Theory Colloquium

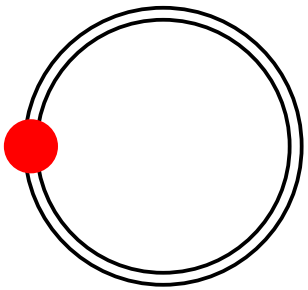
Renormalizability of Yang-Mills theories in extra dimensions

Holger Gies, Heidelberg U.

extra dimensional renormalization:

- Why ?
- How ?
- What ?

$$\partial_t \Gamma_k = \frac{1}{2}\partial_t R_k$$



(PRD 68:085015,2003, hep-th/0305208 & PRD 66:025006,2002, hep-th/0202207)

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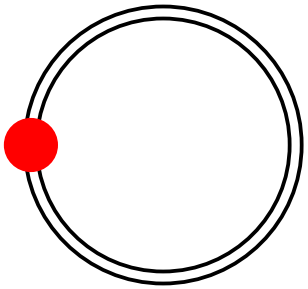

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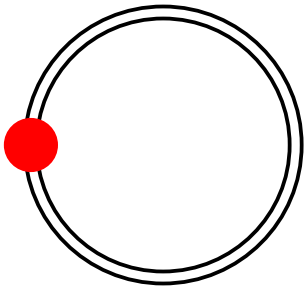

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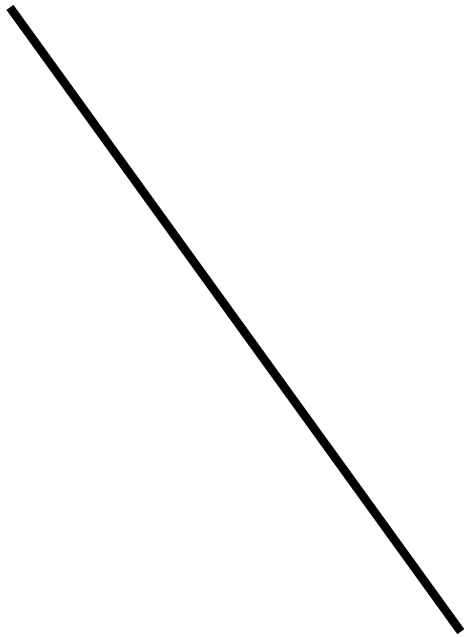
Why extra dimensions ?

- appear in abundance in string theory



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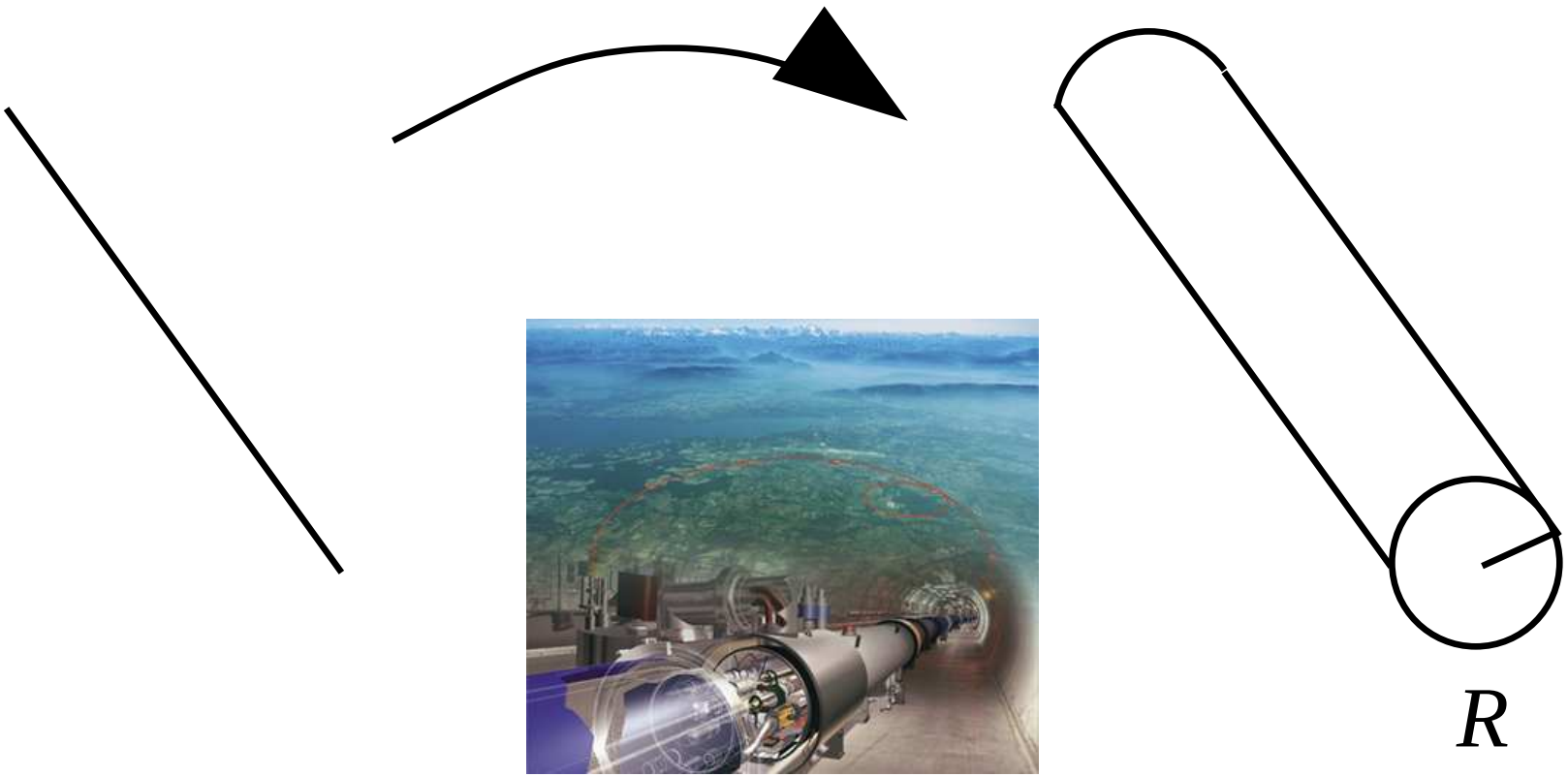

Why extra dimensions ?

- "explains" weakness of gravity

$$F_4 \sim -G\,\frac{m_1\,m_2}{r^2}$$

- $\hbar = c = 1$

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▷ "decompose" Planck scale: (Arkani-Hamed,Dimopoulos,Dvali'98)

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▷ for instance, $N = 6$: $(1/R) \sim 10$ TeV, $M_\star \sim 10^7$GeV

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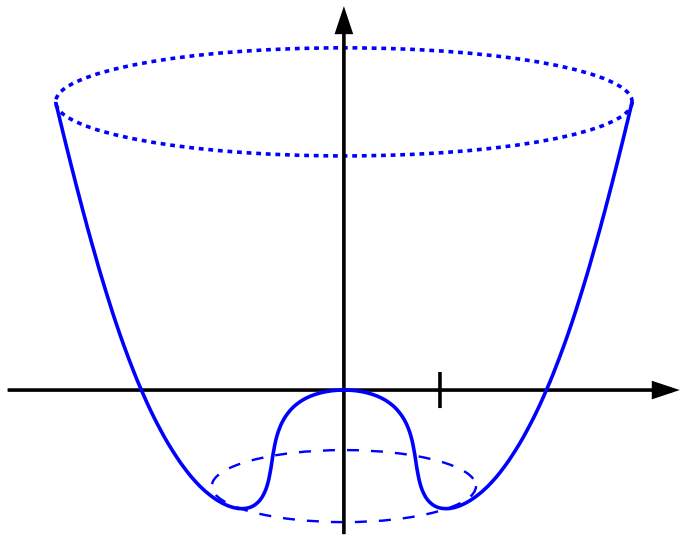
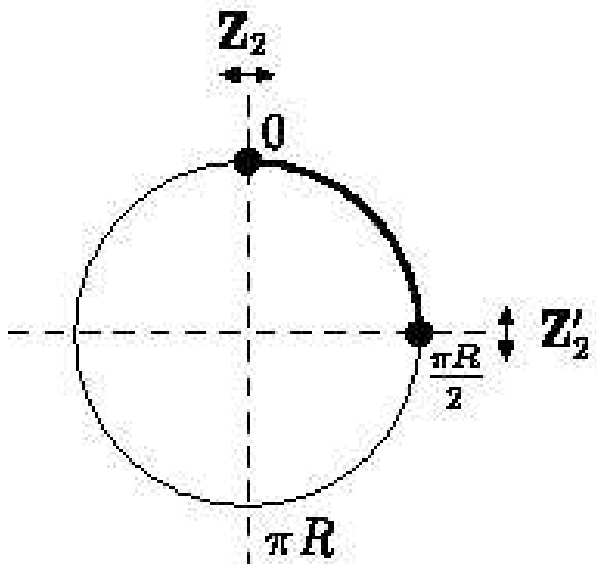
Why extra dimensions ?

▷ more natural symmetry-breaking mechanisms

$$\text{GUT} \longrightarrow SU(3)_{\text{color}} \times SU(2)_L \times U(1)_Y$$

Higgs mechanism vs. "orbifolding"





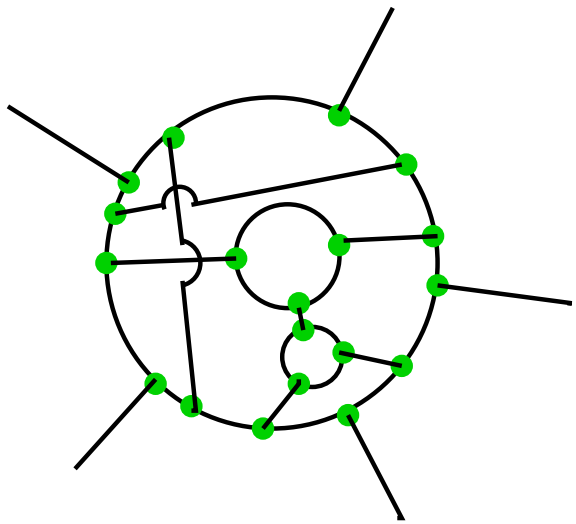
(Kawamura'99; Hebecker,March-Russell'02)

Perturbative QFT in extra dimensions

▷ perturbative expansion

$$\langle \phi \dots \phi \rangle = \sum_{n=0}^{\infty} a_n \, g^n$$

$a_n \quad =$



$\xrightarrow{D>4} \quad \infty$

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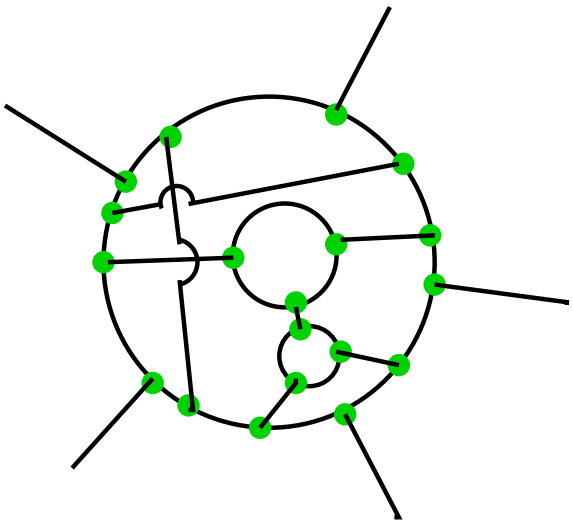
$$a_n \Big|_{\Lambda} =$$



$$\overset{D>4}{\sim} \quad \Lambda^p$$

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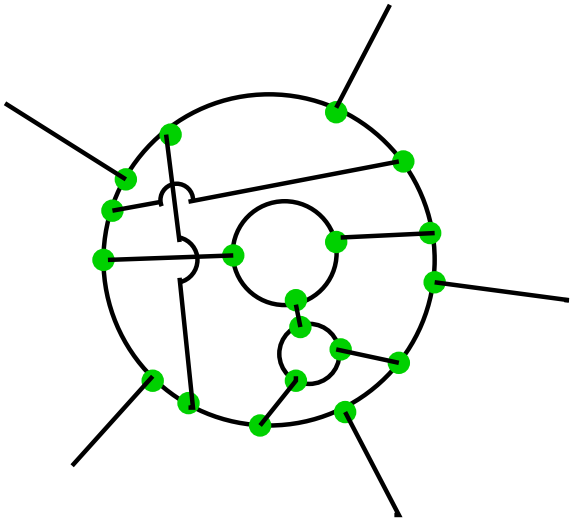

$$\xrightarrow{D>4} \quad \Lambda^p$$

Question: *"How many physical parameters do we have to fix for obtaining Λ-independent predicitions ?"*

Perturbative QFT in extra dimensions

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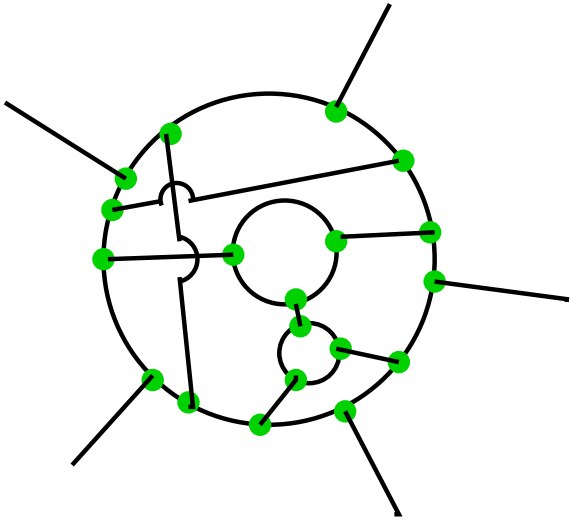

$$\xrightarrow{D>4} \quad \Lambda^p$$

Answer:

$$\#\text{ of physical parameters } = \Delta \; : \; \begin{cases} \Delta < \infty & \text{for } D = 4 \text{ (p. renormalizable)} \\ \Delta \to \infty & \text{for } D > 4 \text{ (p. nonrenormalizable)} \end{cases}$$

Scenarios with extra dimensions

string theory

↑

ExtraD pQFT

↑

$D = 4$ standard model

Scenarios with extra dimensions

string theory

↑

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↑

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g=0

"free theory"

(triviality)

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ExtraD

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QFT

?

↑

$D = 4$ standard model

Why renormalizability ?

- IR physics well separated from UV physics (cutoff Λ independence)
- $\#$ of physical parameters $\Delta < \infty$
- QFT should be predictive

$$\Longrightarrow \text{ realized in perturbative QFT}$$

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$\Longrightarrow$ realized in perturbative QFT

- beyond perturbation theory?

$\Longrightarrow$ Scenario of "Asymptotic Safety"

(Weinberg'76)

(Gell-Mann, Low'54)

Why extra dimensional renormalizability ?

coincidence

$D = 4$	$D = 4$
spacetime manifold	critical dimension of pQFT

Why extra dimensional renormalizability ?

coincidence

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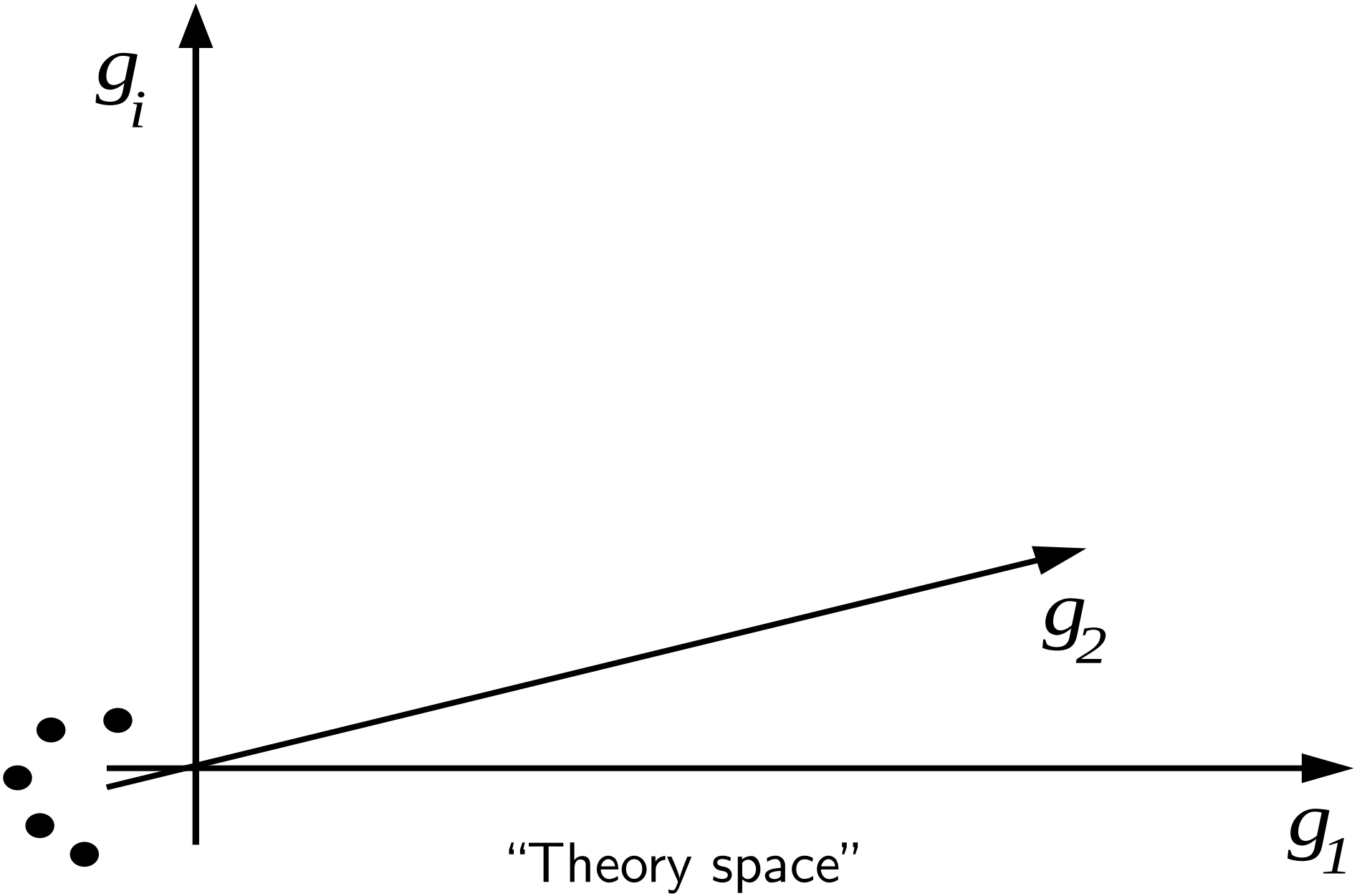
spacetime manifold

$D = 4$

critical dimension of pQFT

How ? Asymptotic Safety



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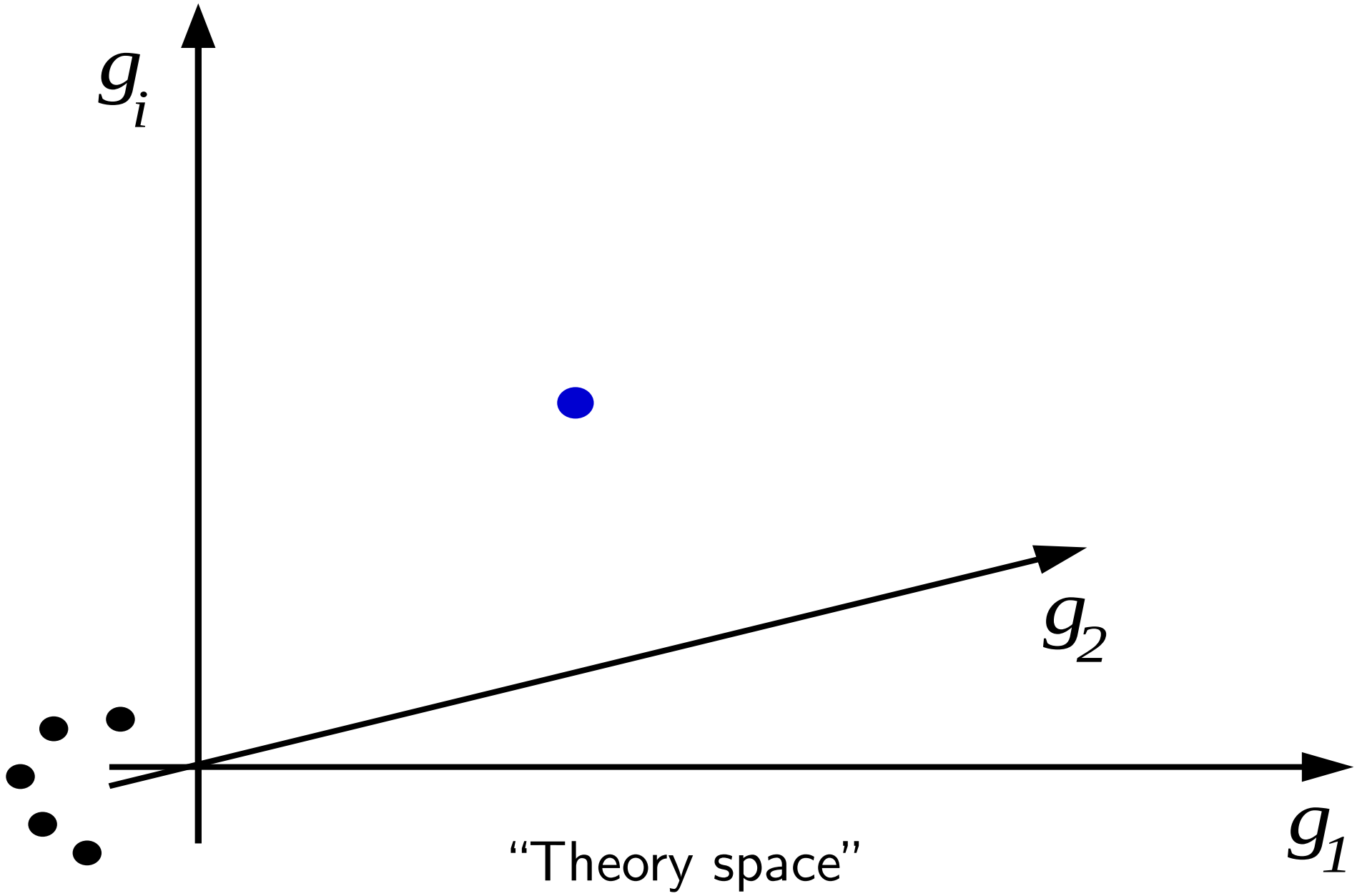

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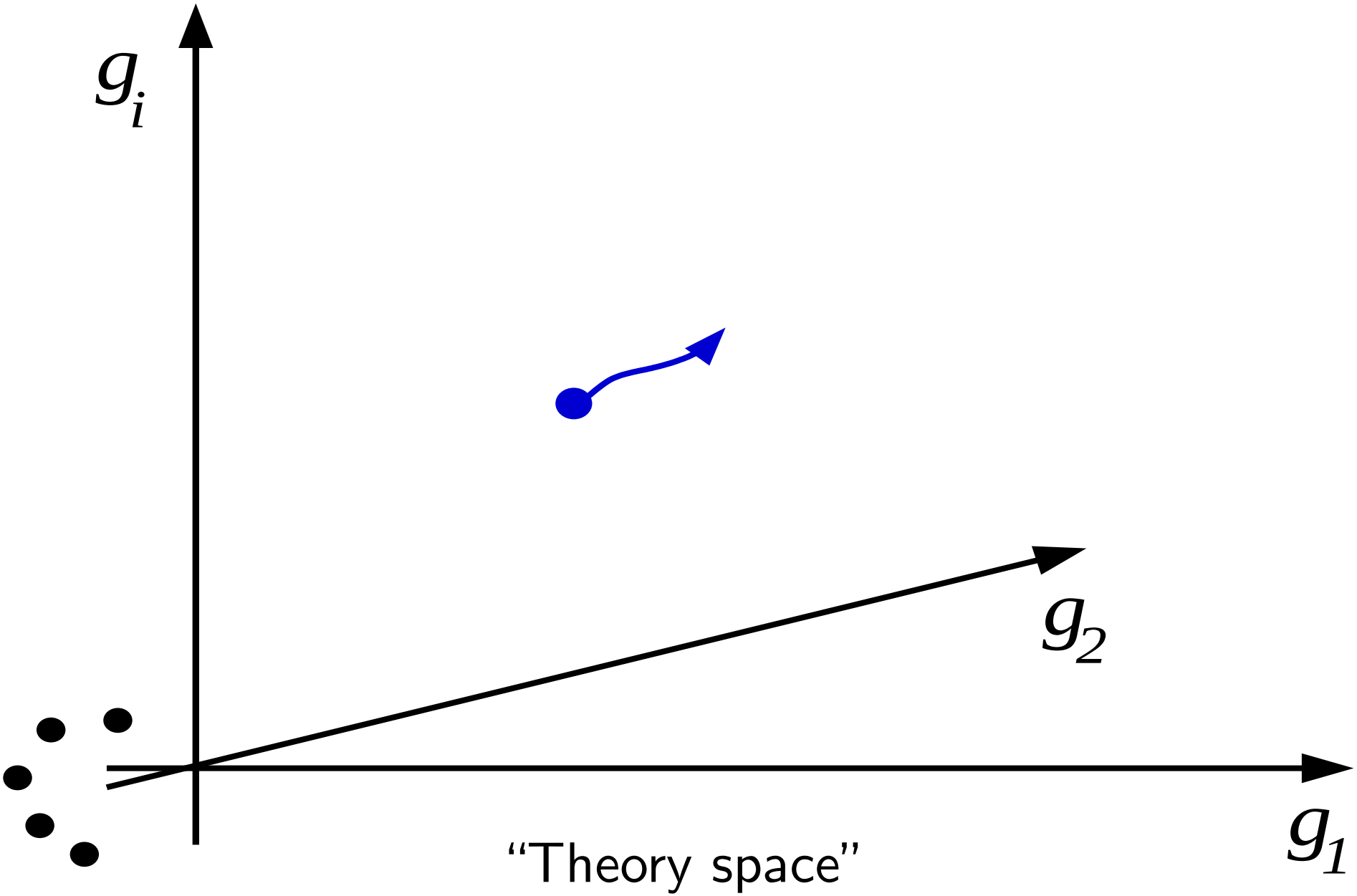

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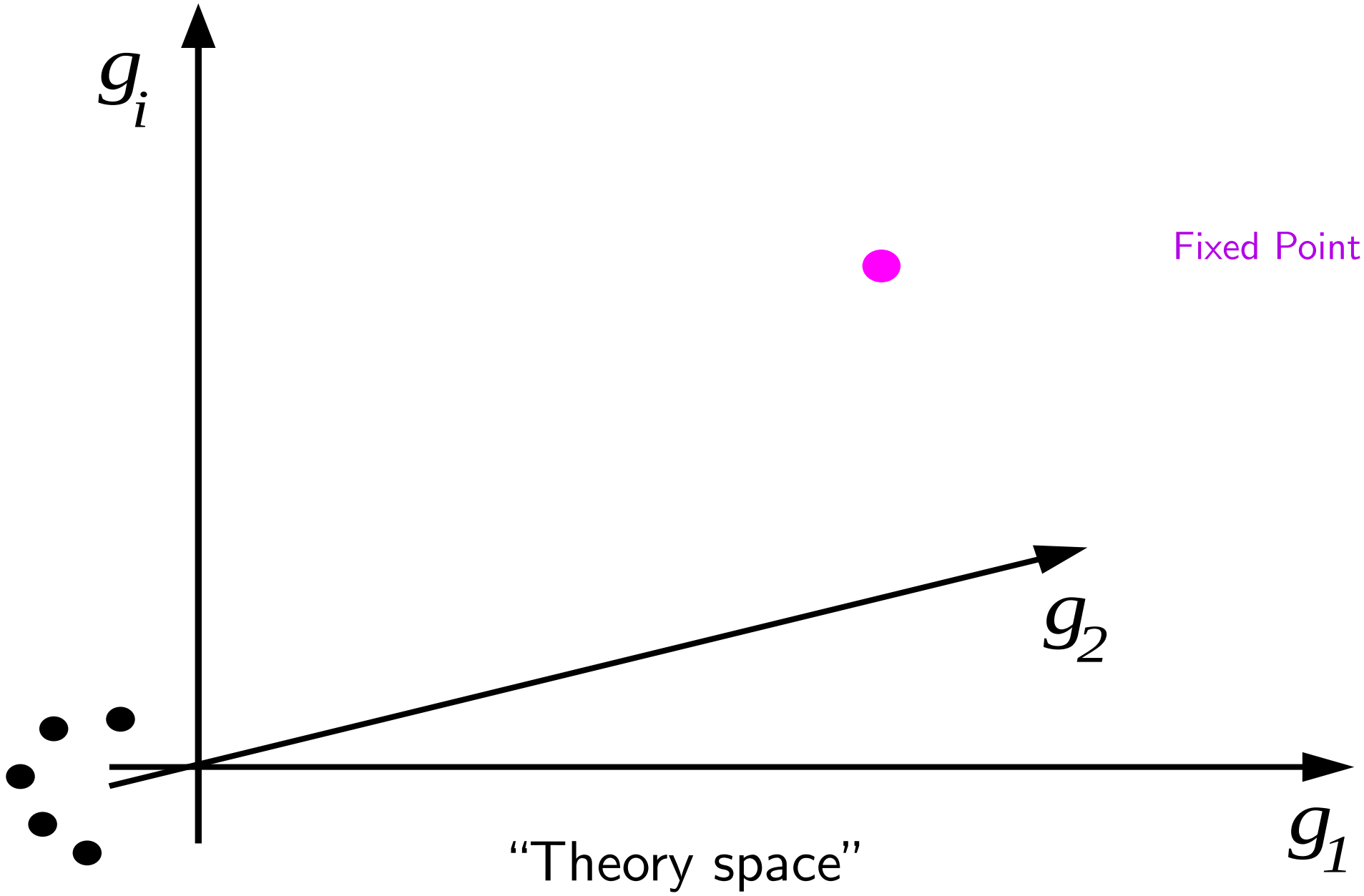

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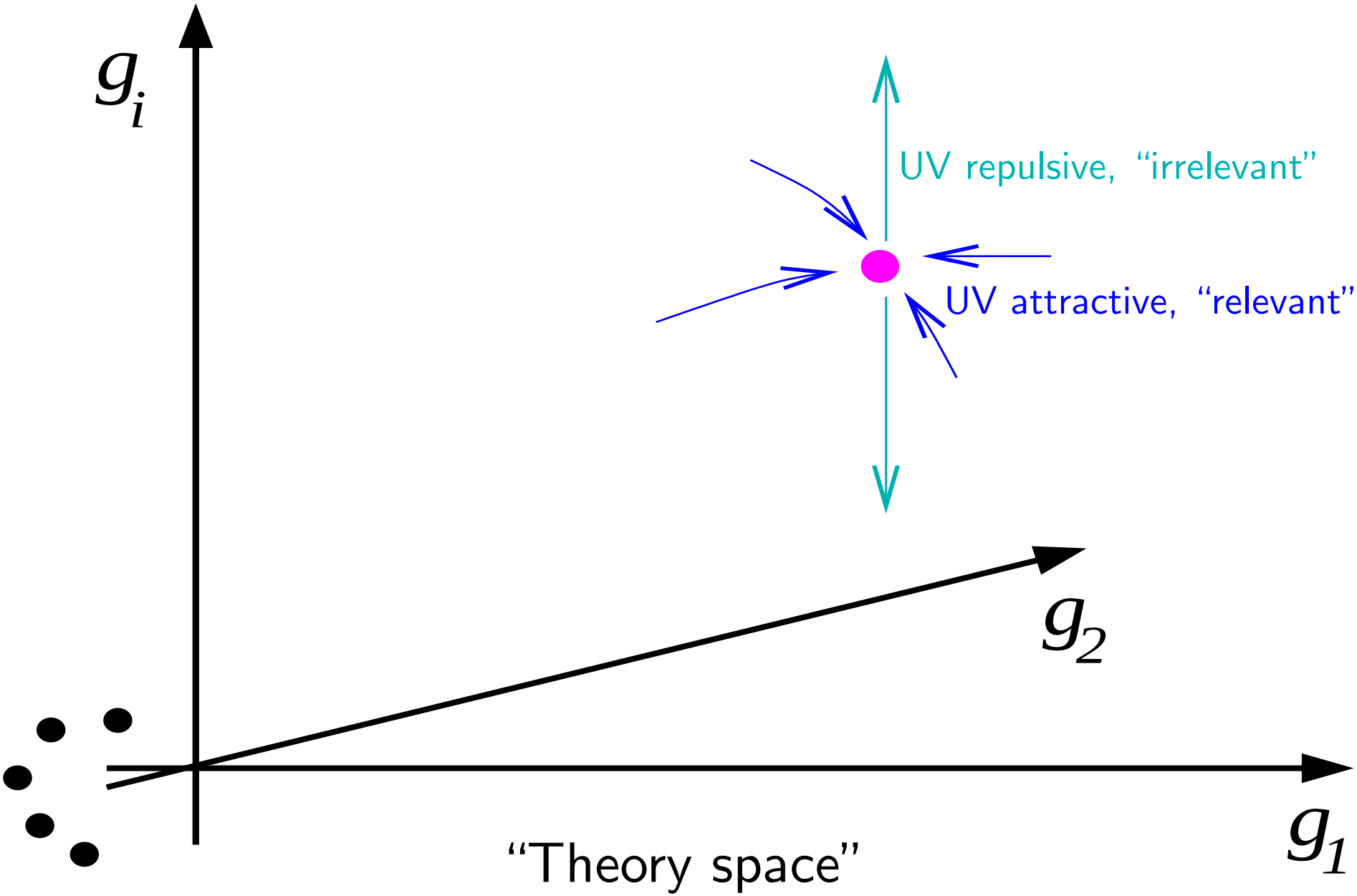

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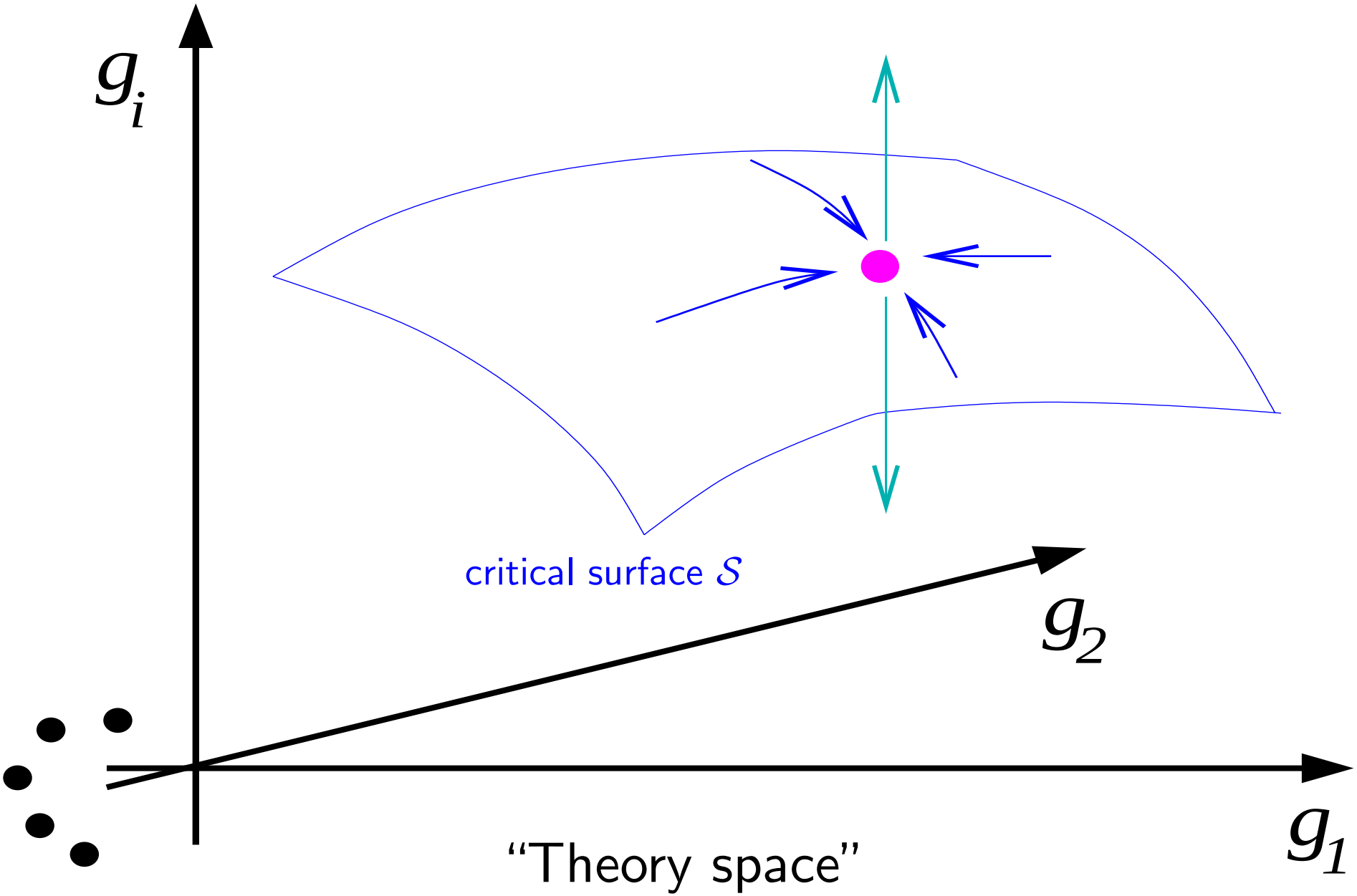

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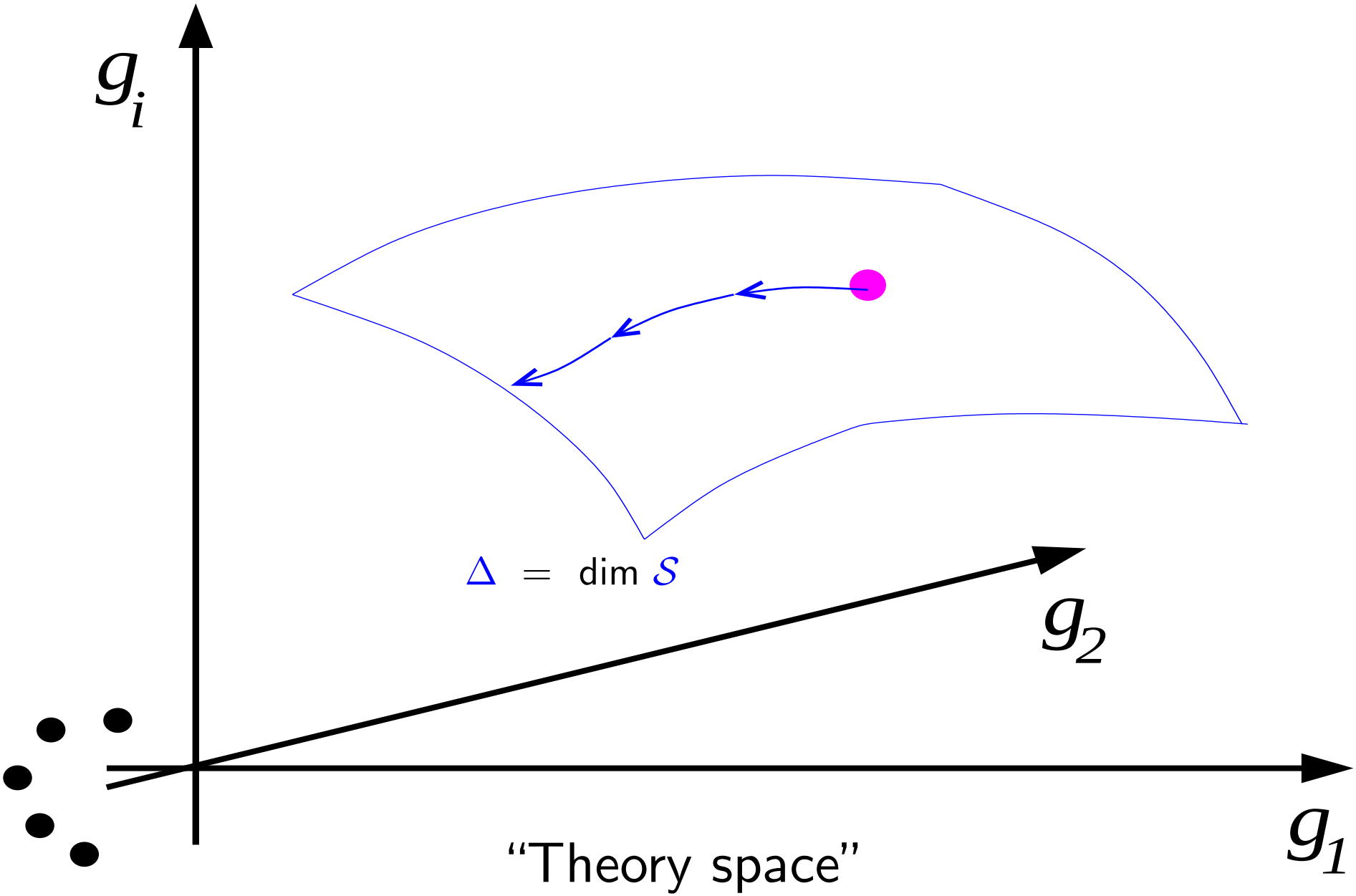

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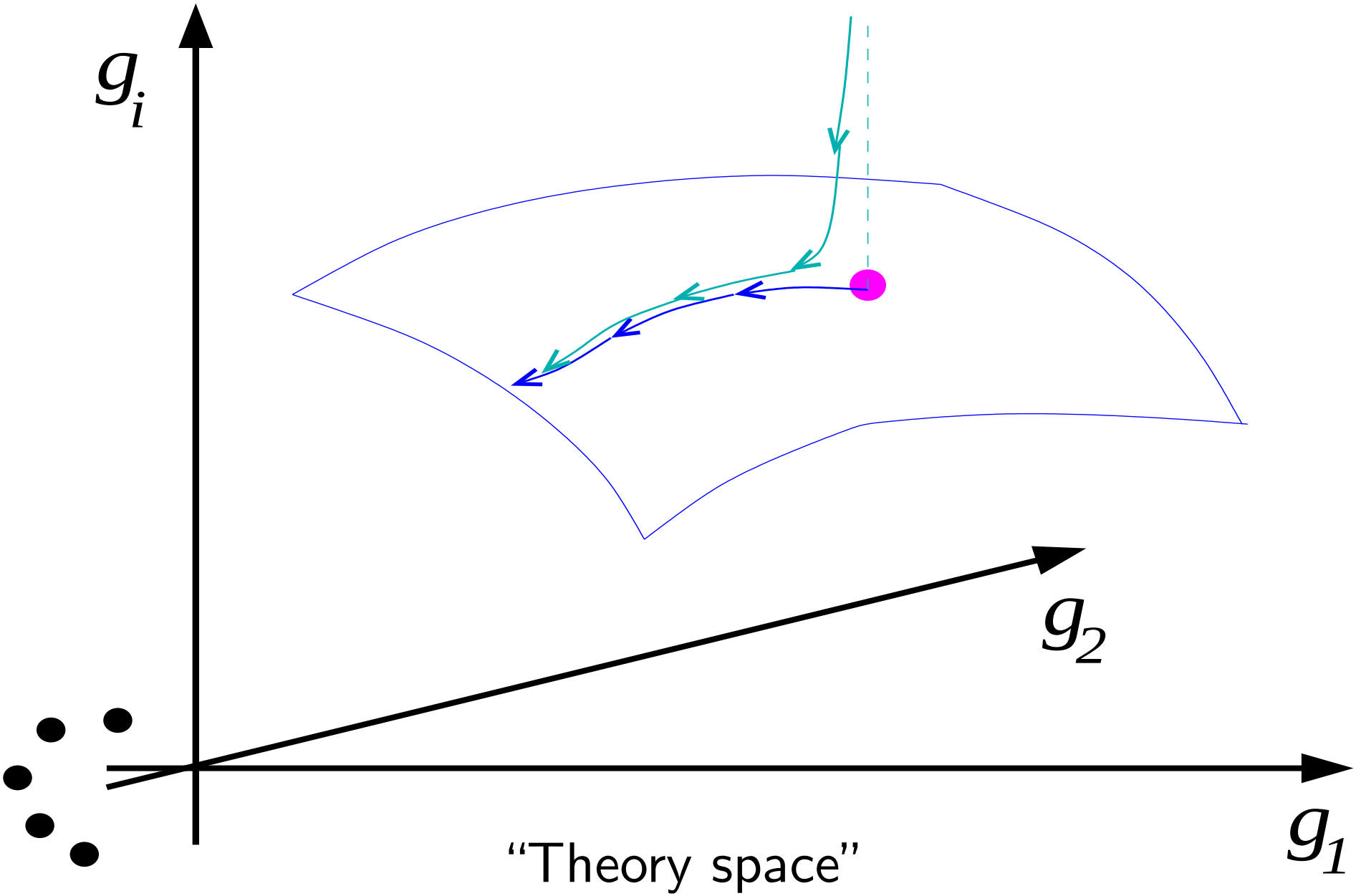

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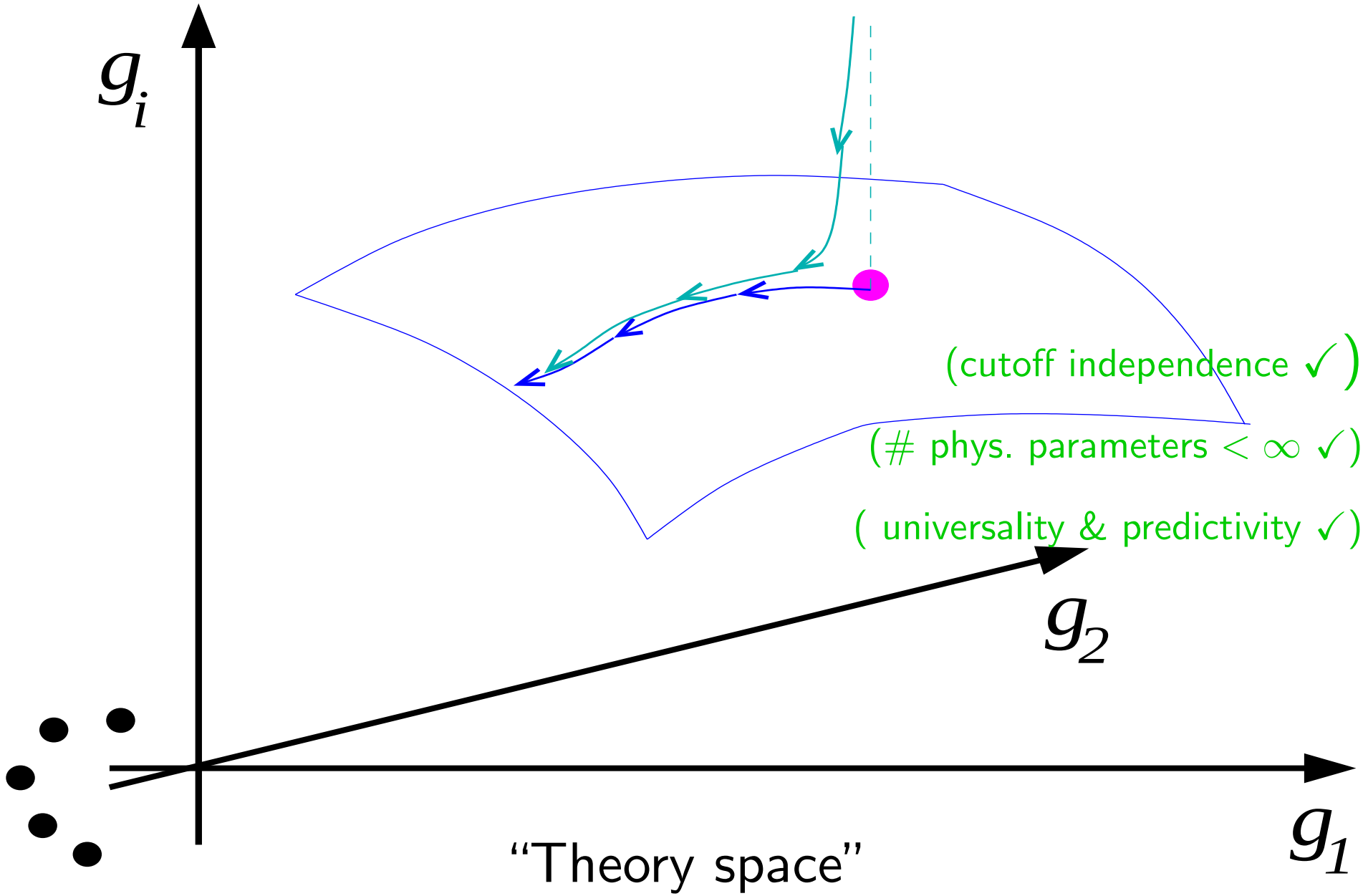

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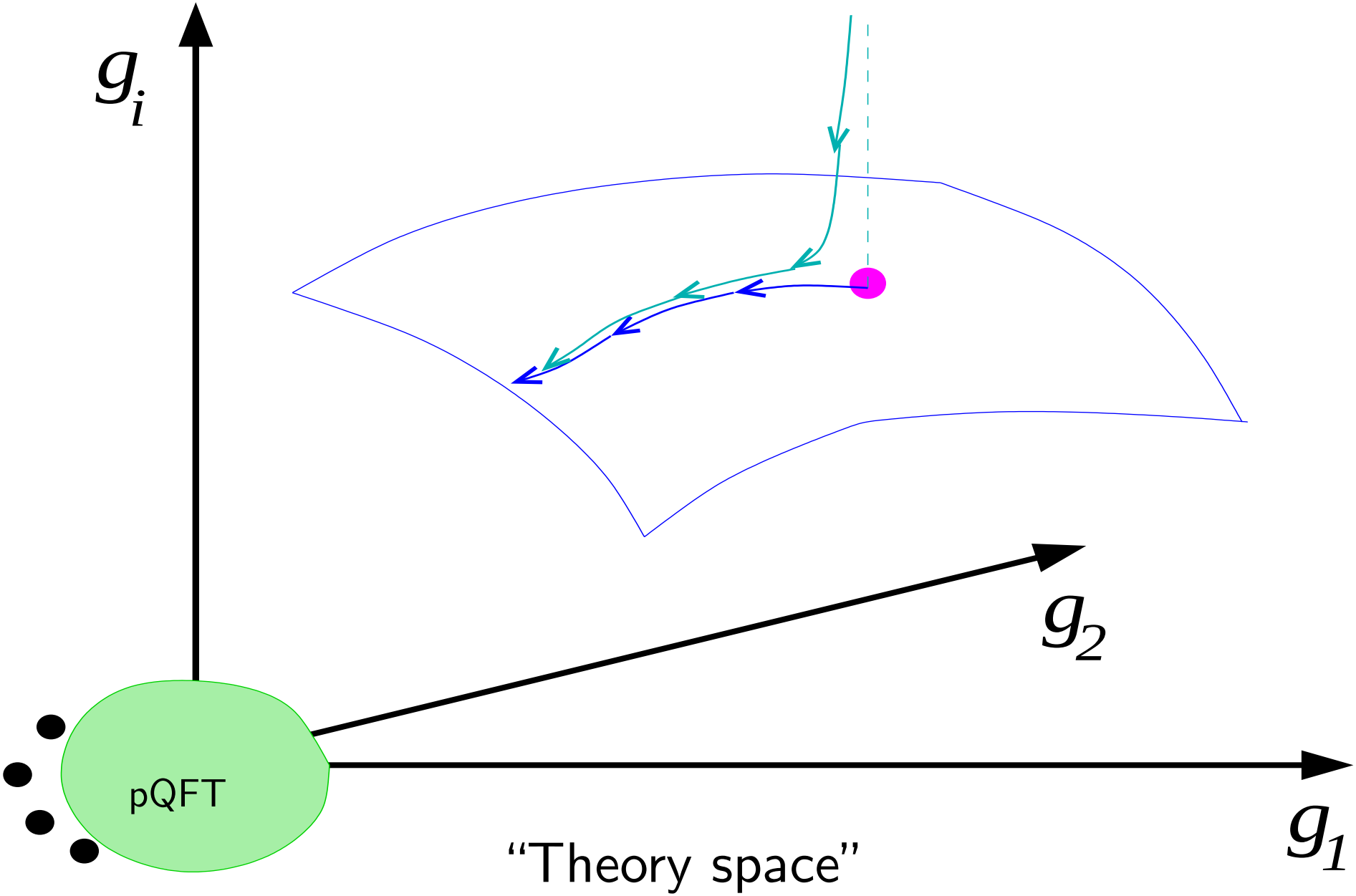


An Example

▷ Nambu–Jona-Lasinio / Gross-Neveu in $D = 3$ dimensions, $[\bar{\lambda}] = -1 \equiv (2 - D)$:

$$\Gamma_k = \int \bar{\psi} \mathrm{i} \partial\!\!\!/ \, \psi + \frac{1}{2} \bar{\lambda} (\bar{\psi}\psi)^2 + \ldots$$

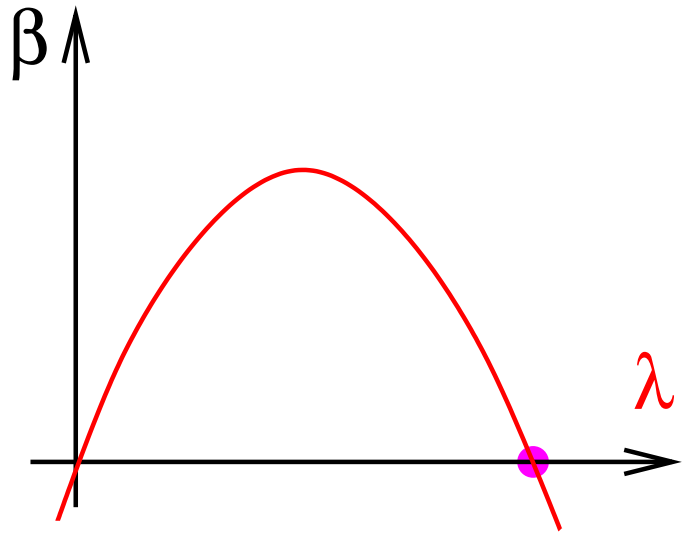
▷ flow of the dim'less coupling $\lambda = k\,\bar{\lambda}$

$$\beta := k\partial_k \lambda = \underbrace{\epsilon}_{D-2=1} \lambda - c\,\lambda^2 + \mathcal{O}(\epsilon^2)$$

ϵ expansion

▷ UV fixed point $\lambda_* = 1/c$

$\Longrightarrow$ asymptotically safe



$\longrightarrow$ "χSB"

(Gawedzki, Kupiainen'85; Rosenstein, Warr, Park'89; de Calan et al.'91)

ϵ-expanded Yang-Mills theory

▷ $D > 4$: bare coupling $\bar{g}_D$ has negative mass dimension $[\bar{g}_D] = (4 - D)/2$:

$$S = \int_x d^D x \, \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \bar{g}_D f^{abc} A^b_\mu A^c_\nu$$

▷ $D - 4 = \epsilon$ expansion, dim'less coupling: $g^2 \sim k^{D-4} \bar{g}_D{}^2$ (Peskin'80)

$$\partial_t g^2 \equiv \beta_{g^2} = (D - 4) g^2 - \frac{22N}{3} \frac{g^4}{16\pi^2} + \ldots, \quad \partial_t \equiv k \frac{d}{dk}$$

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(Gross&Wilczek'73, Politzer'73)

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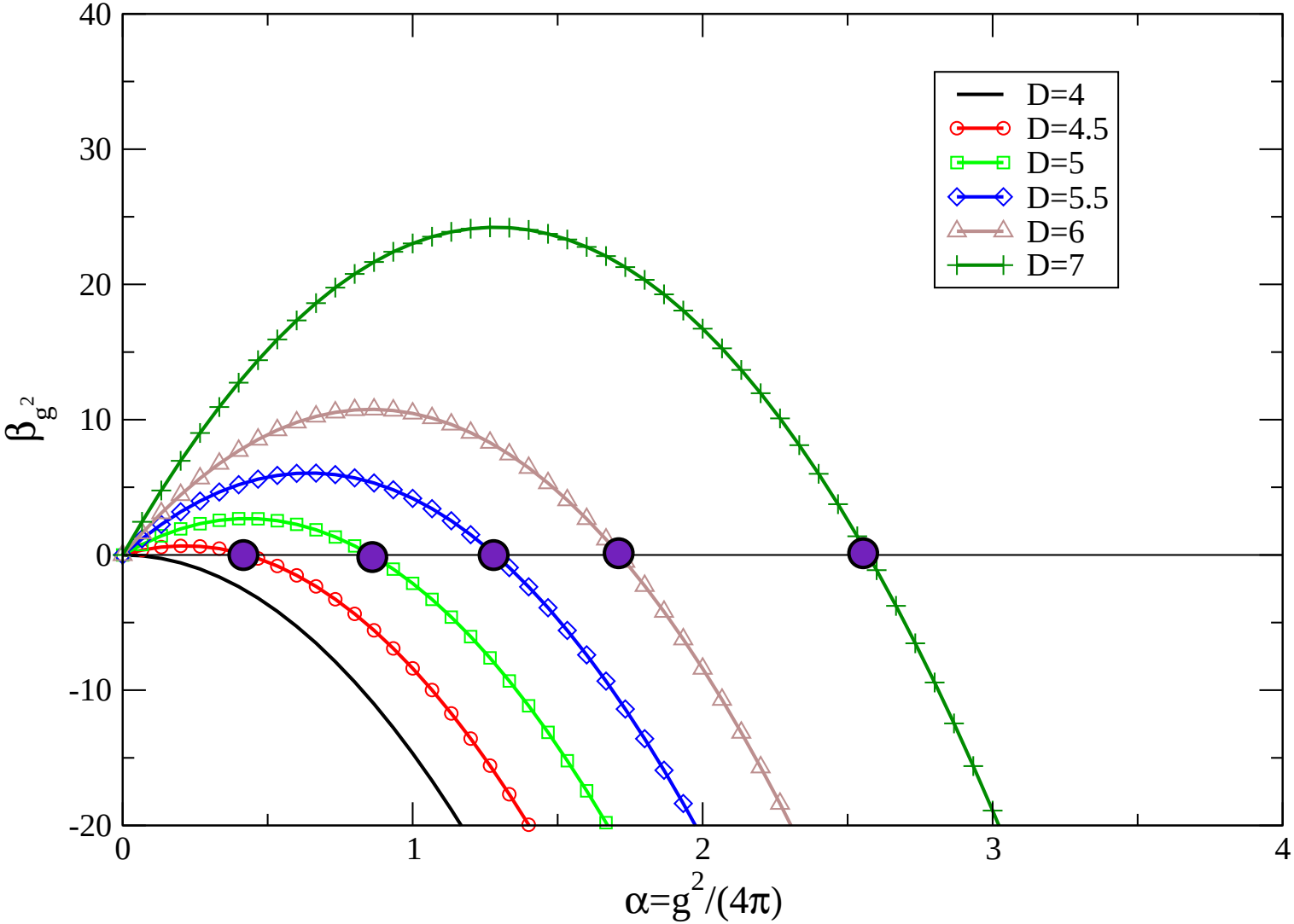
▷ UV fixed point:

$$g_*^2 = (24\pi^2/11N)\epsilon$$

for all ϵ . . . ?

naive ϵ expansion



nonperturbative problem for $\epsilon \gtrsim 1$

A $D \geq 4$ scenario

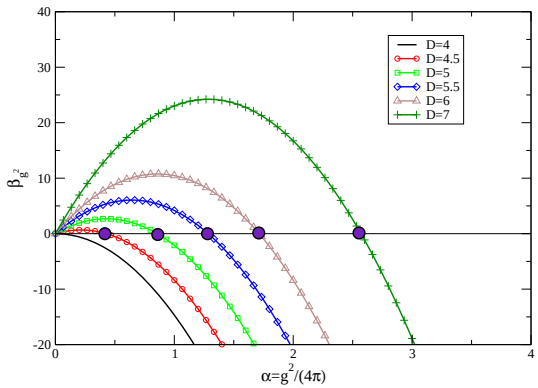
▷ β function:

$$\partial_t g^2 \equiv \beta_{g^2} = (D-4)g^2 - \beta^D_{\text{fluct}}(g^2), \qquad \beta^D_{\text{fluct}}(g^2) = -b_0^D g^4 + \mathcal{O}(g^6)$$

dimensional running $\longleftrightarrow$ fluctuational running

▷ assume analyticity in D:

$\Longrightarrow$ UV in $D > 4$ $\leftrightarrow$ IR in $D = 4$



▷ CAVEAT I: definition dependence of the running coupling

$\Longrightarrow$ running coupling from RG flow of gauge-invariant operators

A $D \geq 4$ scenario cont'd

▷ scale-dependent effective action

$$\Gamma_k = \int \frac{Z_{\mathsf{F}}}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{Y}{2} (D^{ab}_\mu F^b_{\kappa\lambda})^2 + \frac{W_2}{2} \frac{1}{16} (F^a_{\mu\nu} F^a_{\mu\nu})^2 + \frac{\widetilde{W}_2}{2} \frac{1}{16} (\widetilde{F}^a_{\mu\nu} F^a_{\mu\nu})^2 \ldots,$$

▷ running coupling:

$$g^2 = k^{D-4} \; {Z_{\mathsf{F}}}^{-1} \; {\bar{g}_D}^2$$

$\Longrightarrow$ UV fixed point in g^2 $\rightarrow$ renormalizable operator $\sim F^2$

▷ CAVEAT II: running coupling is regulator-scheme dependent

$\Longrightarrow$ mass-dependent regulator scheme required
for nonperturbative problems
(e.g. threshold behavior, mass generation, etc.)

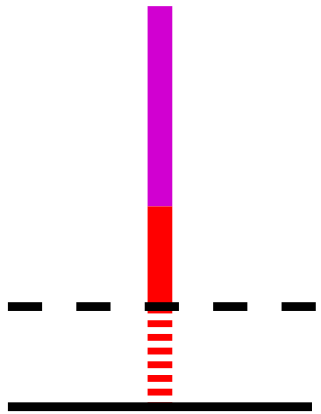
A $D \geq 4$ scenario: $D = 4$ vs. $D > 4$

▷ $D = 4$: Yang-Mills mass gap M

$\Longrightarrow$ threshold behavior for $k^2 \ll M$

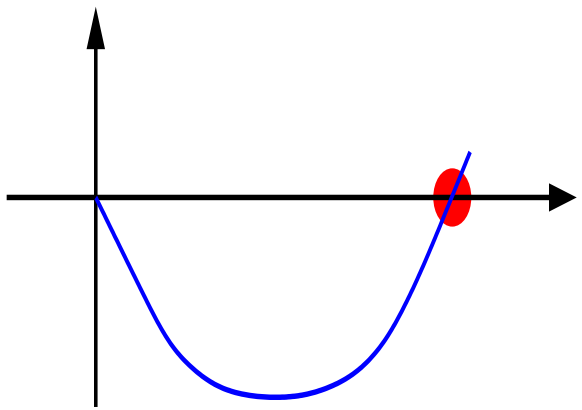
$\Longrightarrow$ freeze-out of couplings in the IR

$\Longrightarrow$ IR fixed point expected

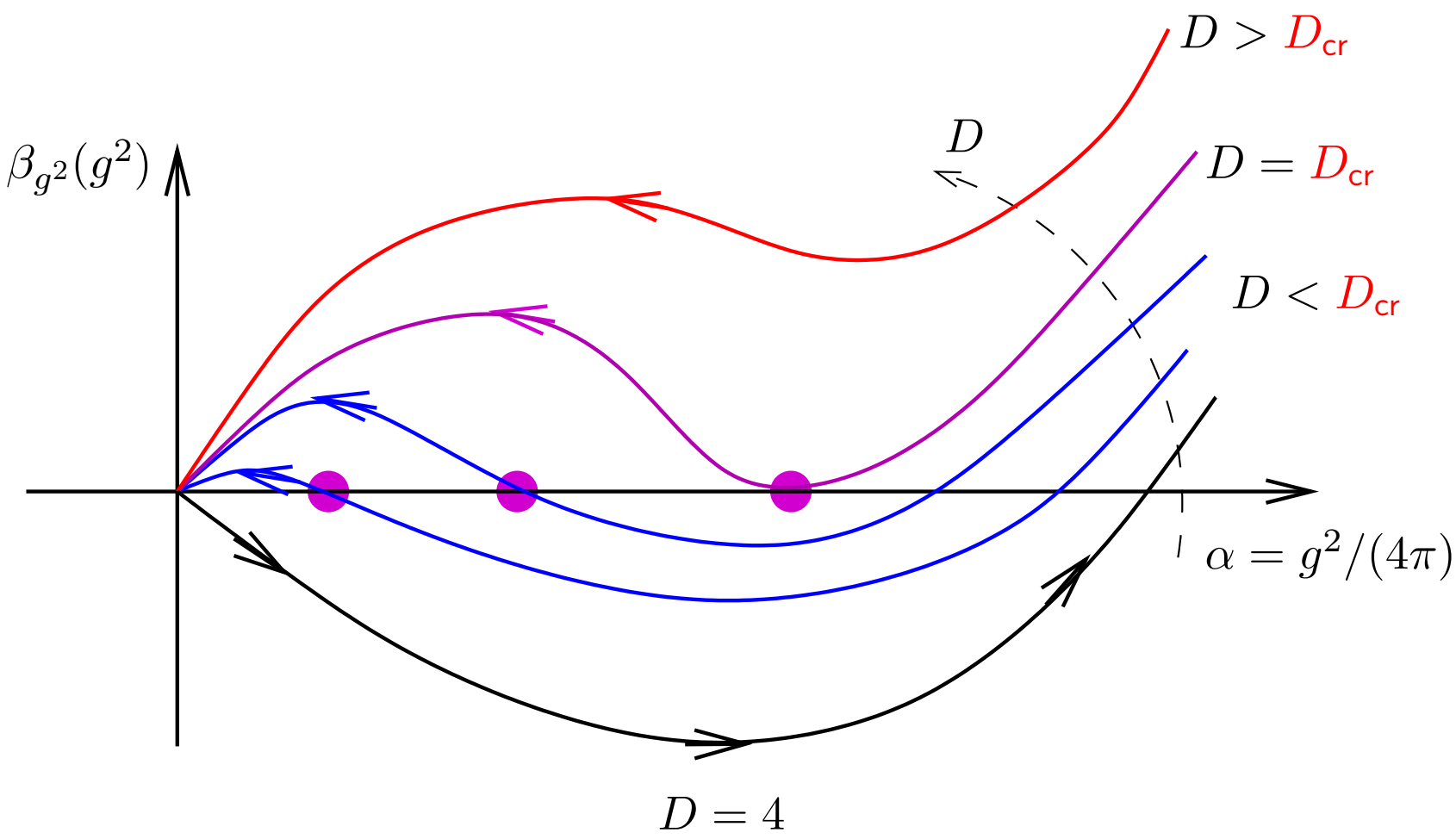


▷ combined evidence for $D = 4$ SU(3) Yang-Mills theory

- DSE in Landau gauge: $\alpha_* \simeq 2.97$
  (Lerche,Smekal'02; Alkofer,Fischer'02)
- ERG in Landau gauge: $\alpha_* \simeq 2.97$
  (Pawlowski et al.'04; Fischer, HG'04)
- lattice in Landau gauge: $\alpha_* \simeq 2.74$
  (Oliveira,Silva'04)
- ERG in background gauge: $\alpha_* \simeq 7.7$
  (HG'02)



A $D \geq 4$ scenario: $D = 4$ vs. $D > 4$, cont'd

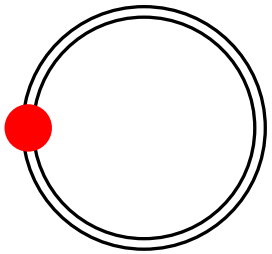


▷ D-analyticity & $(D = 4)$ IR behavior $\Longrightarrow$ existence of D_{cr}

$$D_{cr} = ?$$

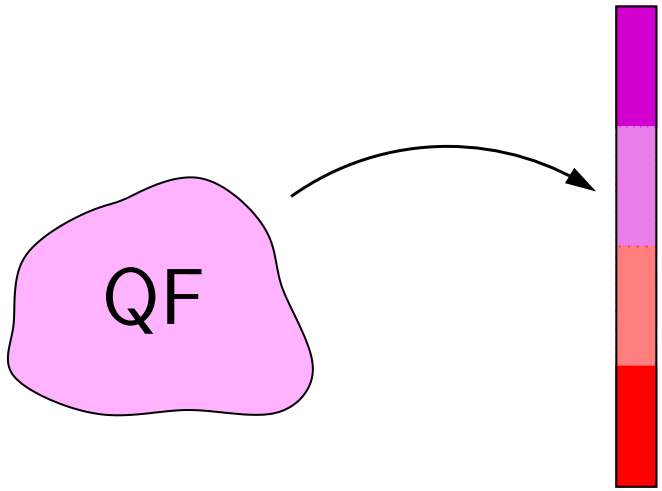
Exact RG Flow Equation

$$\partial_t \Gamma_k \;=\; \frac{1}{2}\,\mathrm{Tr}\;\partial_t R_k (\Gamma_k^{(2)} + R_k)^{-1} =$$

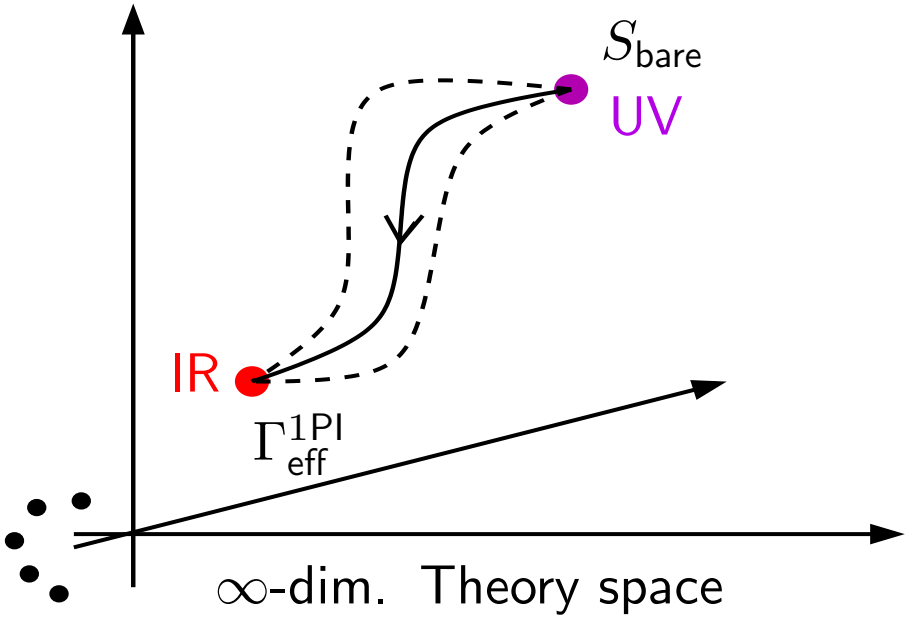


(Wetterich'93)

▷ quantum fluctuations



▷ RG trajectory: $\Gamma_{k=\Lambda} = S_{\text{bare}} \;\to\; \Gamma_{k=0} = \Gamma^{\text{1PI}}_{\text{eff}}$



Operator expansion

- Standard ghost and gauge-fixing terms
- Gluonic truncation: $\Gamma_k[A] = \int d^d x \; W_k(F^2), \qquad F^2 \equiv F^a_{\mu\nu} F^a_{\mu\nu}$

$$W_k(F^2) = \frac{Z_{\mathsf{F}}}{4} \; F^2 + \frac{1}{16} W_2 \; (F^2)^2 + \frac{1}{3! \cdot 4^3} W_3 \; (F^2)^3 + \frac{1}{4! \cdot 4^4} W_4 \; (F^2)^4 + \ldots$$

(cf. Savvidy model of confinement)

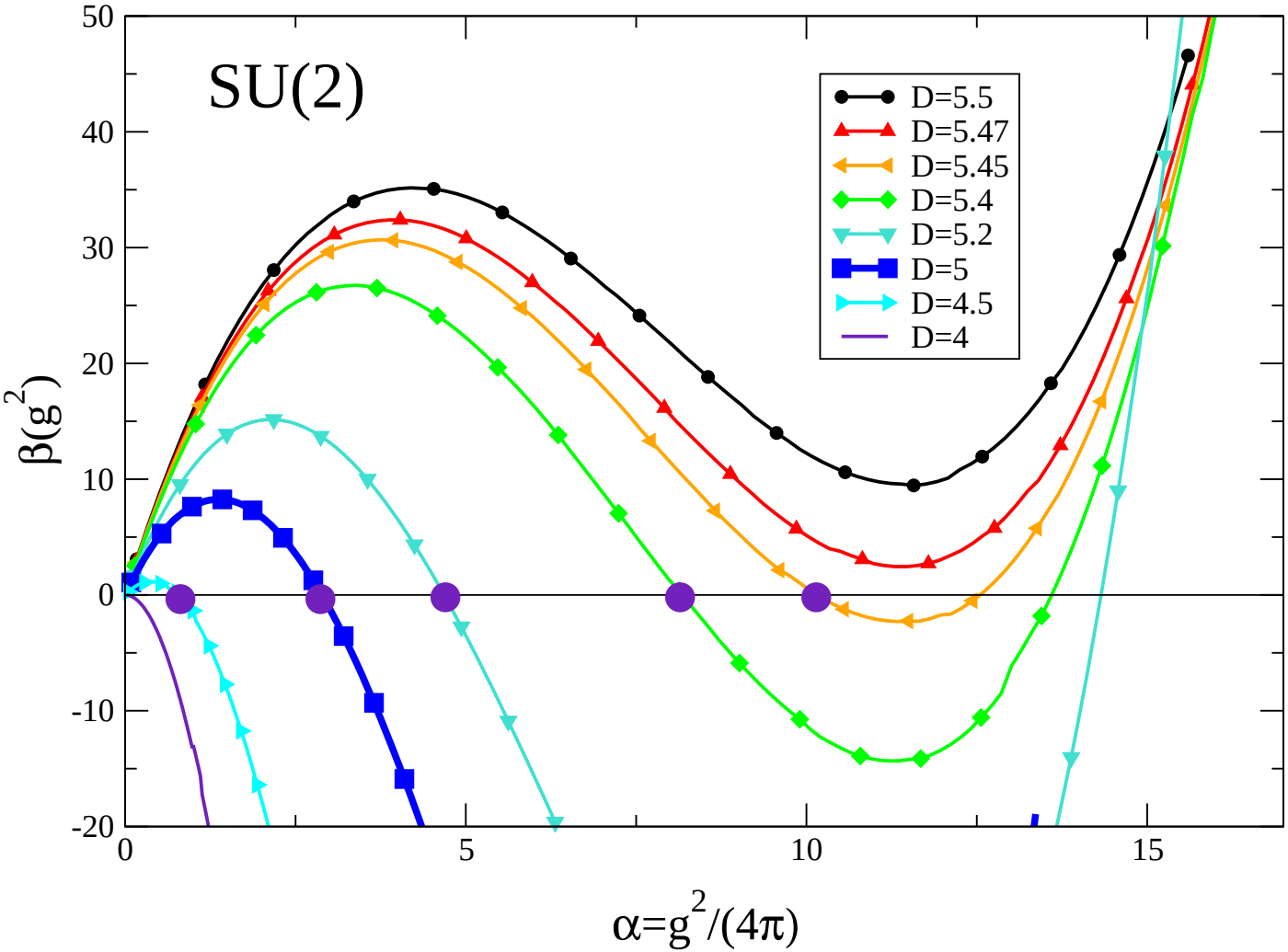
- spectrally adjusted flow equation:

$$\partial_t Z_{\mathsf{F}} \curvearrowleft \partial_t W_2 \curvearrowleft \partial_t W_3 \curvearrowleft \partial_t W_4 \curvearrowleft \partial_t W_5 \ldots$$

- running coupling: $g^2 = k^{D-4} \, Z_{\mathsf{F}}^{-1} \, \bar{g}^2$
- β function: $\boxed{\partial_t g^2 \equiv \beta_{g^2} = (D - 4 + \eta) \; g^2}$ $\qquad \eta = -\partial_t \, \ln Z_{\mathsf{F}}$

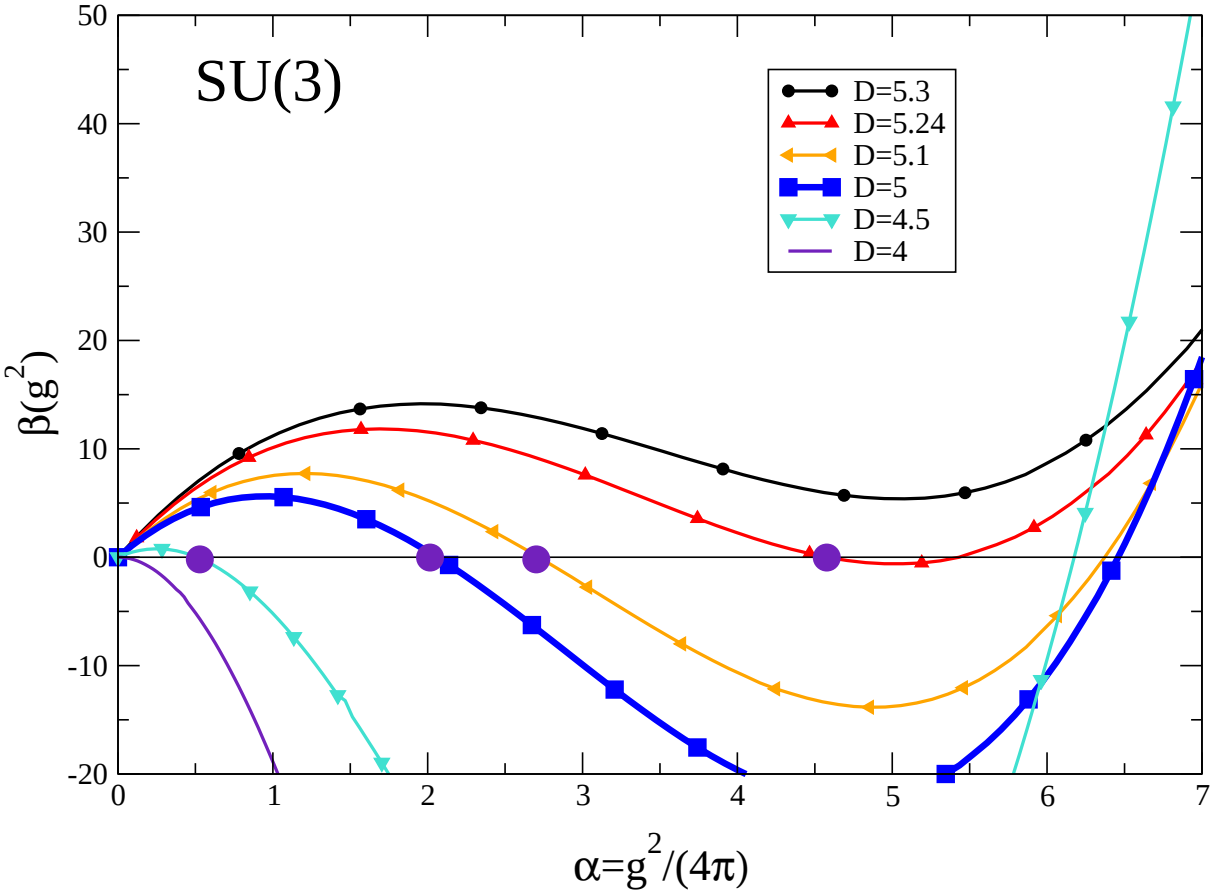
β function for SU(2)

$D_{\text{cr}} = 5.46$



β function for SU(3)

$$D_{\text{cr}} = 5.26 \pm 0.01$$



▷ SU(5): $D_{\text{cr}} = 5.05 \pm 0.05$

Comments I

$$D_{\text{cr}}{}^{\text{SU(2)}} \simeq 5.46, \quad D_{\text{cr}}{}^{\text{SU(3)}} \simeq 5.26 \pm 0.01, \quad D_{\text{cr}}{}^{\text{SU(5)}} \simeq 5.05 \pm 0.05,$$

▷ $D = 4$ analogue:

upper bound: $D_{\text{cr}}{}^{\text{true}} \lesssim D_{\text{cr}}{}^{\text{trunc}}$

▷ lattice studies (Creutz'79; Kawai,Nio,Okamoto'92; Nishimura'96)

spin-wave phase $\overset{\text{1st order}}{\longleftrightarrow}$ confining phase

$\Longrightarrow$ no QFT continuum limit found (with Wilson action)

Comments II

▷ conservative viewpoint

$D_{\text{cr}}{}^{\text{true}} < 5$

▷ alternative viewpoint

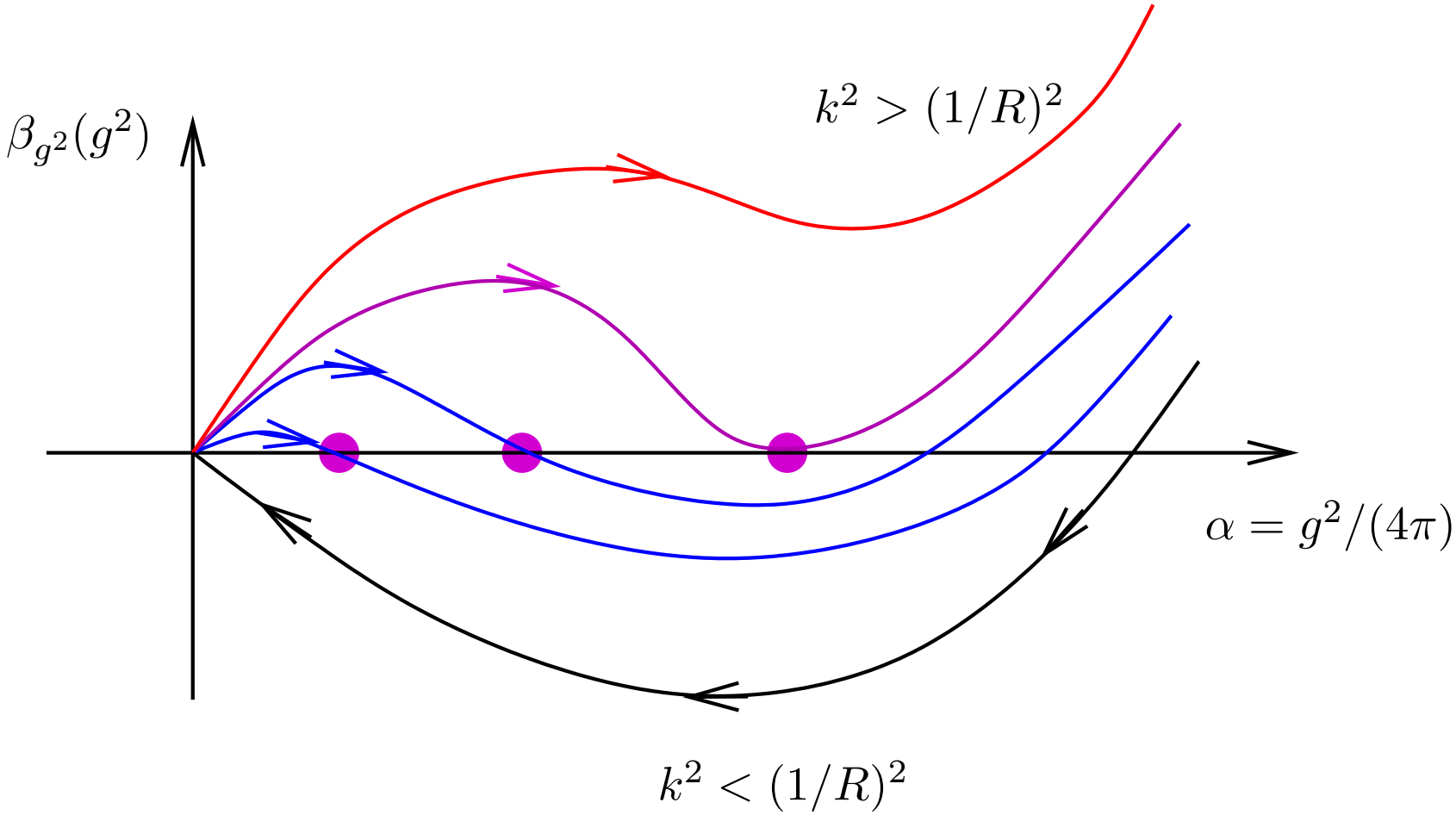
UV fixed point exists, but $\Delta > 1$

▷ naive inclusion of fermions: 1-loop running

$$\beta_{g^2}^{\text{fermion}} = +\frac{4}{3} N_{\text{f}} \frac{g^4}{16\pi^2} + \dots$$

$\Longrightarrow$ with fermions: $4 < D_{\text{cr}} < 5$

Compactified extra dimensions ?



Conclusions

- $4 \leq D < D_{\text{cr}}$ can be asymptotically safe

  $=$ (nonperturbatively) renormalizable

- $D_{\text{cr}} \lesssim 5$ for pure Yang-Mills theories

  no asymptotic safety for $D \geq 6$

- model building based on (partial) UV fixed points ?

  . . . not recommended

Coincidence ?

$D = 4$

spacetime manifold

$D \simeq 4$

RG critical dimension

▷ ϵ expansion to 4-loop order: (Morris'04)

- UV fixed point exists formally for any $D > 4$
- $D = 5$: asymptotic series predictive for <10-loop
- $D \gtrsim 6$: ϵ series in asymptotic regime (loss of predictivity)

β function

- asymptotic series with R_k-dependent coefficients a_m^D

$$\eta = \sum_{m=1}^{\infty} a_m^D \left(\frac{g^2}{(4\pi)^2}\right)^m$$

- perturbative beta function, SU(N_c), $D = 4$:

$$\begin{aligned}\beta(g^2) &= -\frac{22N_c}{3}\frac{g^4}{(4\pi)^2} \\ &\quad -\left(\frac{77N_c^2}{3} - \frac{127(3N_c^2-2)}{45} f(R_k)\right)\frac{g^6}{(4\pi)^4} + \dots\end{aligned}$$

- 1 loop: <u>exact</u>

  2 loop: <u>99%</u> for SU(2), 95% for SU(3),
  (for exponential regulator)

1	-29.3333
2	-357.83
3	-191.32
4	15499.6
5	$-1.88776 \cdot 10^{6}$
6	$1.65315 \cdot 10^{7}$
7	$2.79324 \cdot 10^{9}$
8	$-1.37622 \cdot 10^{11}$
9	$-4.21715 \cdot 10^{12}$
10	$8.60663 \cdot 10^{14}$
11	$-8.05611 \cdot 10^{16}$
12	$5.21052 \cdot 10^{19}$
13	$-6.30043 \cdot 10^{22}$
14	$9.35648 \cdot 10^{25}$
15	$-1.78717 \cdot 10^{29}$
16	$4.35314 \cdot 10^{32}$
17	$-1.33397 \cdot 10^{36}$
18	$5.08021 \cdot 10^{39}$
19	$-2.37794 \cdot 10^{43}$
20	$1.35433 \cdot 10^{47}$