

From Fermions to Bosons: RG flow for the relevant degrees of freedom

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&

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What are the **relevant** degrees of freedom ?

What are the **relevant** degrees of freedom ?

micro DoFs



ψ

macro DoFs

ϕ

Identification criterion ?

Identification criterion ?

Simplicity

How does the translation work ?

How does the translation work ?

- partition function transformation

$$\int \mathcal{D}\psi \dots \longrightarrow \int \mathcal{D}\phi \dots$$

- continuous transformation ?

Ultracold Fermionic Atoms

BEC-BCS crossover

(EAGLES'69; LEGGETT'80)

- ▶ Bound molecules of two atoms on microscopic scale

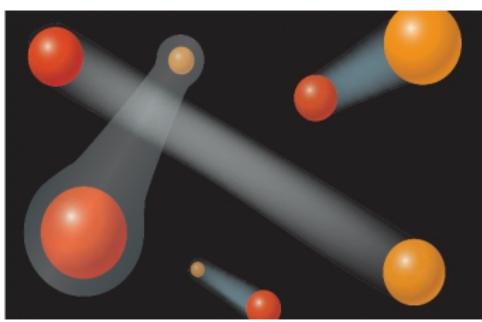
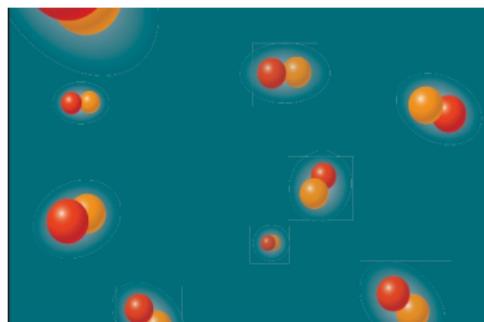
Bose-Einstein condensate (BEC)

- ▶ Fermions with attractive interactions

BCS superfluidity

at low T

at low T

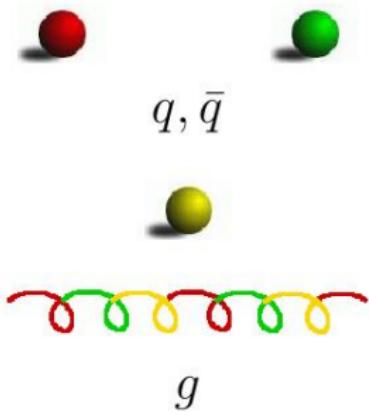


(CHO@SCIENCE'03)

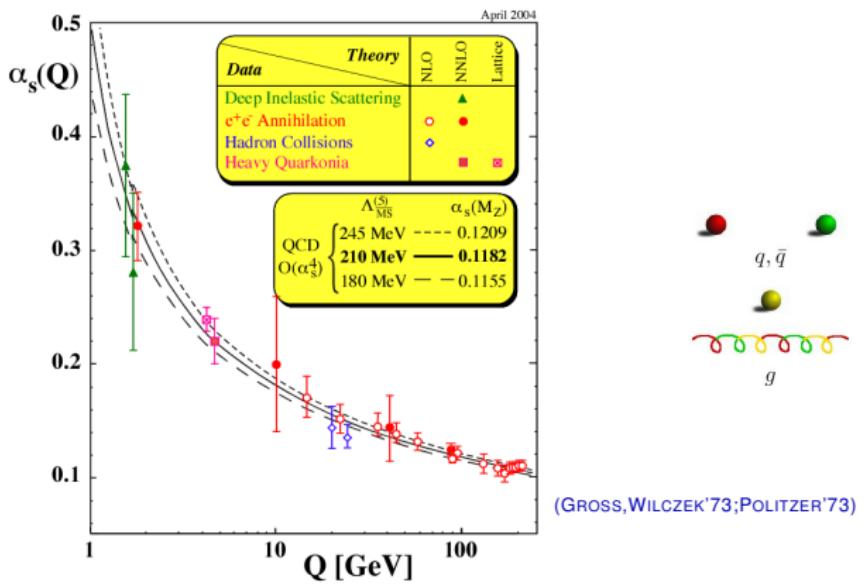
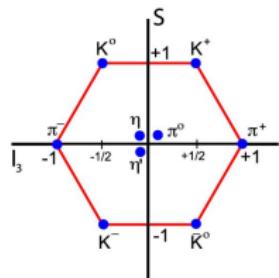
Crossover by means of a Feshbach resonance

(REGAL'04; BARTENSTEIN'04; ZWIERLEIN'04; KINAST'04; BOURDEL'04)

Quantum Chromodynamics

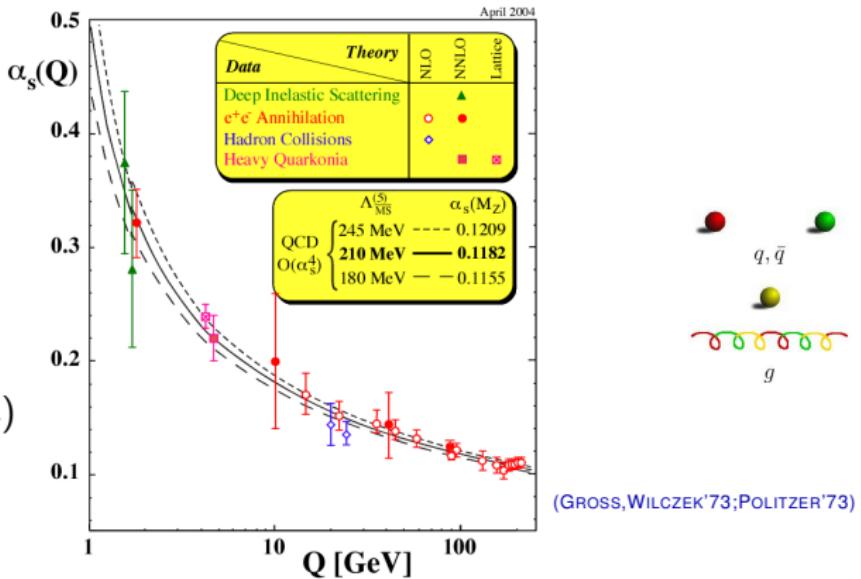


Quantum Chromodynamics



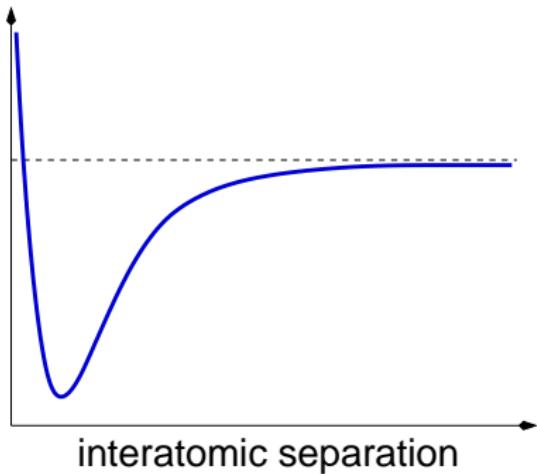
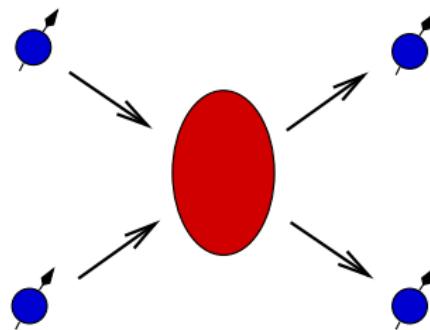
Quantum Chromodynamics

- ▷ running coupling
- ▷ confinement
- ▷ mass gap
- ▷ chiral symmetry breaking
- ▷ phase diagram (T, μ)
- ▷ hadron spectrum



Fermionic Atoms: Microscopic Interactions

▷ 2-atom scattering



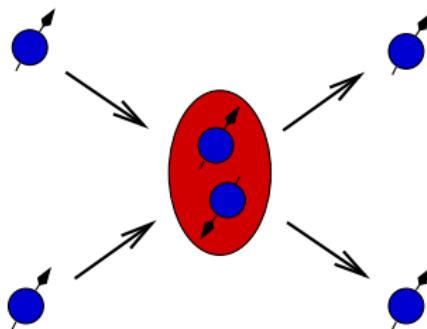
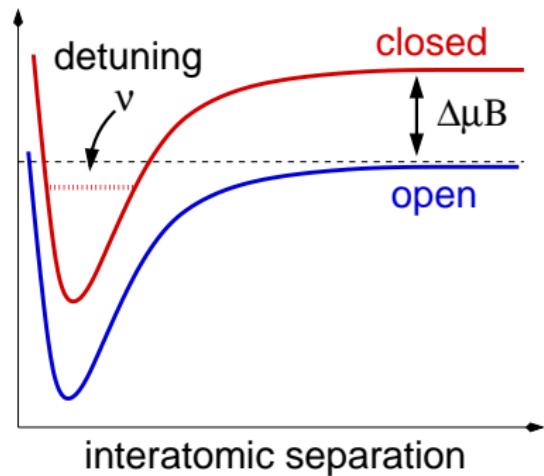
▷ s-wave scattering length

a

Fermionic Atoms: Microscopic Interactions

▷ 2-atom scattering

resonant hyperfine interaction
between interaction channels



▷ s-wave scattering length

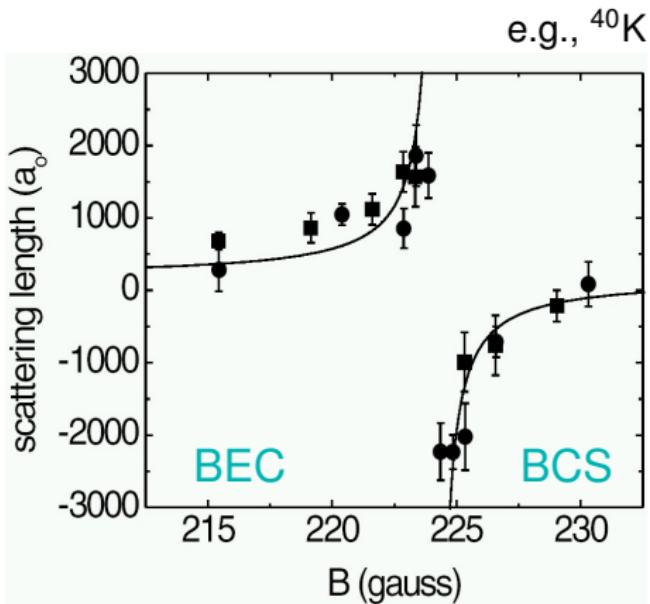
$$a(B)$$

near a Feshbach resonance

Microscopic Interactions

- ▷ tunable interaction strength

$$a(B) = a_{\text{bg}} \left(1 - \frac{\Delta B}{B - B_0} \right)$$



(REGAL,JIN'03)

Many-Body System: QFT with Fermions

- ▶ microscopic action:

$$S_F = \int d\tau \int d^3x \left[\psi^\dagger \left(\partial_\tau - \frac{\nabla^2}{2M} \right) \psi + \frac{1}{2} \lambda_\psi (\psi^\dagger \psi)(\psi^\dagger \psi) \right]$$

- ▶ fermionic interaction parameter (+UV renormalization):

$$\lambda_\psi = \frac{4\pi}{M} a(B)$$

- ▶ further parameters of 2-atom physics encodable in momentum-dependence of λ_ψ

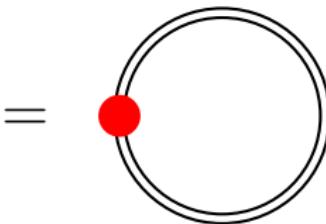
Functional RG Flow Equation

IR: $k \rightarrow 0$



UV: $k \rightarrow \Lambda$

$$\partial_t \Gamma_k \equiv k \partial_k \Gamma_k = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k$$

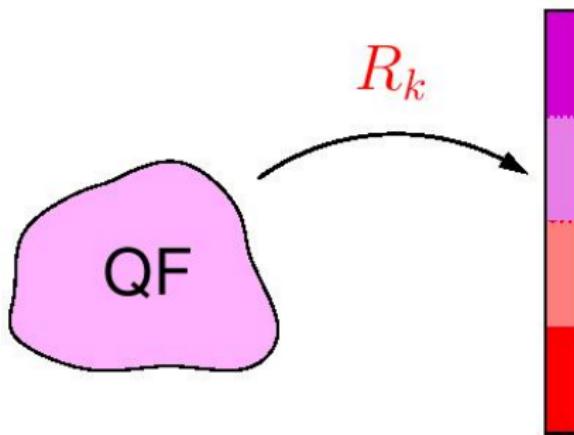


(WILSON'71; WEGNER&HOUGHTON'73; POLCHINSKI'84; WETTERICH'93)

Functional RG Flow Equation

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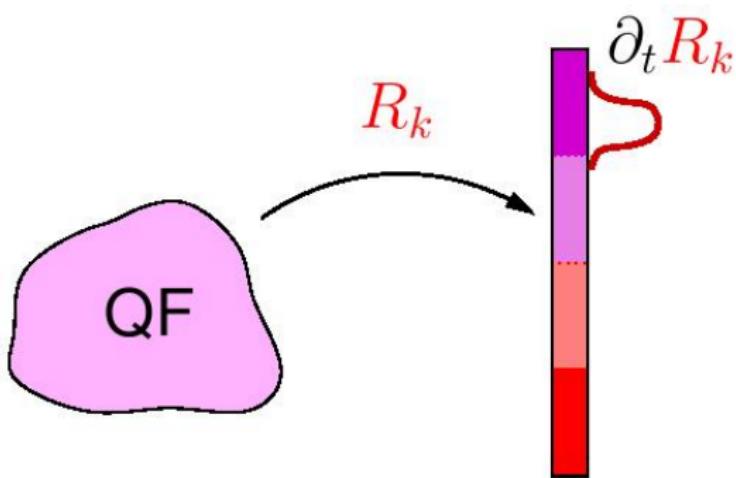
▷ quantum fluctuations:



Functional RG Flow Equation

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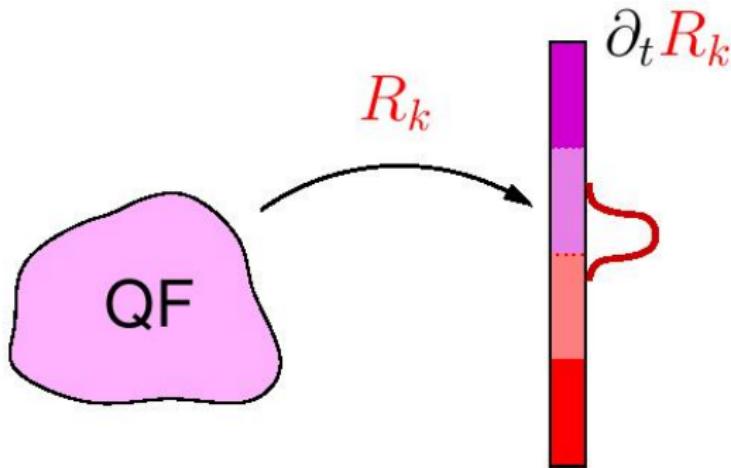
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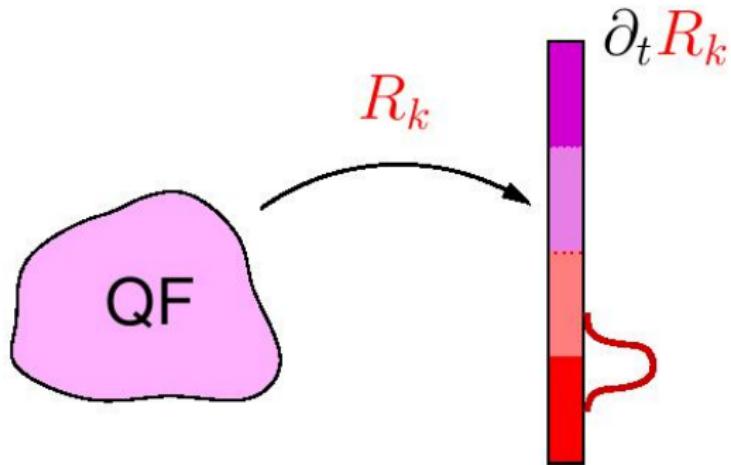
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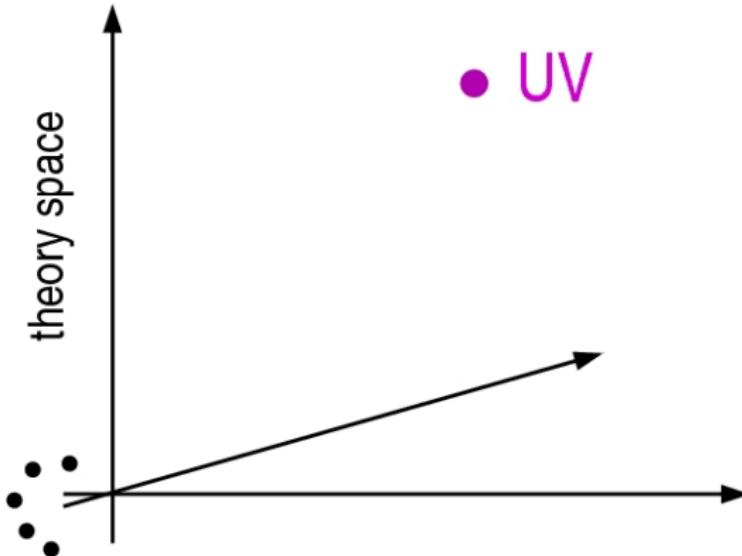
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Functional RG Flow Equation

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k$$

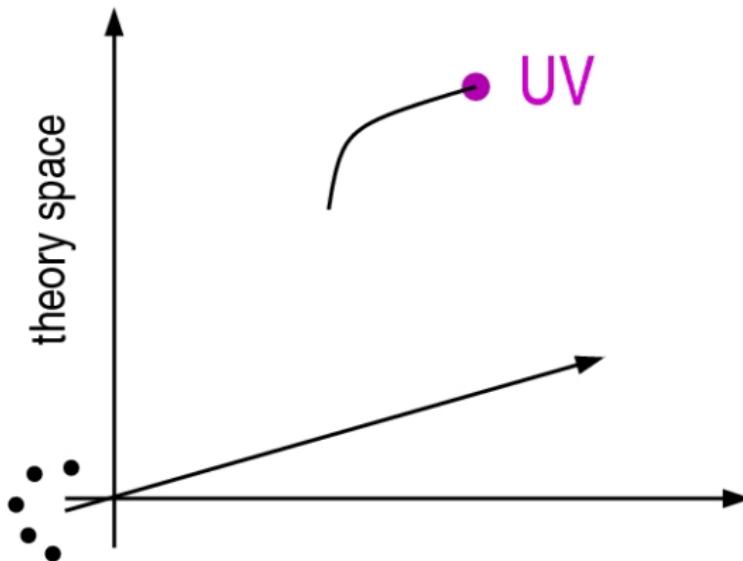
▷ RG trajectory: $\Gamma_{k=\Lambda} = S_{\text{micro}} = \int \psi^\dagger \left(\partial_\tau - \frac{\nabla^2}{2M} \right) \psi + S_{\text{int}}[\psi^\dagger, \psi]$



Functional RG Flow Equation

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k$$

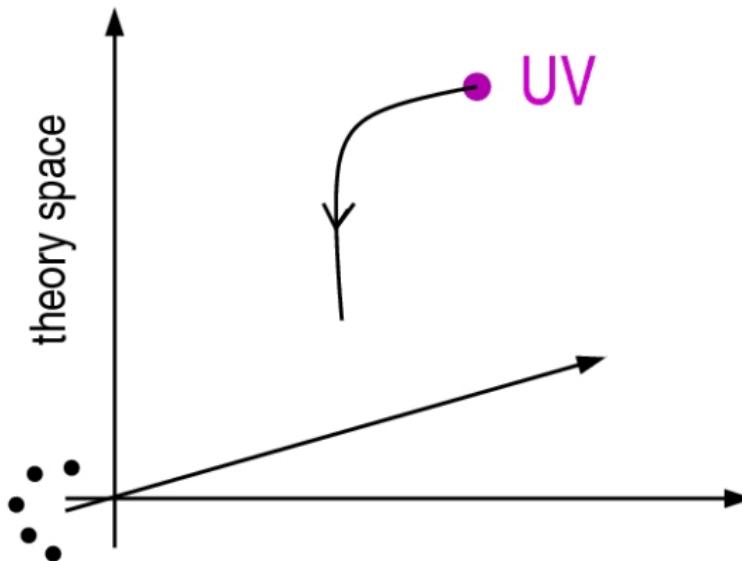
▷ RG trajectory:



Functional RG Flow Equation

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▷ RG trajectory:

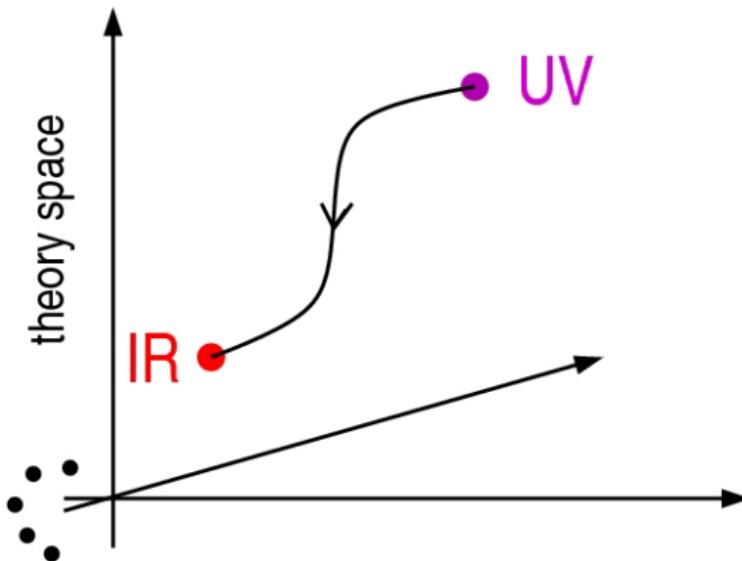


Functional RG Flow Equation

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k$$

▷ RG trajectory:

$$\Gamma_{k \rightarrow 0} = \Gamma$$

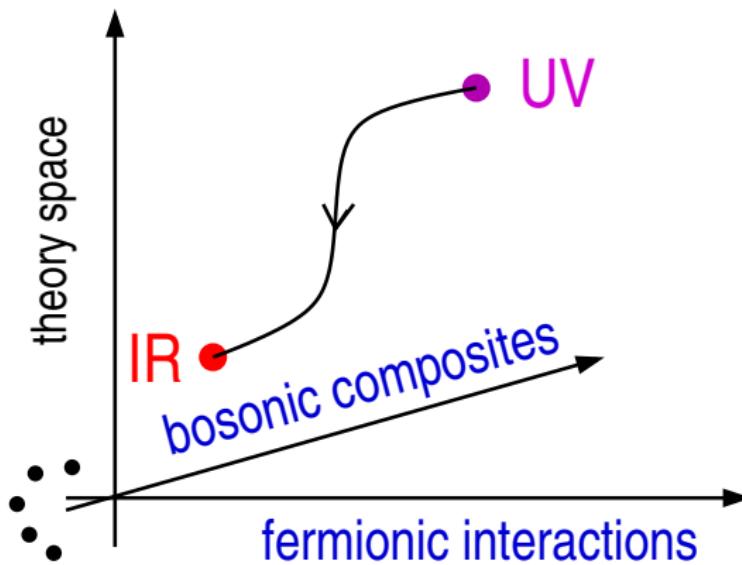


Functional RG Flow Equation

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k$$

▷ RG trajectory:

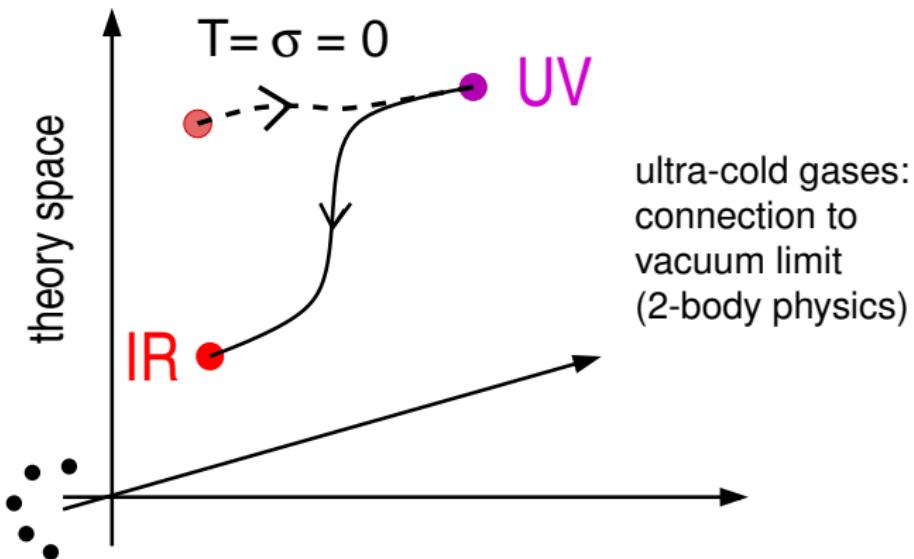
$$\Gamma_{k \rightarrow 0} = \Gamma[\psi, \phi]$$



Functional RG Flow Equation

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k$$

▷ RG trajectory:



Effective Action

- ▷ Guiding principle:

treat all potentially relevant degrees of freedom
on the same footing

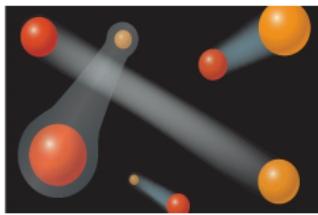
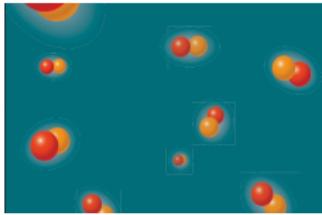
$$\psi \quad \phi$$

Effective Action

- ▷ Systematic and consistent derivative expansion of Γ_k :

$$\begin{aligned}\Gamma_k[\psi, \phi] = & \int d\tau \int d^3x \left\{ \psi^\dagger (\textcolor{red}{Z}_\psi \partial_\tau - \textcolor{red}{A}_\psi \nabla^2 - \sigma) \psi + \lambda_\psi (\psi^\dagger \psi)^2 \right. \\ & \left. + \phi^* (\textcolor{red}{Z}_\phi \partial_\tau - \textcolor{red}{A}_\phi \nabla^2) \phi + \textcolor{red}{U}(\phi) - \frac{\hbar_\phi}{2} (\phi^* \psi^\top \epsilon \psi - \phi \psi^\dagger \epsilon \phi^*) + \dots \right\}\end{aligned}$$

- ▷ ψ : stable fermionic atom field
- ▷ ϕ : bosonic molecule field / Cooper pair



Effective Action

- ▶ Systematic and consistent derivative expansion of Γ_k :

$$\begin{aligned}\Gamma_k[\psi, \phi] = & \int d\tau \int d^3x \left\{ \psi^\dagger (\textcolor{red}{Z}_\psi \partial_\tau - \textcolor{red}{A}_\psi \nabla^2 - \sigma) \psi + \lambda_\psi (\psi^\dagger \psi)^2 \right. \\ & \left. + \phi^* (\textcolor{red}{Z}_\phi \partial_\tau - \textcolor{red}{A}_\phi \nabla^2) \phi + \textcolor{red}{U}(\phi) - \frac{h_\phi}{2} (\phi^* \psi^\dagger \epsilon \psi - \phi \psi^\dagger \epsilon \phi^*) + \dots \right\}\end{aligned}$$

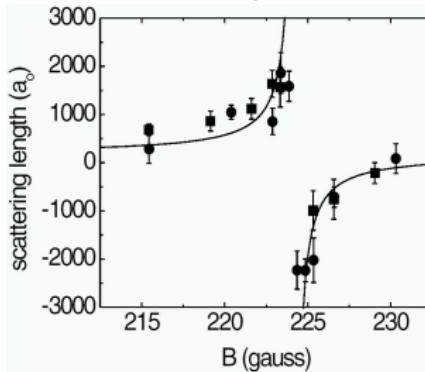
- ▶ bosonic potential: $\textcolor{red}{U}(\phi) = m_\phi^2 \phi^* \phi + \frac{\lambda_\phi}{2} (\phi^* \phi)^2 + \dots$
- ▶ relation to microphysics (via Hubbard-Stratonovich):

$$\lambda_{\psi, \text{eff}} = \lambda_{\psi,0} + \frac{1}{2} \frac{h_{\phi,0}^2}{m_{\phi,0}^2}$$

$$\lambda_{\psi,0} = \frac{4\pi a_{\text{bg}}}{M}$$

$$m_{\phi,0}^2 = 2\bar{\mu}(\textcolor{green}{B} - B_0) - 2\sigma$$

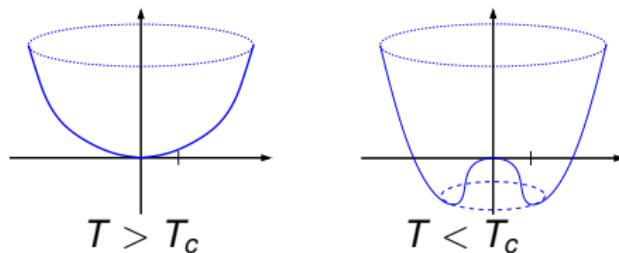
$$h_{\phi,0}^2 \sim \Delta B$$



IR physics from RG Flow

► e.g., from effective potential $U(\phi)|_{k \rightarrow 0}$:

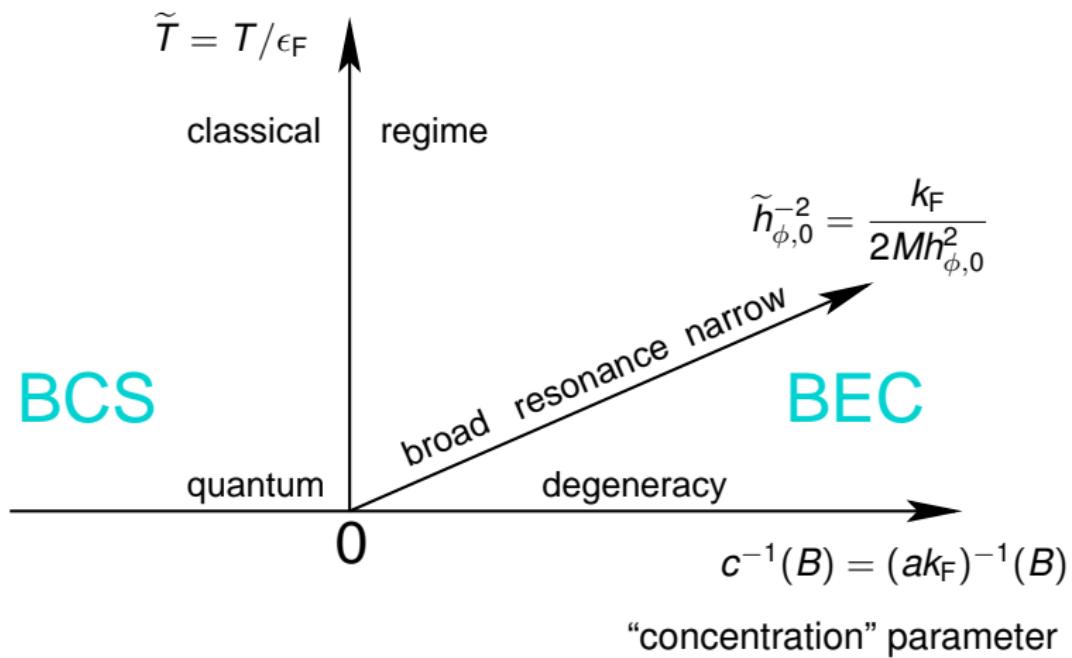
- determines symmetry status



- potential minimum $\rho_0 = \phi_{\min}^* \phi_{\min}$: order parameter
 - ... determines gap Δ and condensate fraction Ω_c
- U'' determines correlation length ξ
- determines density $n = -\frac{\partial U(\phi_{\min})}{\partial \sigma}$

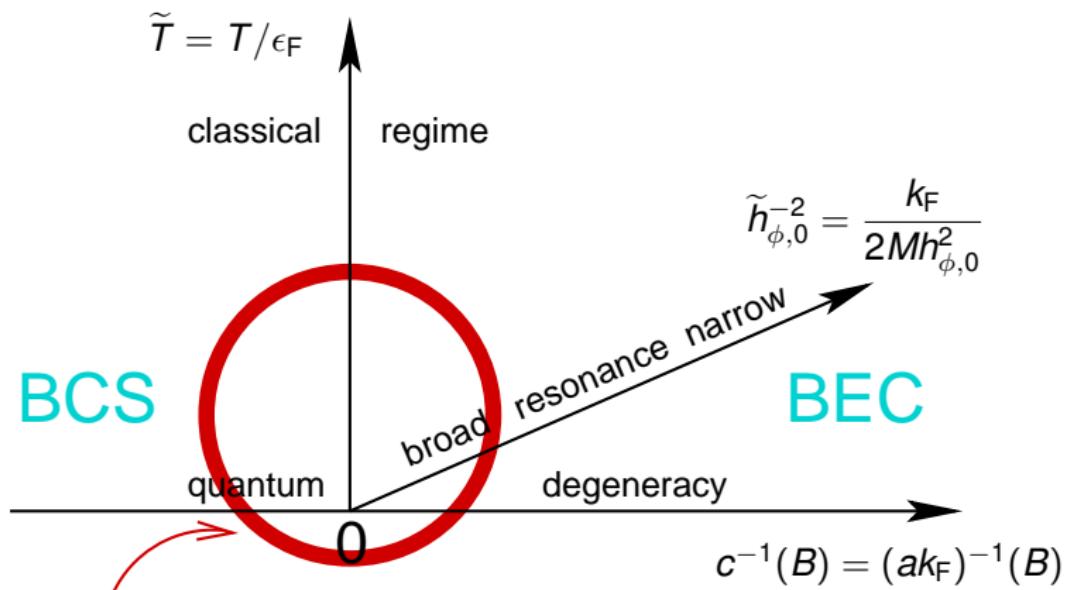
Crossover Diagram

- in units of Fermi momentum $k_F = (3\pi^2 n)^{1/3}$ and energy $\epsilon_F = k_F^2/(2M)$



Crossover Diagram

► in units of Fermi momentum $k_F = (3\pi^2 n)^{1/3}$ and energy $\epsilon_F = k_F^2/(2M)$



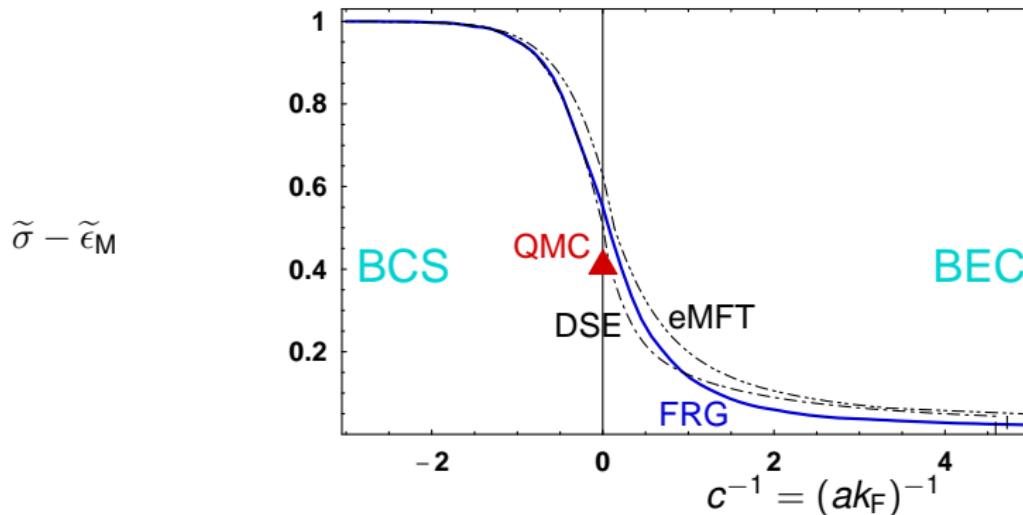
strong coupling, “broad-resonance universality” $h_{\phi,0} \sim \Delta B \rightarrow \infty$

Universal long-distance physics for ${}^6\text{Li}$ and ${}^{40}\text{K}$?

(DIEHL, WETTERICH'05; NICOLIĆ, SACHDEV'06)

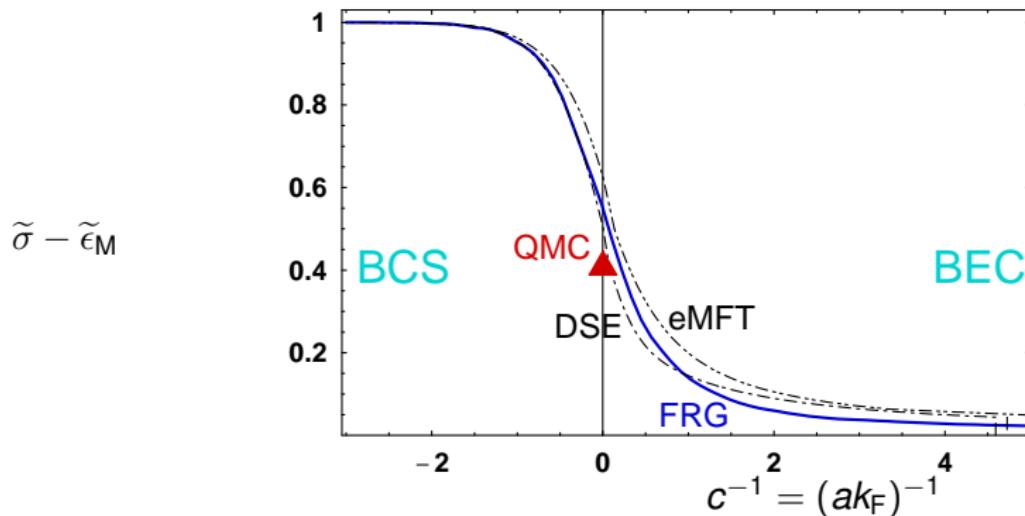
Estimating Many-Body Effects

- ▶ chemical potential $\tilde{\sigma}$ minus molecular binding energy $\tilde{\epsilon}_M$ at $T = 0$



Estimating Many-Body Effects

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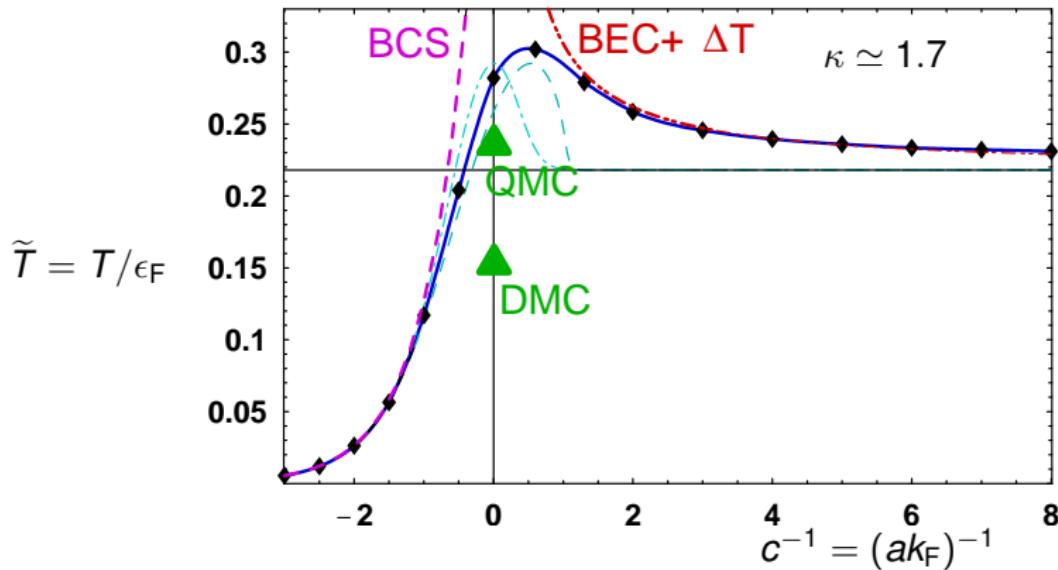


QMC (GIORGINI & '04)	FRG (THIS WORK)	ϵ NLO (SON & '06)	DSE (DIEHL & '05)	MFT	2PI (ZWINGER & '06)	Padé (BAKER '99)	NSR (HU & '06)	Exp.
0.42(2)	0.55	0.475	0.50	0.63	0.36	0.33	0.40	0.32 -0.51

Phase Diagram

▷ broad resonance limit

(DIEHL,HG,PAWLOWSKI,WETTERICH'07)



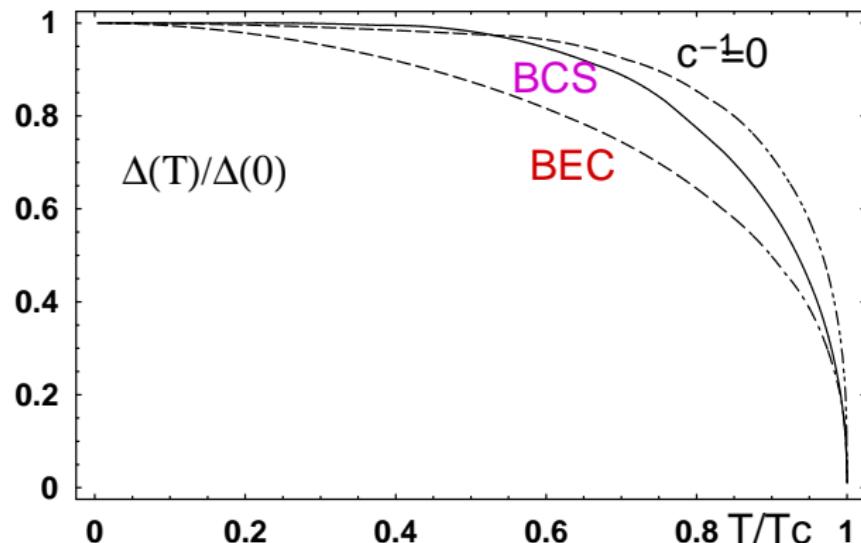
▷ shift of T_c in BEC regime:

(BAYM,BLAIZOT,HOLZMANN,LALOË,VAUTHERIN'99)
(HASSELMANN,LEDOWSKI,KOPIETZ'04)
(BLAIZOT,MENDEZ-GALAIN,WSCHEBOR'05,06)

$$\frac{T_c - T_c^{\text{BEC}}}{T_c^{\text{BEC}}} = \kappa a_B n^{1/3}, \quad \kappa \simeq 1.3$$

Fermionic Gap

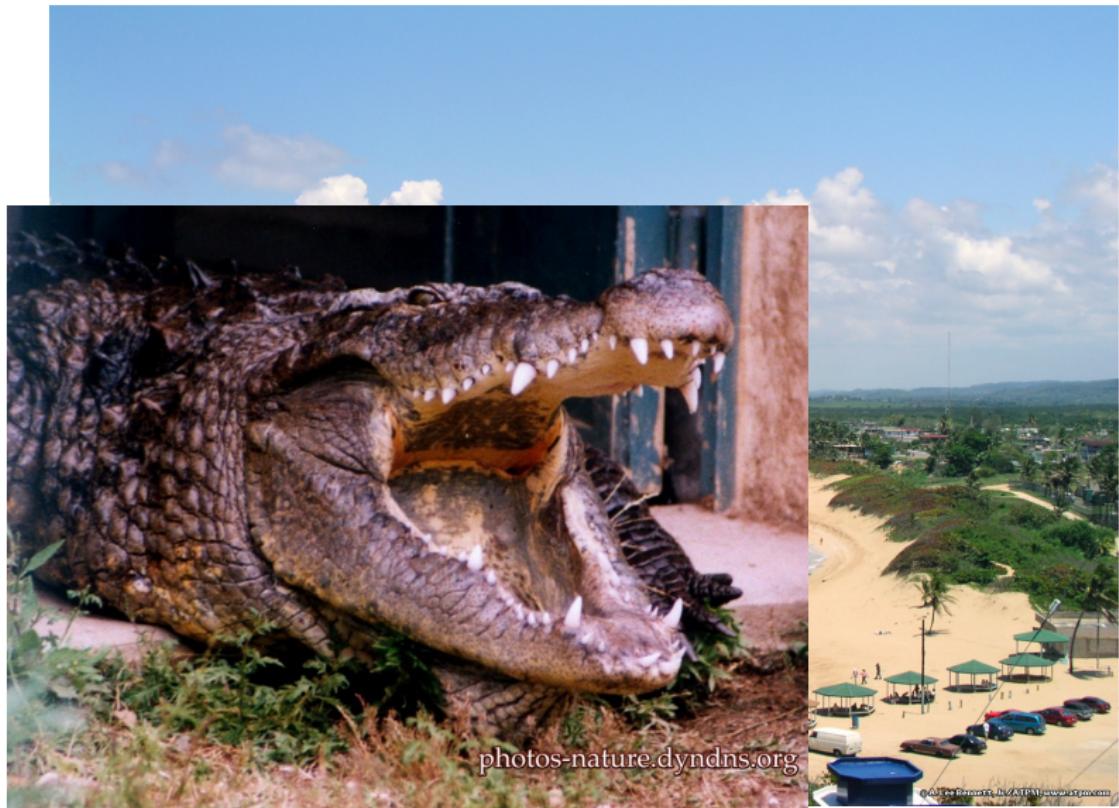
► $\Delta = h_\phi \sqrt{\rho_{\min}}$



2nd order phase transition



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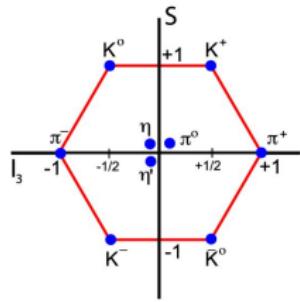


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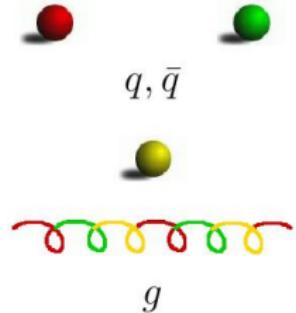
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Translating degrees of freedom

macro IR DoF



micro UV DoF



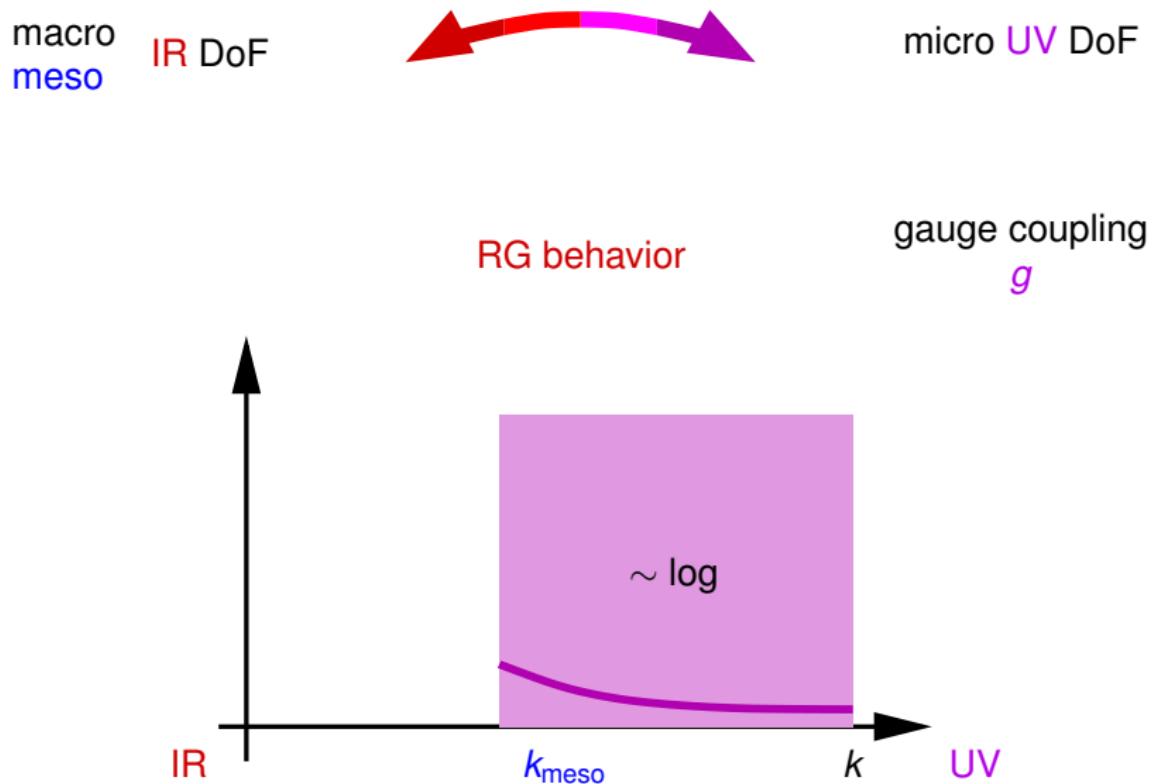
mesoscopic DoF

$$k \rightarrow 0$$

$$k = k_{\text{meso}}$$

$$k \rightarrow \Lambda$$

Translating degrees of freedom



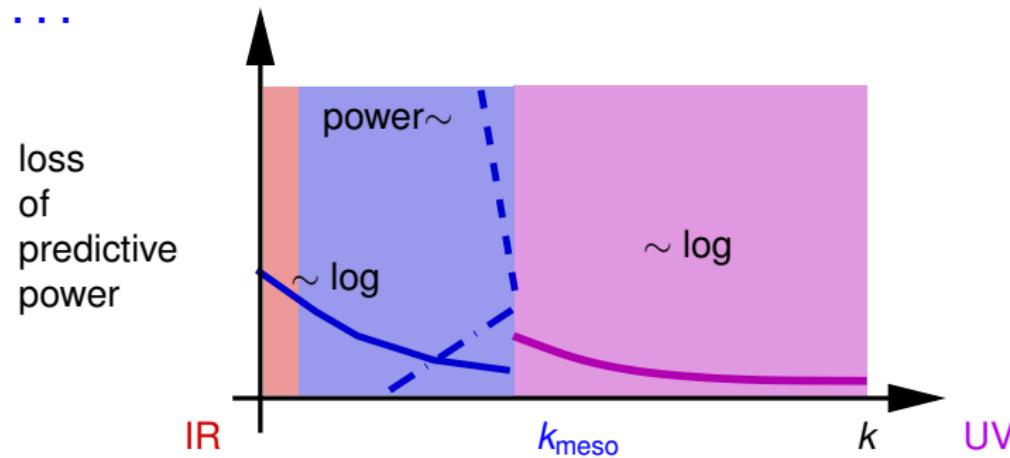
Translating degrees of freedom

macro
meso IR DoF

micro UV DoF

Yukawa h
masses m^2
self-coup. λ_ϕ

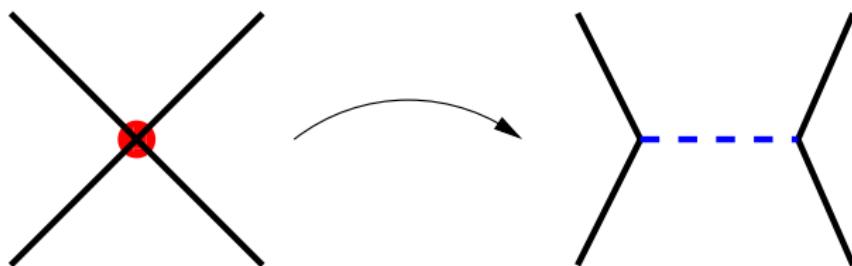
gauge coupling
 g



Translating degrees of freedom

▷ Hubbard-Stratonovich transformation

(STRATONOVICH'57, HUBBARD'59)



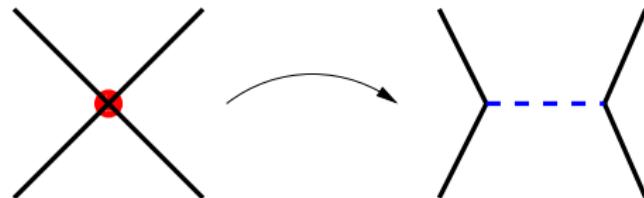
$$2 \lambda_\sigma \bar{\psi}_R \psi_L \bar{\psi}_L \psi_R \rightarrow h (\bar{\psi}_R \psi_L \phi - \bar{\psi}_L \psi_R \phi^*) + m^2 \phi^* \phi$$

$$\lambda_\sigma \Big|_{k=k_{\text{meso}}} = \frac{1}{2} \frac{h^2}{m^2}$$

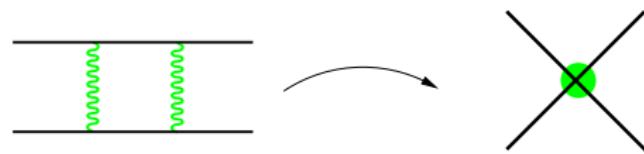
Translating degrees of freedom

BUT:

at k :



at $k - \Delta k$:



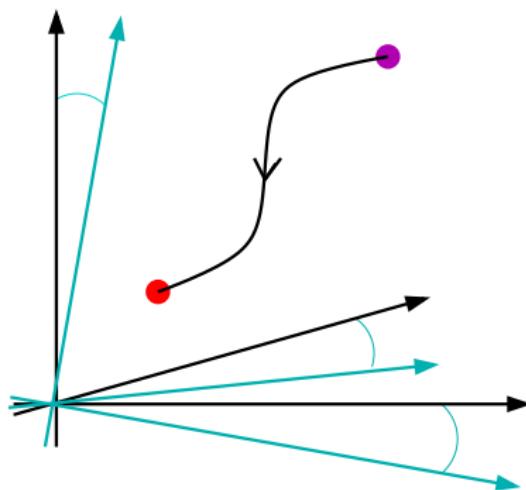
ϕ is only “perfect” at k

Scale-Dependent Field Transformation

► Scale-dependent field transformations

(HG,WETTERICH'01)

$$\phi \rightarrow \phi_k : \quad \phi_k = F_k[\phi, \psi, \dots]$$



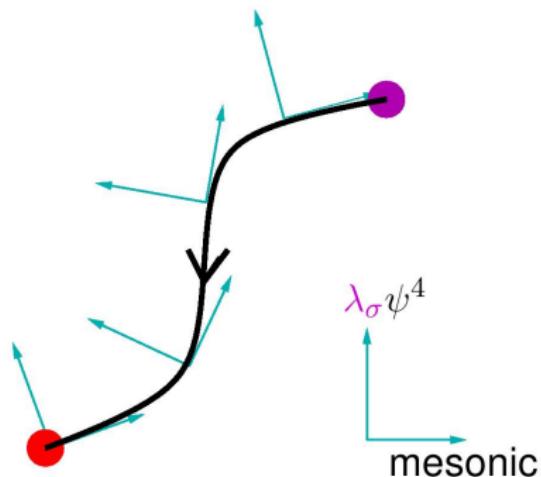
coordinate transformations in field space

Scale-Dependent Field Transformation

► Scale-dependent field transformations

(HG,WETTERICH'01)

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fermion-to-boson translation:

$$\partial_t \phi_k \sim \bar{\psi}_L \psi_R$$

$$\lambda_\sigma \psi^4 \rightarrow 0$$

“re-bosonization”

Scale-Dependent Field Transformation

▷ passive transformations:

(HG,WETTERICH'01)

$$\Gamma_k[\phi_k] = \Gamma_k[\phi[\phi_k]], \quad \partial_t \Gamma_k[\phi_k]|_{\phi_k} = \partial_t \Gamma_k[\phi]|_{\phi=\phi[\phi_k]} - \int \frac{\delta \Gamma_k[\phi_k]}{\delta \phi_k} \partial_t \phi_k$$

▷ active transformations:

(HG,WETTERICH'01; PAWLOWSKI'05)

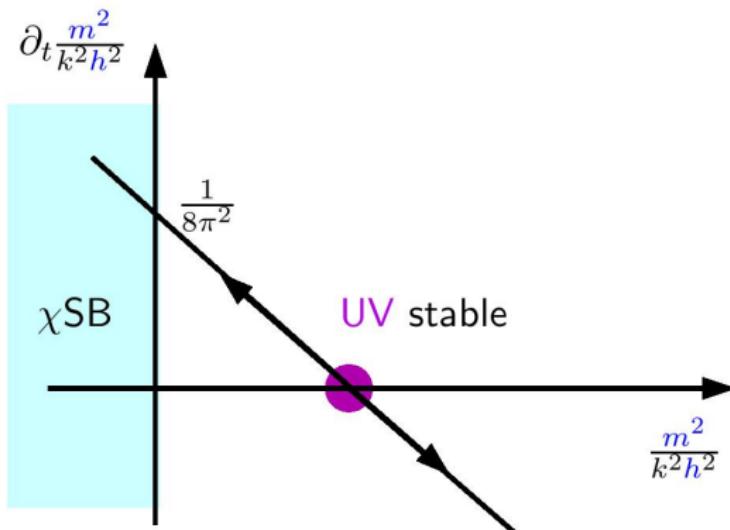
$$(\exp \int J\varphi) \rightarrow (\exp \int J\varphi_k)$$

$$\partial_t \Gamma_k[\phi_k]|_{\phi_k} = \frac{1}{2} \text{Tr} \partial_t R_k G_k + \int \left(G_k \frac{\delta}{\delta \phi_k} \right) R_k \partial_t \phi_k - \int \frac{\delta \Gamma_k[\phi_k]}{\delta \phi_k} \partial_t \phi_k,$$

A simple example a la NJL



$$2 \lambda_\sigma \bar{\psi}_R \psi_L \bar{\psi}_L \psi_R \rightarrow h (\bar{\psi}_R \psi_L \phi - \bar{\psi}_L \psi_R \phi^*) + m^2 \phi^* \phi$$



A simple example: gauged NJL



$$\partial_t \lambda_\sigma = -\frac{9}{8\pi^2} \frac{1}{k^2} e^4$$



A simple example: gauged NJL

- ▷ scale-dependent field transformation

$$\partial_t \phi_{\mathbf{k}} = -(\bar{\psi}_{\mathbf{L}} \psi_{\mathbf{R}}) \partial_t \alpha_{\mathbf{k}}$$

$$\implies \partial_t \lambda_{\sigma} = -\frac{9}{8\pi^2} \frac{1}{k^2} e^4 - h \partial_t \alpha_{\mathbf{k}}$$

- ▷ Idea: choose $\alpha_{\mathbf{k}}$ such that $\partial_t \lambda_{\sigma} \stackrel{!}{=} 0$ (simplicity)

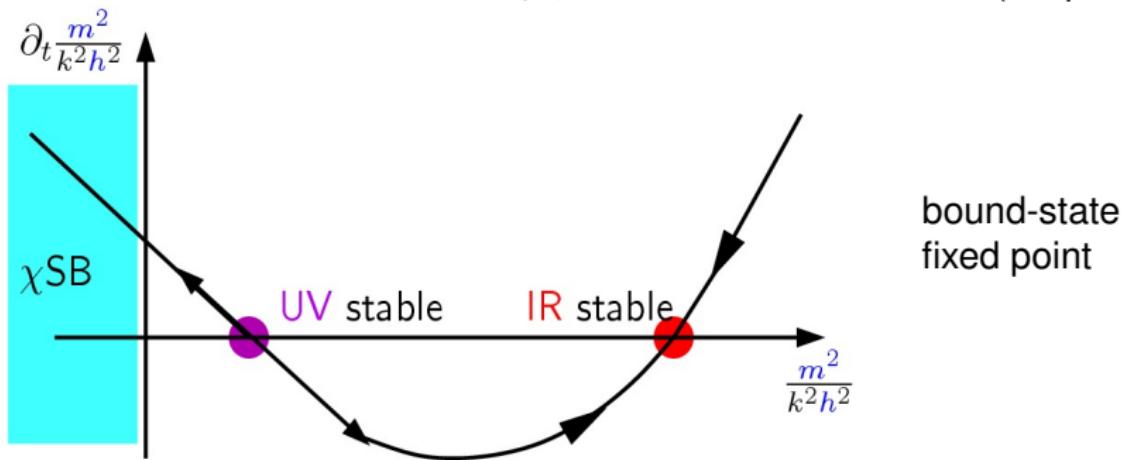
A simple example: gauged NJL

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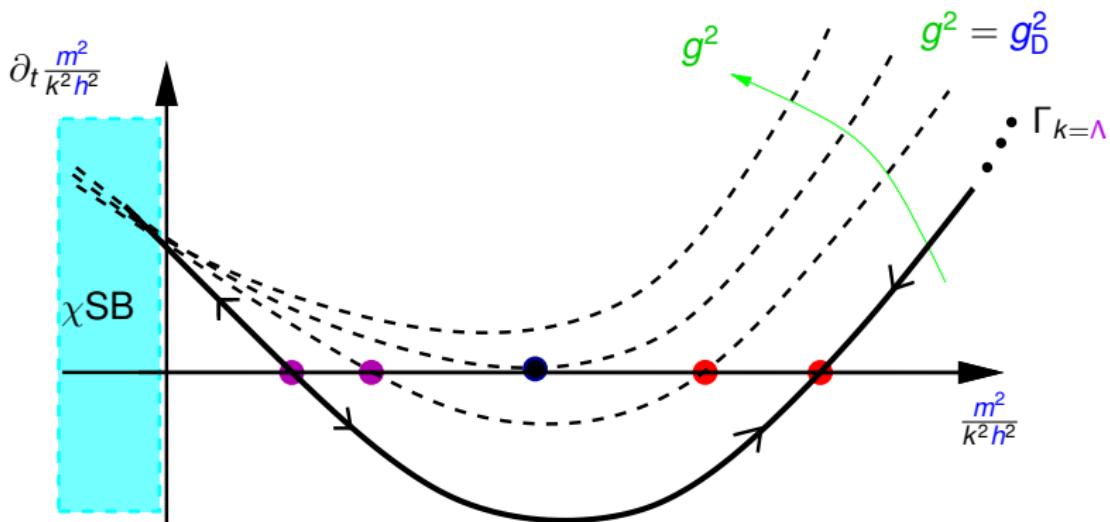
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- ▷ Idea: choose $\alpha_{\mathbf{k}}$ such that $\partial_t \lambda_{\sigma} \stackrel{!}{=} 0$ (simplicity)



Approach to criticality

- ▷ destabilization of the bound-state fixed point



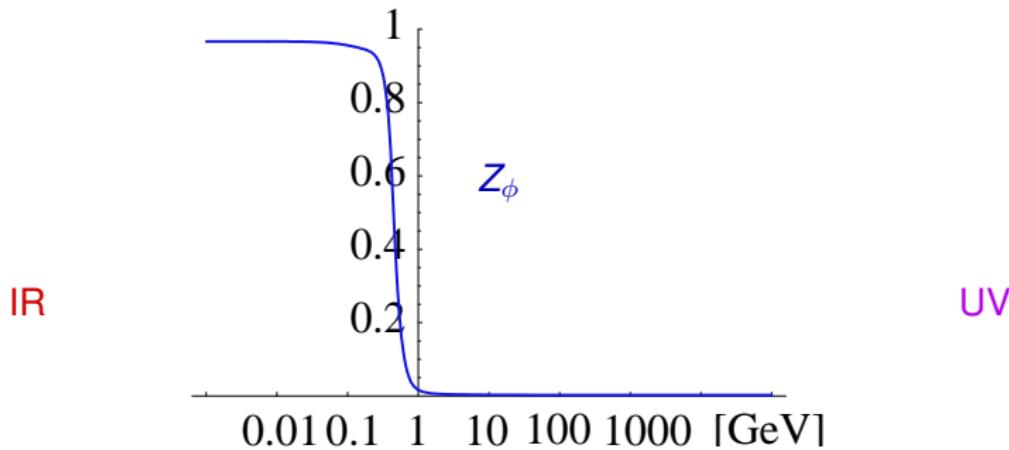
$$\text{e.g., } \alpha_{D,\text{QCD}} \equiv \frac{g_D^2}{4\pi} \simeq 0.7$$

χ SB Flow

▷ mesonic wave function renormalization:

(HG,WETTERICH'04)

$$\Gamma_k^{\text{meson}} = \int Z_\phi |\partial_\mu \phi|^2 + V(\phi)$$

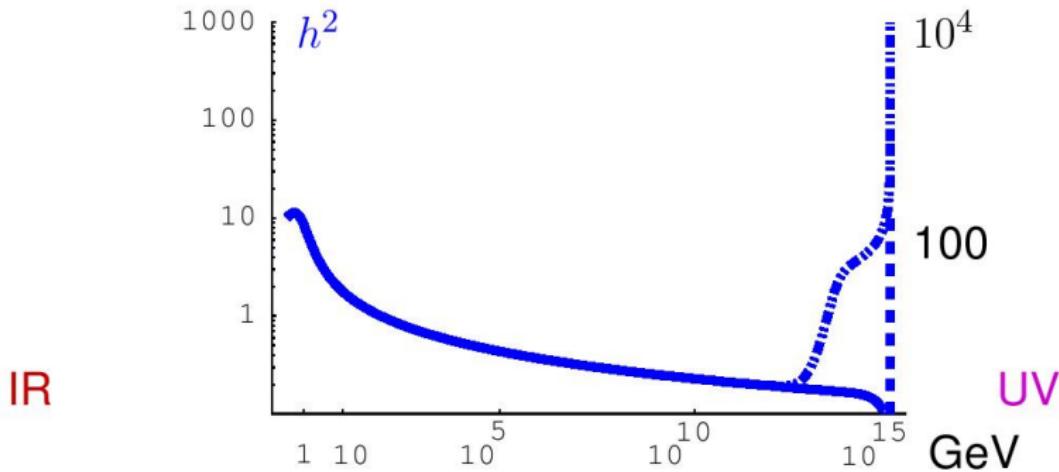


ϕ dynamical for $k < 1\text{GeV}$

χ SB Flow

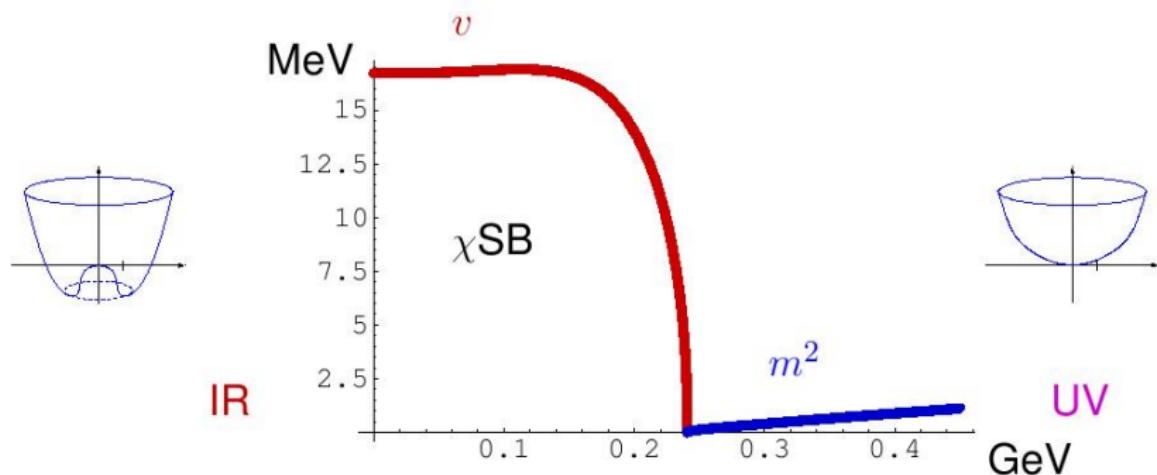
- ▷ (In-)dependence of bosonic initial conditions:

e.g., Yukawa coupling



χ SB Flow

- approach to χ SB in SU(3), $N_f = 1$ QCD:

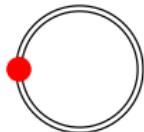


- IR prediction:

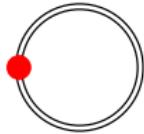
$$f_\pi = 2v = 33 \text{ MeV}$$

(without $U_A(1)$ anomaly)

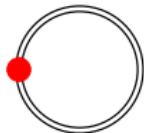
Conclusion



functional RG as a universal theoretical tool
for complex fermionic systems



ultracold fermionic atoms
as a benchmark for precision methods
in nonperturbative field theory



Scale-dependent field transformations:
“choose **relevant** DoFs . . . and let the flow decide”

RG: “Physics of Scales”

Length		Momentum	
atom radius	r_A/a_B	2-4	Λ/eV
vdW length	l_{vdW}/a_B	50-200	k_{vdW}/eV
density	d/a_B	600-2800	k_F/eV
T de Broglie	λ_{dB}/a_B	8800-39000	k_{dB}/eV
trap	l_{trap}/a_B	2900-290000	$k_{\text{trap}}/\text{eV}$

► Bohr's radius: $a_B \simeq 5.3 \times 10^{-11} \text{ m}$

► $k_F \simeq 1 \text{ eV} \implies n \simeq 4.4 \times 10^{12} \text{ cm}^{-3}$

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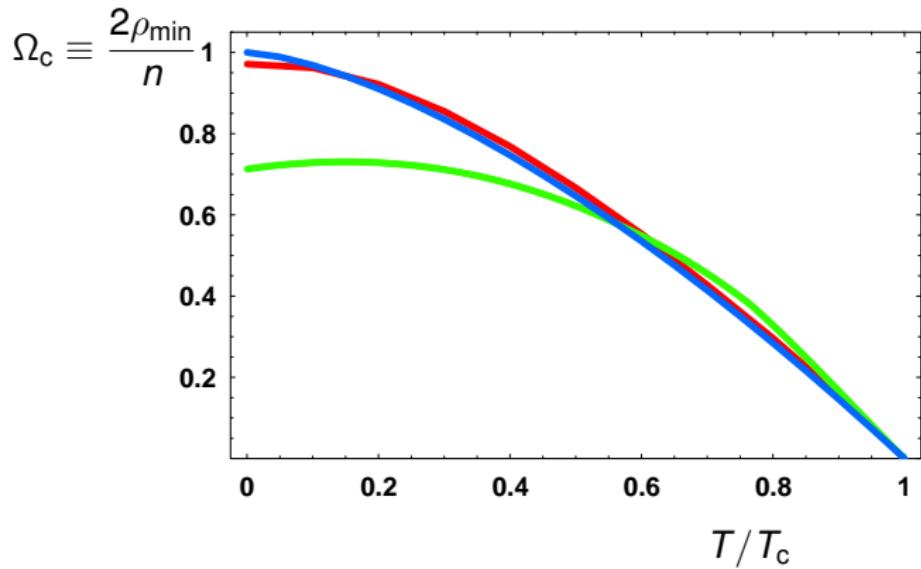
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Second-Order Phase Transition

- ▷ condensate fraction

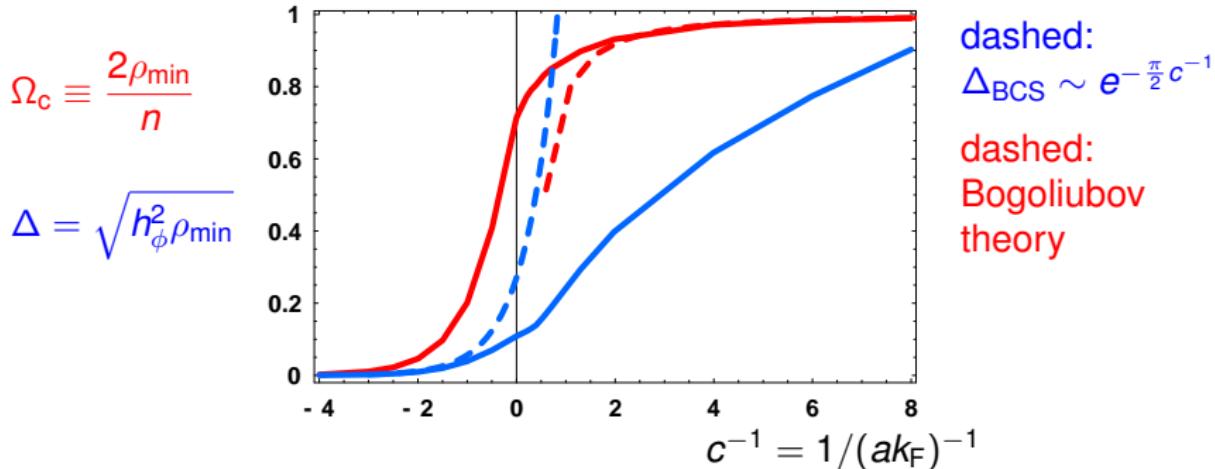


BEC
 $c^{-1} \rightarrow \infty$
 $c^{-1} > 0$
 $c^{-1} = 0$

2nd order
phase
transition

BCS and BEC limiting cases

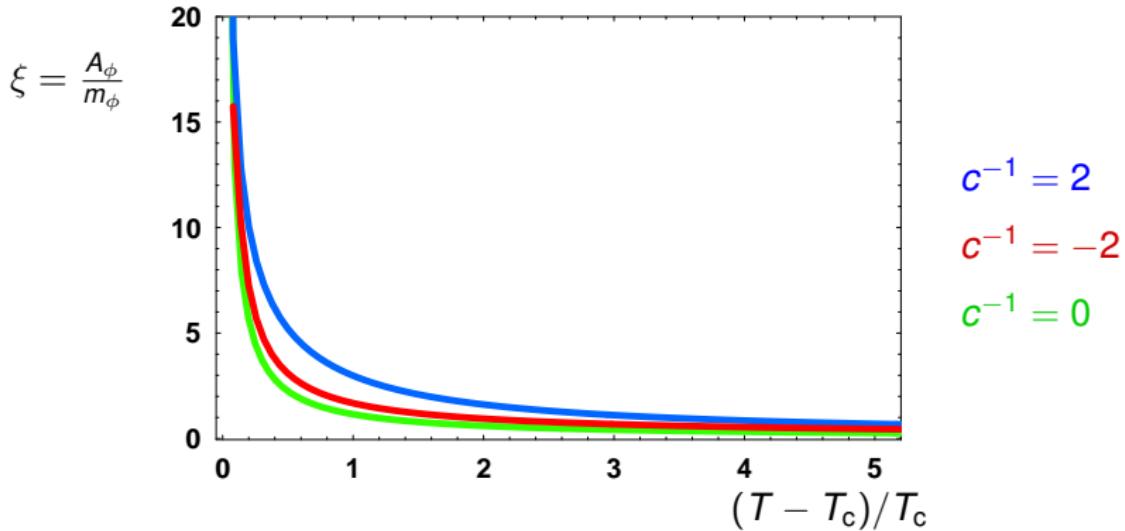
- condensate fraction and gap



dashed:
 $\Delta_{\text{BCS}} \sim e^{-\frac{\pi}{2}c^{-1}}$

dashed:
Bogoliubov
theory

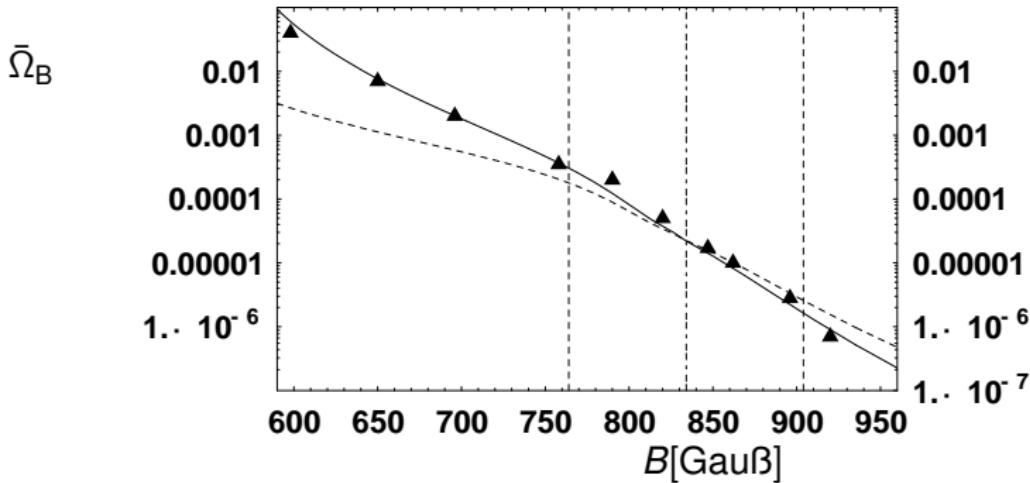
Correlation Length



Non-Universal Aspects

- ▷ Bare Molecule Fraction for ${}^6\text{Li}$

$$\bar{\Omega}_B = Z_\phi^{-1}(\Omega_M + \Omega_c) \equiv \bar{h}_{\phi,0}^{-2}(\Delta B, \lambda_{bg}) h_\phi^2(\Omega_M + \Omega_c)$$

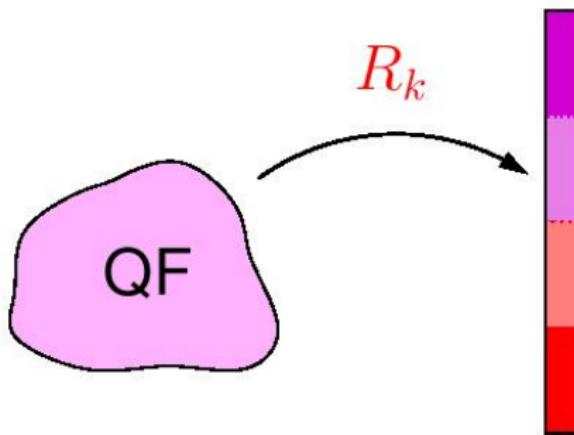


(PARTRIDGE ET AL.'05; DIEHL, WETTERICH'05)

Functional RG Flow Equation

$$\partial_t \Gamma_k = \frac{1}{2} \operatorname{Tr} \frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k$$

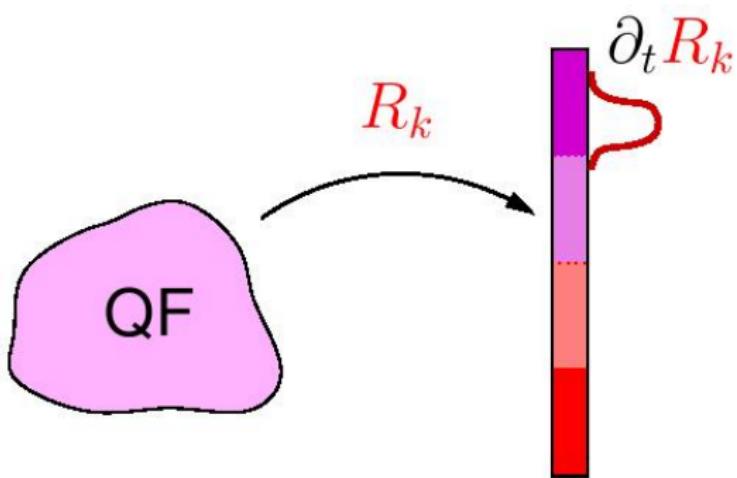
▷ quantum fluctuations:



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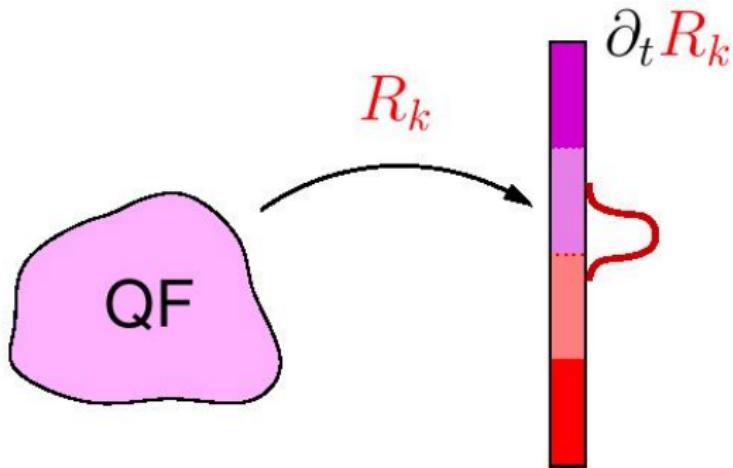
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