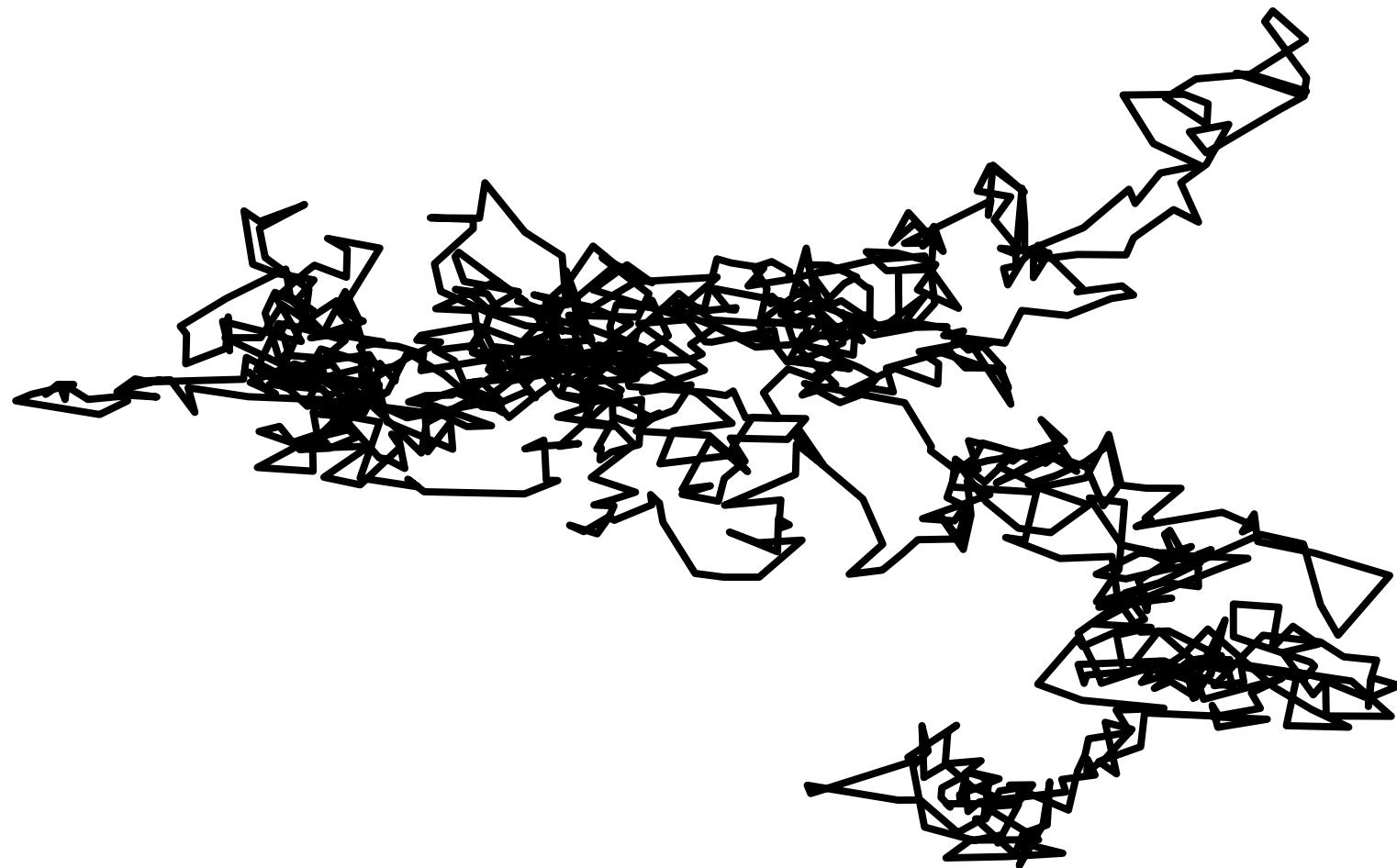
For instance,  $N = 40$  ppl



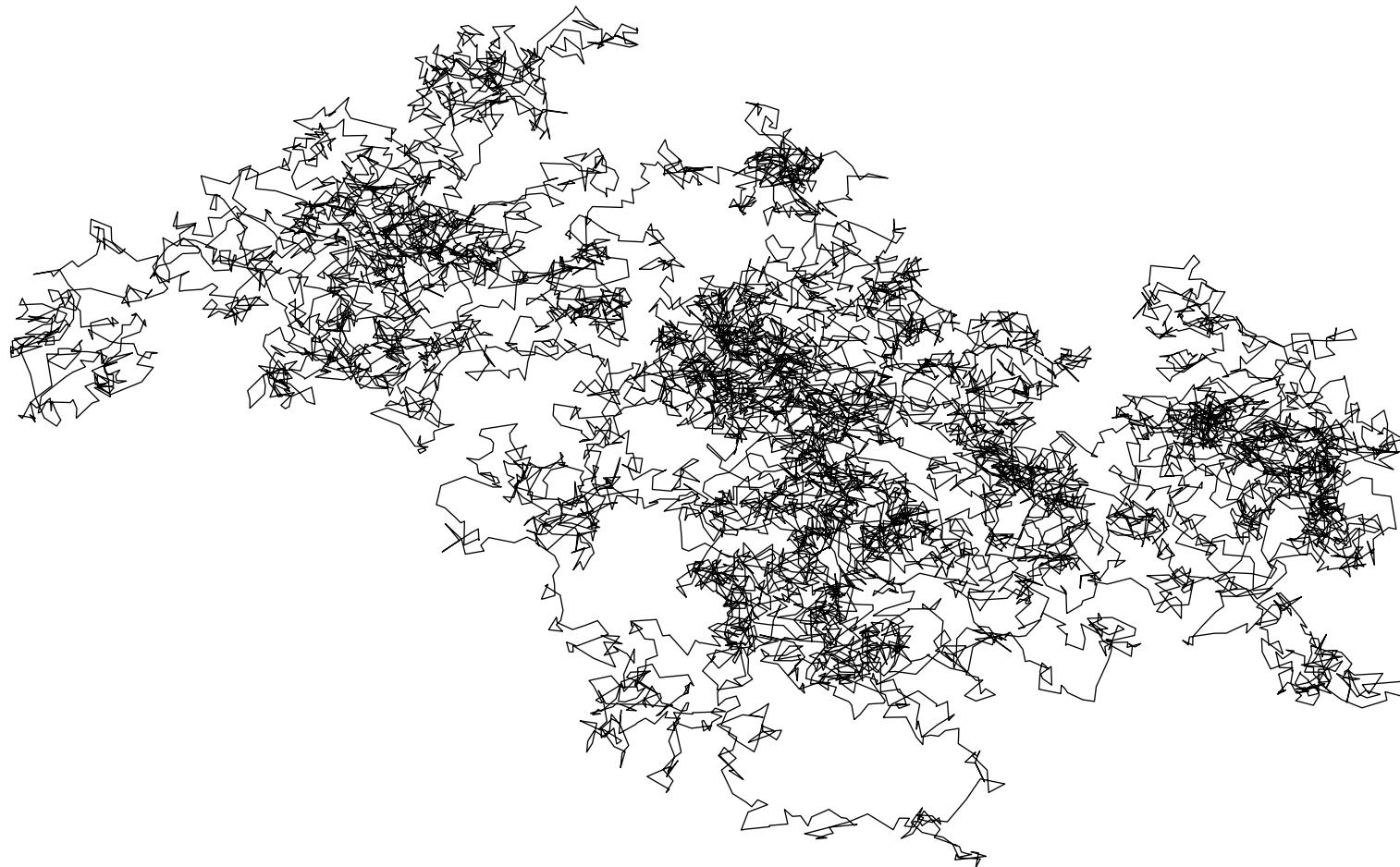
For instance,  $N = 100$  ppl



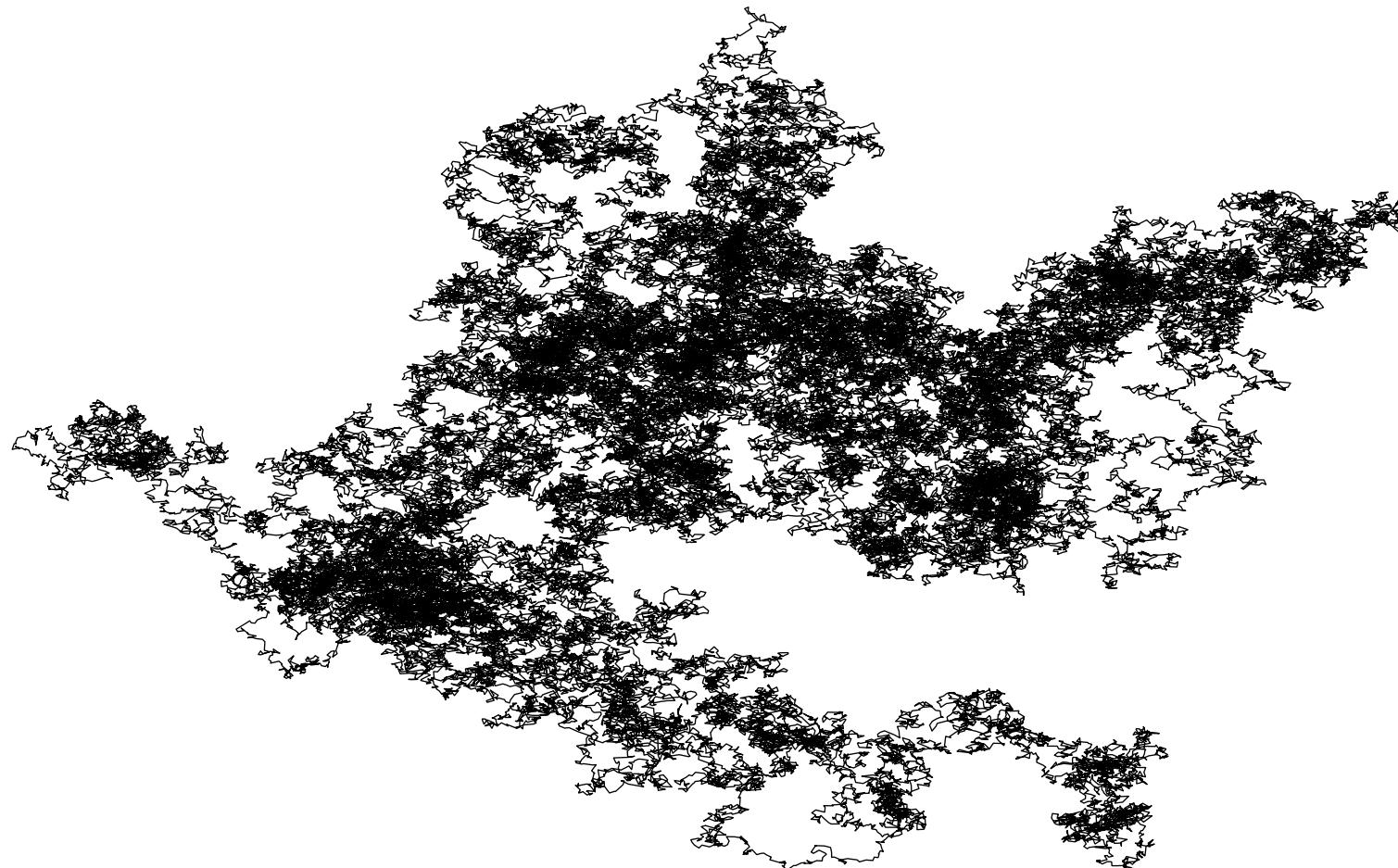
For instance,  $N = 1000$  ppl



For instance,  $N = 10000$  ppl



For instance,  $N = 100000$  ppl

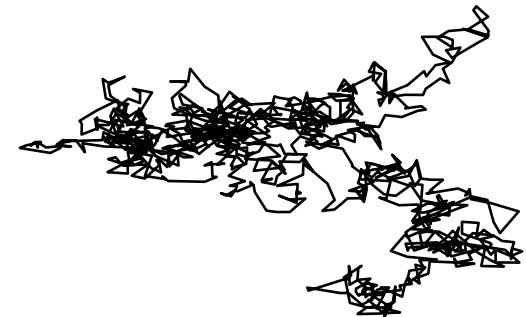


# Numerical challenges

- ▷ discretization ( $\textcolor{blue}{x}(\tau) = \sqrt{T} \textcolor{blue}{y}(T t)$ ,  $t \in [0, 1]$ ):

$$V[\textcolor{blue}{x}] = T^{3-D/2} \int_0^1 dt_1 dt_2 \frac{1}{|\textcolor{blue}{y}_1 - \textcolor{blue}{y}_2|^{D-2}} =: T^{3-D/2} v[\textcolor{blue}{y}]$$

$$v[\textcolor{blue}{y}] \rightarrow \frac{1}{N^2} \sum_{i,j} \frac{1}{|\textcolor{blue}{y}_i - \textcolor{blue}{y}_j|^{D-2}}, \quad , i, j = 0, \dots N,$$



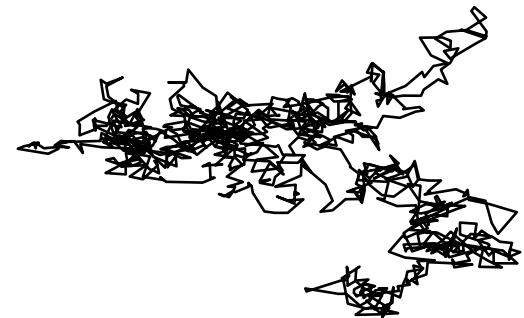
$\implies i = j: v[\textcolor{blue}{y}] \rightarrow \infty !$

regularization required

# Numerical challenges

- ▷ discretization

$$v[\mathbf{y}] \rightarrow \frac{1}{N^2} \sum_{i \neq j} \frac{1}{|\mathbf{y}_i - \mathbf{y}_j|^{D-2}}, \quad , i, j = 0, \dots N,$$



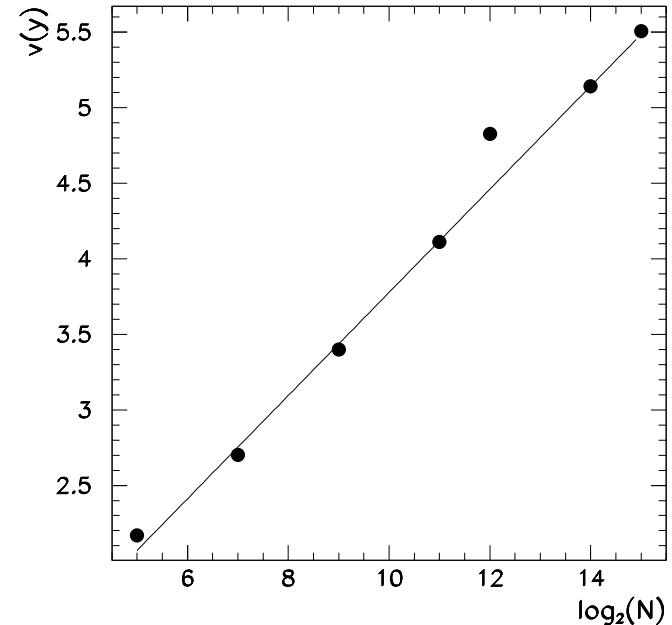
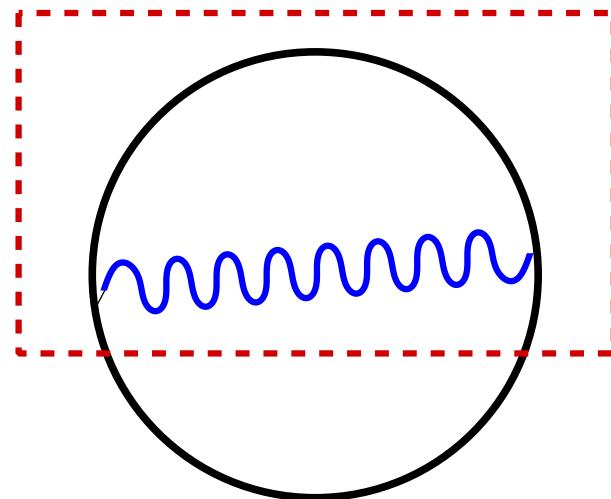
- ▷ UV divergence → divergence with  $N$

## Numerical challenges II

- ▷ worldline average of  $v[\textcolor{blue}{y}]$

$$\langle v[\textcolor{blue}{y}] \rangle = a + b \ln N, \quad a \simeq 0.363, b \simeq 0.341,$$

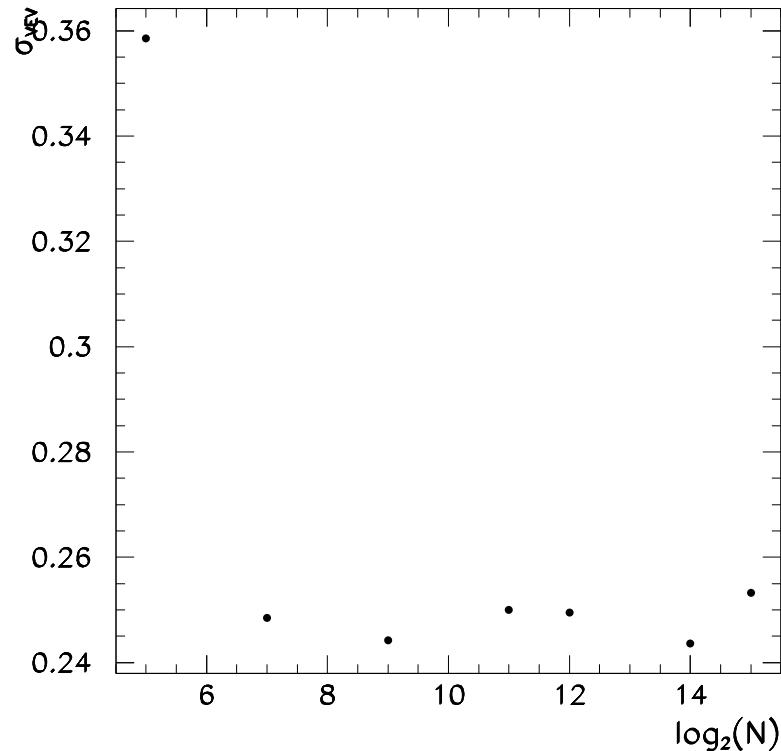
- ▷ UV divergence from mass operator



# Numerical challenges II

- ▷ root mean square of  $v[\textcolor{blue}{y}]$

$$\sigma^2 := \langle v^2[\textcolor{blue}{y}] \rangle - \langle v[\textcolor{blue}{y}] \rangle^2 \rightarrow \text{const.}$$



➡ super-renormalizability

(no independent divergencies in mass operator at higher loops)

## Numerical challenges III

- ▷ for instance ( $D = 4$ )

$$\left\langle e^{-\textcolor{violet}{g}^T v[\textcolor{blue}{y}]} \right\rangle \Big|_{\textcolor{violet}{g}, T, N^{\text{fixed}}} \simeq 0.3 \pm 0.5$$

⇒ only a few worldlines dominate the expectation value !

... “overlap problem?”



# Harmonic oscillator

▷ propagator

$$G(T, R) = \int_{x(0)=x_l}^{x(T)=x_F} \mathcal{D}\mathbf{x} \exp\left\{-\frac{1}{2} \int_0^T d\tau \dot{\mathbf{x}}^2 + \omega^2 \mathbf{x}^2\right\}, \quad R = |x_F - x_l|.$$

▷ persistence amplitude

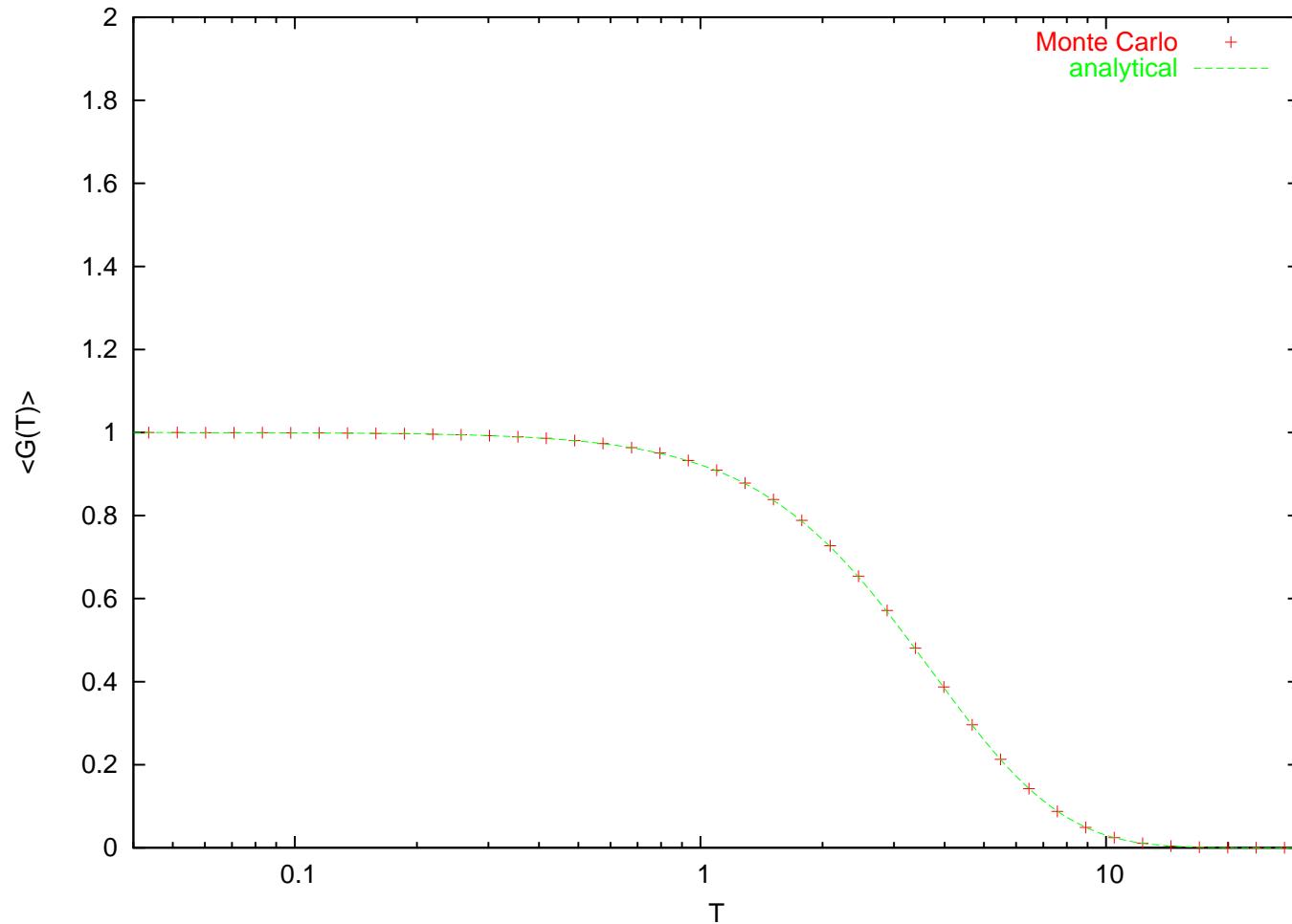
$$G(T, R=0) = \left\langle \exp\left[-\frac{1}{2}\omega^2 T^2 v_2[\mathbf{y}]\right] \right\rangle_{\mathbf{y}} \stackrel{\text{exact}}{=} \sqrt{\frac{\omega T}{\sinh \omega T}}$$

▷ “self-interaction” potential

$$v_2[\mathbf{y}] := \int_0^1 dt \mathbf{y}^2$$

# Harmonic oscillator

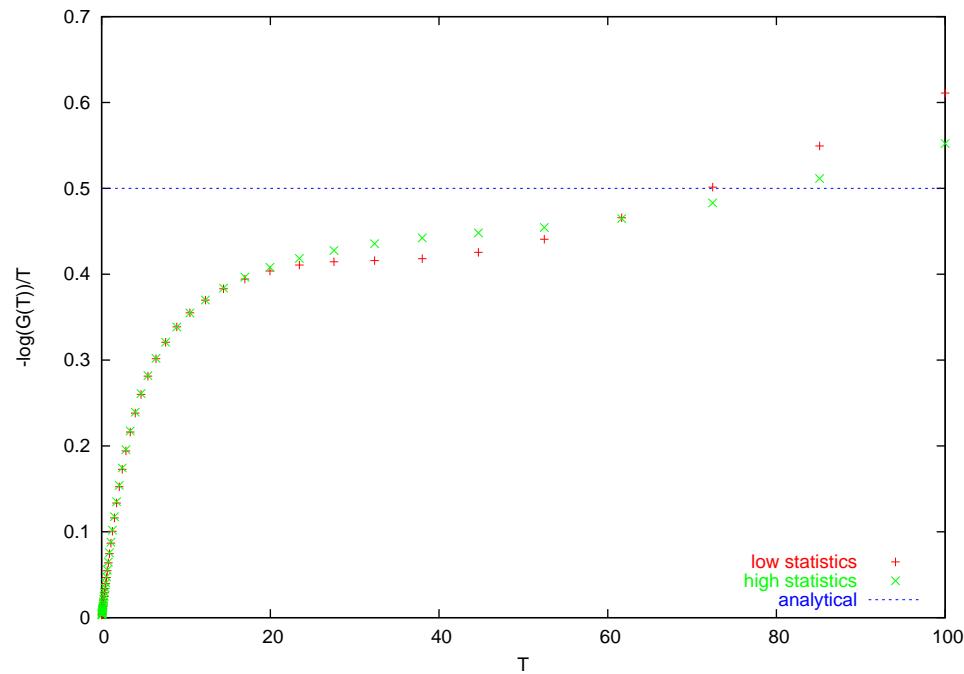
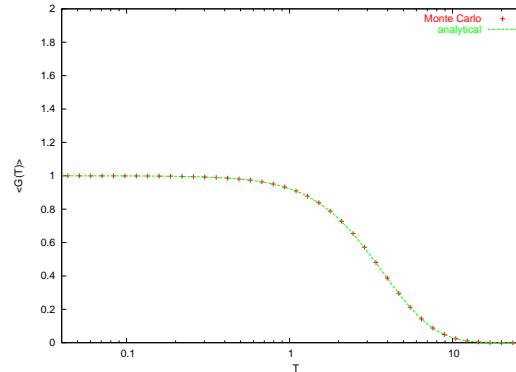
▷ persistence amplitude



# Harmonic oscillator

- ▷ ground state energy:

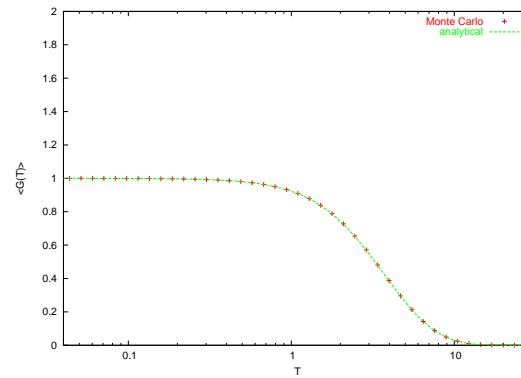
$$E_0 = \lim_{T \rightarrow \infty} -\frac{\log(G(T))}{T}$$



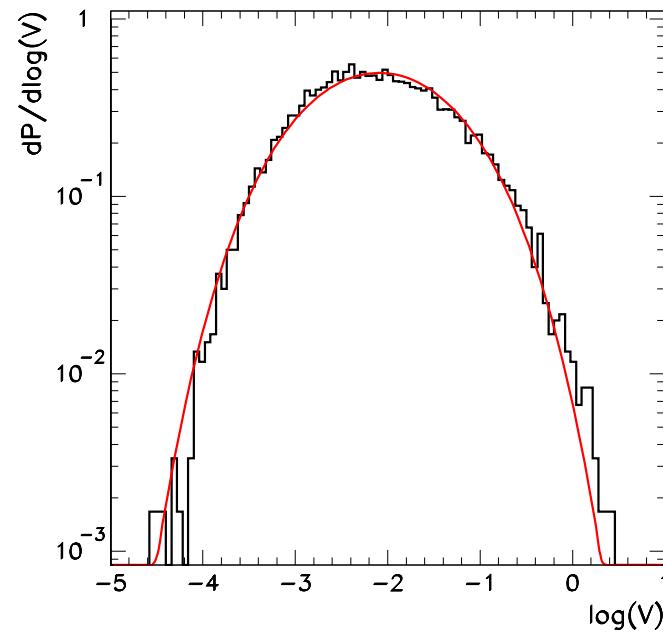
# Harmonic oscillator

- ▷ “classical collaps”:

$$G(T, R = 0) = \left\langle \exp \left[ -\frac{1}{2} \omega^2 T^2 v_2[\mathbf{y}] \right] \right\rangle_{\mathbf{y}}$$



- ▷ large  $T$  behavior dominated by small  $v_2[\mathbf{y}]$
- ⇒ bad (“overlap”) statistics for small  $v_2[\mathbf{y}]$



# Harmonic oscillator

- ▷ solution: probability distribution function (PDF)

$$P(v_2) = \mathcal{N} \int \mathcal{D}\mathbf{y} \delta(v_2 - v_2[\mathbf{y}]) e^{-\int \frac{\mathbf{y}^2}{2}}, \quad \int dv_2 P(v_2) \equiv 1$$

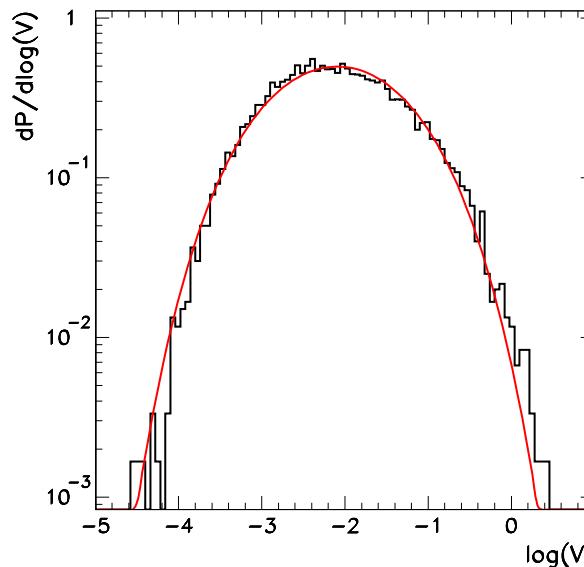
- ▷ propagator:

$$G(T, R = 0) = \int dv_2 P(v_2) e^{-\frac{1}{2}\omega^2 T^2 v_2} \equiv \left\langle \exp \left[ -\frac{1}{2}\omega^2 T^2 v_2[\mathbf{y}] \right] \right\rangle_{\mathbf{y}}$$

- ▷ PDF fit

$$P_{\text{ansatz}}(v_2) = \mathcal{N} e^{-(\mathbf{a}/v_2 + \mathbf{b}v_2)}$$

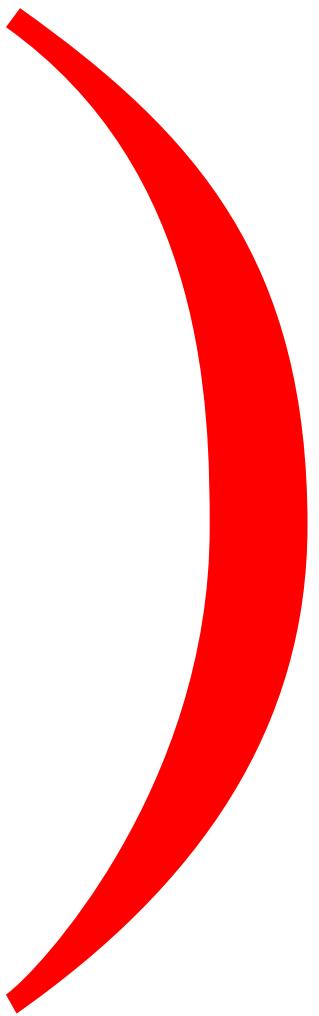
$$\implies \mathbf{a} \simeq 0.11, \mathbf{b} \simeq 18.4$$



## Harmonic oscillator

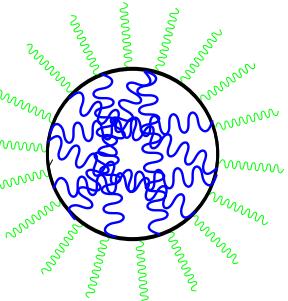
- ▷ ground state energy

$$E_0 = \sqrt{2} \textcolor{teal}{a} \omega \simeq 0.47\omega$$



# Quenched effective action

- ▷ soft-photon effective action,  $A \simeq \text{const.}$  ( . . . á la Heisenberg-Euler)

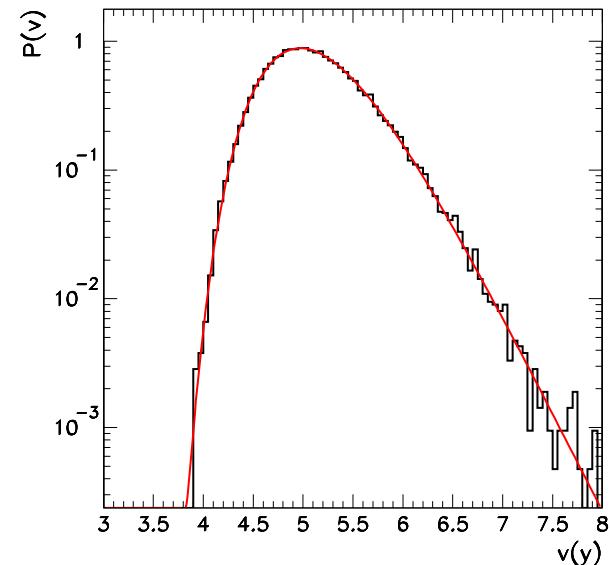
$$\Gamma_{\text{QA}}[A] = -\frac{1}{2(4\pi)^{D/2}} \int_0^\infty \frac{dT}{T^{1+D/2}} e^{-m^2 T} e^{i h A T} \left\langle e^{-g T v[y]} \right\rangle_y =$$


$$\left\langle e^{-g T v[y]} \right\rangle_y = \int dv P(v) e^{-g T v}$$

- ▷ PDF fit, e.g.,

$$P(v) = \frac{\beta^{1+\alpha}}{\Gamma(\alpha+1)} (v - v_0)^\alpha e^{-\beta(v-v_0)} \theta(v - v_0)$$

$$\alpha \simeq 0.79, \quad \beta \simeq 13.2, \quad v_0 \simeq 0.34 \ln N + 0.23$$



# Renormalization

- ▷ renormalization conditions ( . . . no divergent  $\delta Z$ 's)

$$\underbrace{\frac{\delta \Gamma[A=0]}{\delta A(x)}}_{\text{"no tadpole"}} = 0, \quad \underbrace{\frac{\delta^2 \Gamma[A=0]}{\delta A(-p) \delta A(p)} \Big|_{p^2=0}}_{\text{photon mass }=0} = 0$$

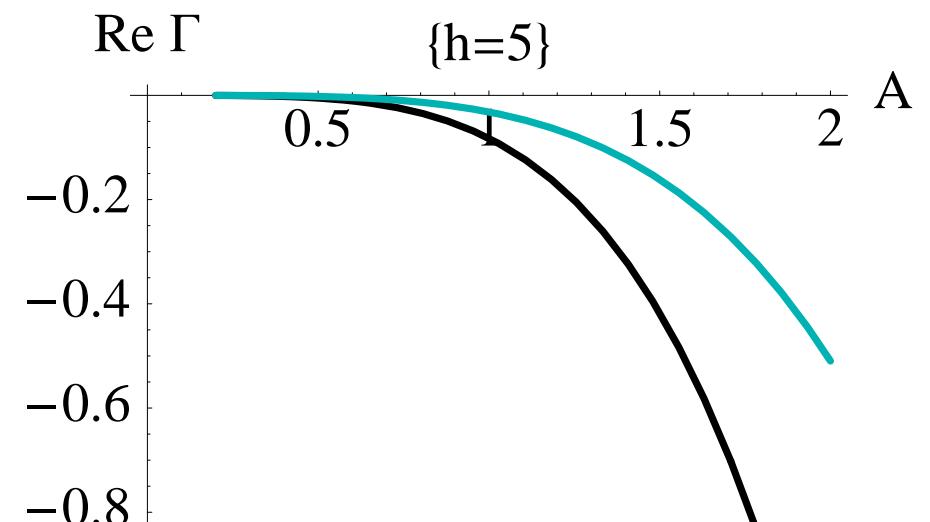
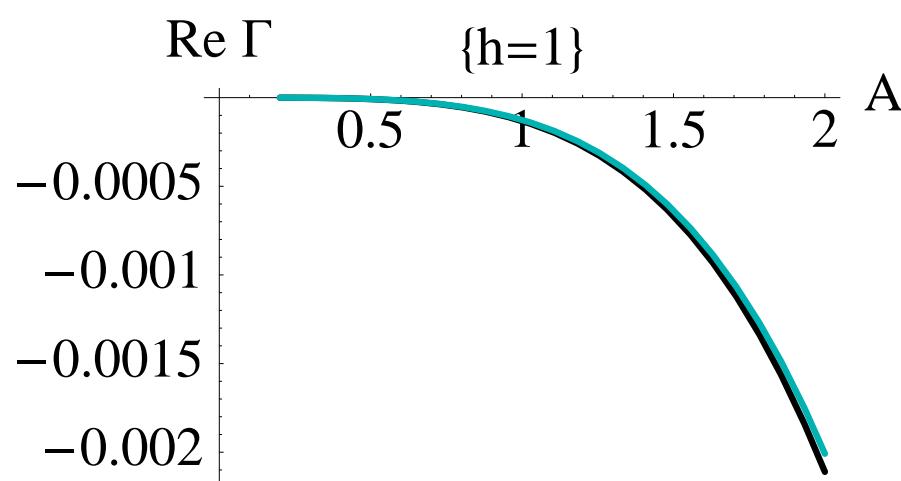
- ▷  $\phi$  mass renormalization

$$\left. \text{T integrand} \right|_{T \rightarrow \infty} \sim f(T) e^{-m_R^2 T} \quad (\text{instead of } G_\phi(p^2 \rightarrow 0)^{-1} = m_R^2)$$

$$\implies m_R^2 = m^2 + \frac{\hbar^2}{8\pi^2} v_0 \simeq m^2 + \frac{\hbar^2}{8\pi^2} (0.34 \ln N + 0.23)$$

# Renormalized effective action

$$\begin{aligned}\Gamma_{\text{QA,R}}[\textcolor{red}{A}] = & -\frac{1}{32\pi^2} \int d^4x \int_0^\infty \frac{dT}{T^3} e^{-m_R^2 T} \left( e^{ih\textcolor{red}{A}T} - 1 - i\textcolor{blue}{h}\textcolor{red}{A}T + \frac{(\textcolor{blue}{h}\textcolor{red}{A}T)^2}{2} \right) \\ & \times \left( \frac{\beta}{\beta + \frac{\textcolor{blue}{h}^2}{8\pi^2} T} \right)^{1+\alpha}\end{aligned}$$



# Massless Limit?

- ▷ one-loop small- $\phi$ -mass limit:

$$\Gamma_{\text{1-loop}}[A] \Big|_{\frac{hA}{m_R^2} \gg 1} \simeq -\frac{1}{64\pi^2} \int d^4x (\hbar A)^2 \ln \frac{\hbar A}{m_R^2}$$

- ▷ quenched small- $\phi$ -mass limit:

$$\Gamma_{\text{QA,R}}[A] \Big|_{m_R=0} = -\frac{[-\Gamma(-2-\alpha)] \cos \frac{\pi}{2}\alpha}{2^{5-3\alpha} \pi^{2(1-\alpha)} \beta^\alpha} \int d^4x (\hbar A)^2 \left(\frac{A}{\hbar}\right)^\alpha [1 + \mathcal{O}((A/\hbar))]$$

- ⇒ break-down of massless limit  $\sim$  artifact of perturbation theory

... large log's summable

# Conclusions . . .



Nonperturbative worldline numerics:

. . . a useful tool under development

. . . and Outlook



Wish list:

propagators

renormalizable theories

gauge invariant discretization

(quenched) all-order  $\beta$  functions