Quantum Vacuum on the Worldline

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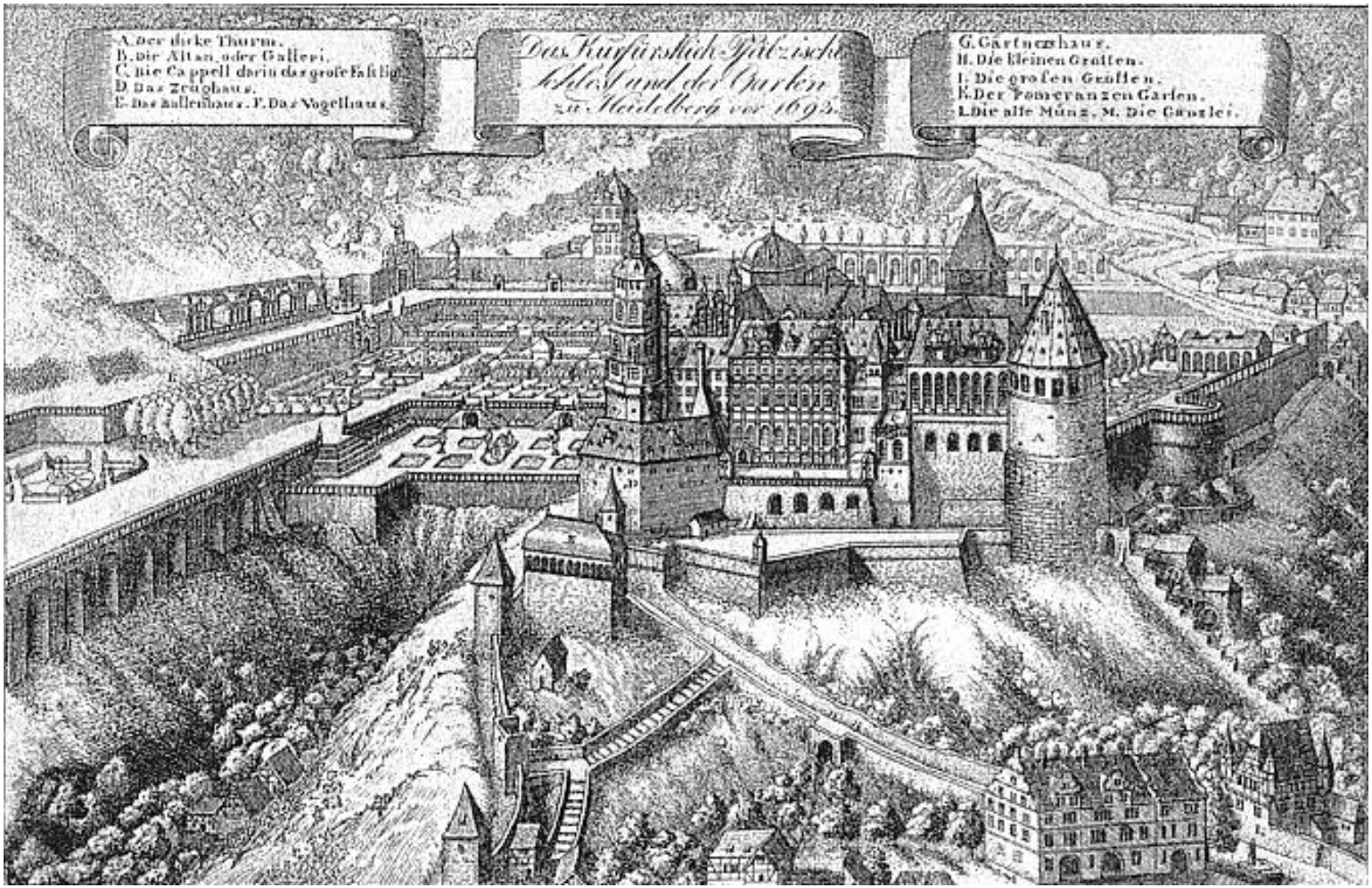
Topography of QFT.

(MERIAN 1620)



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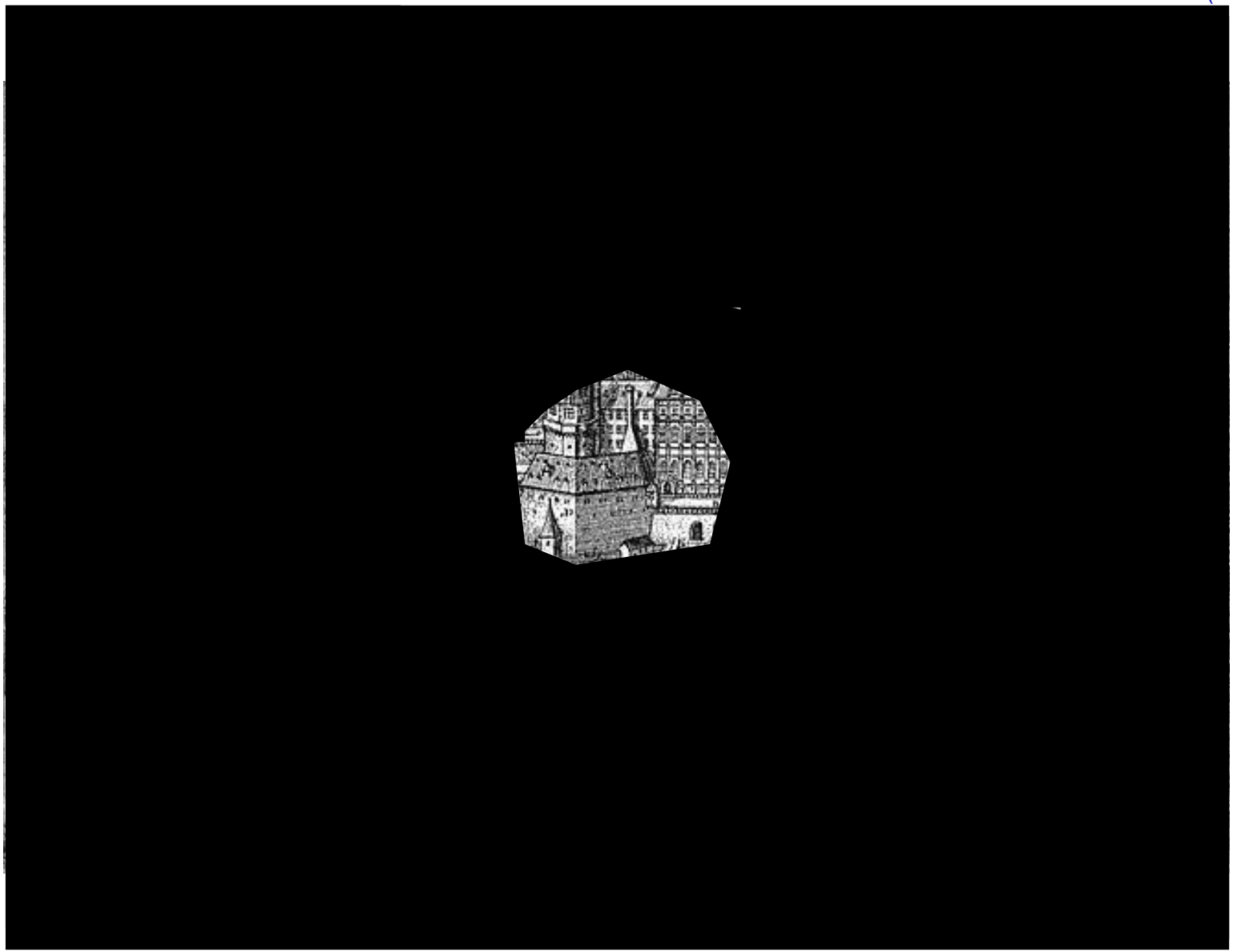

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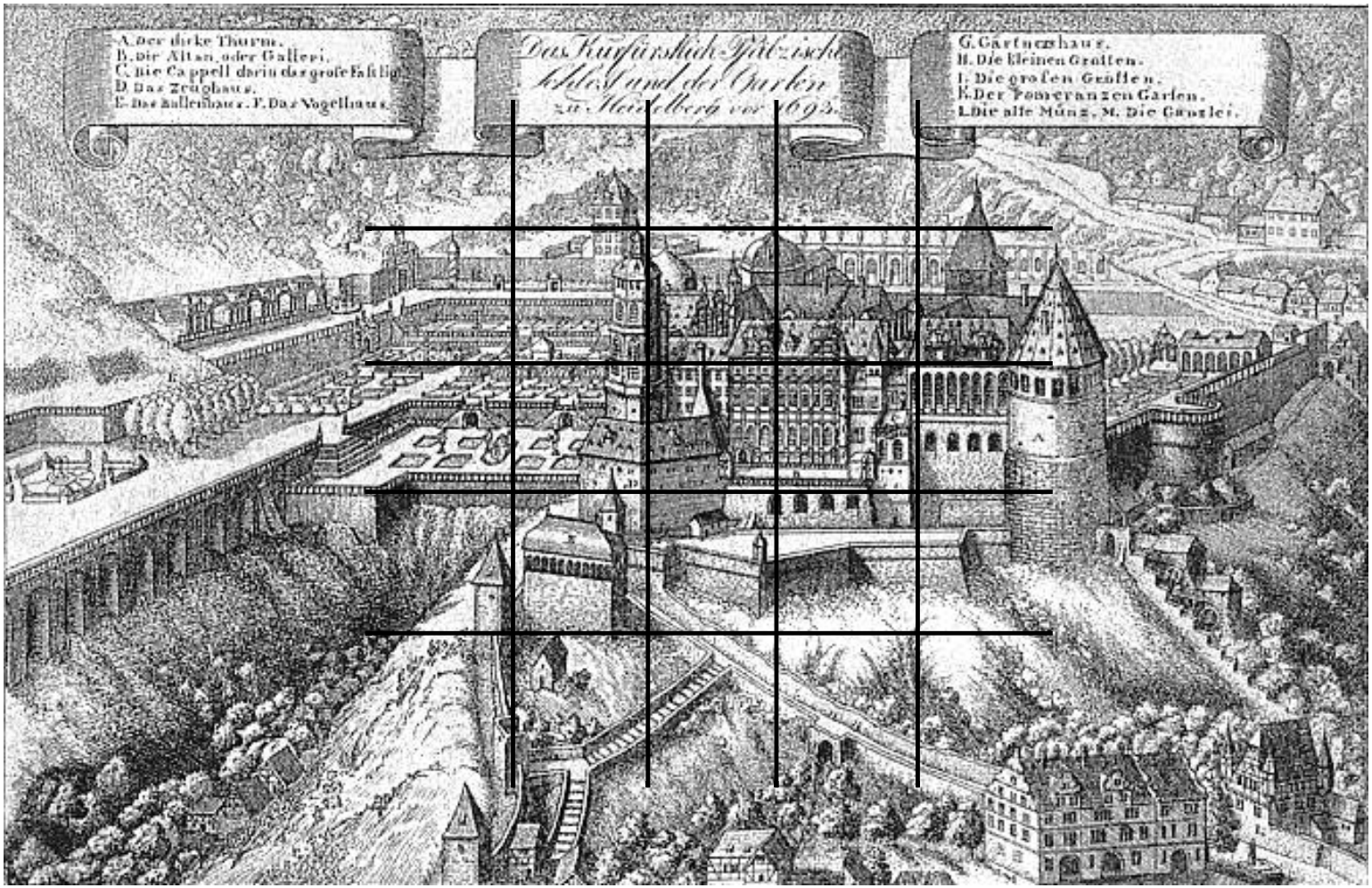

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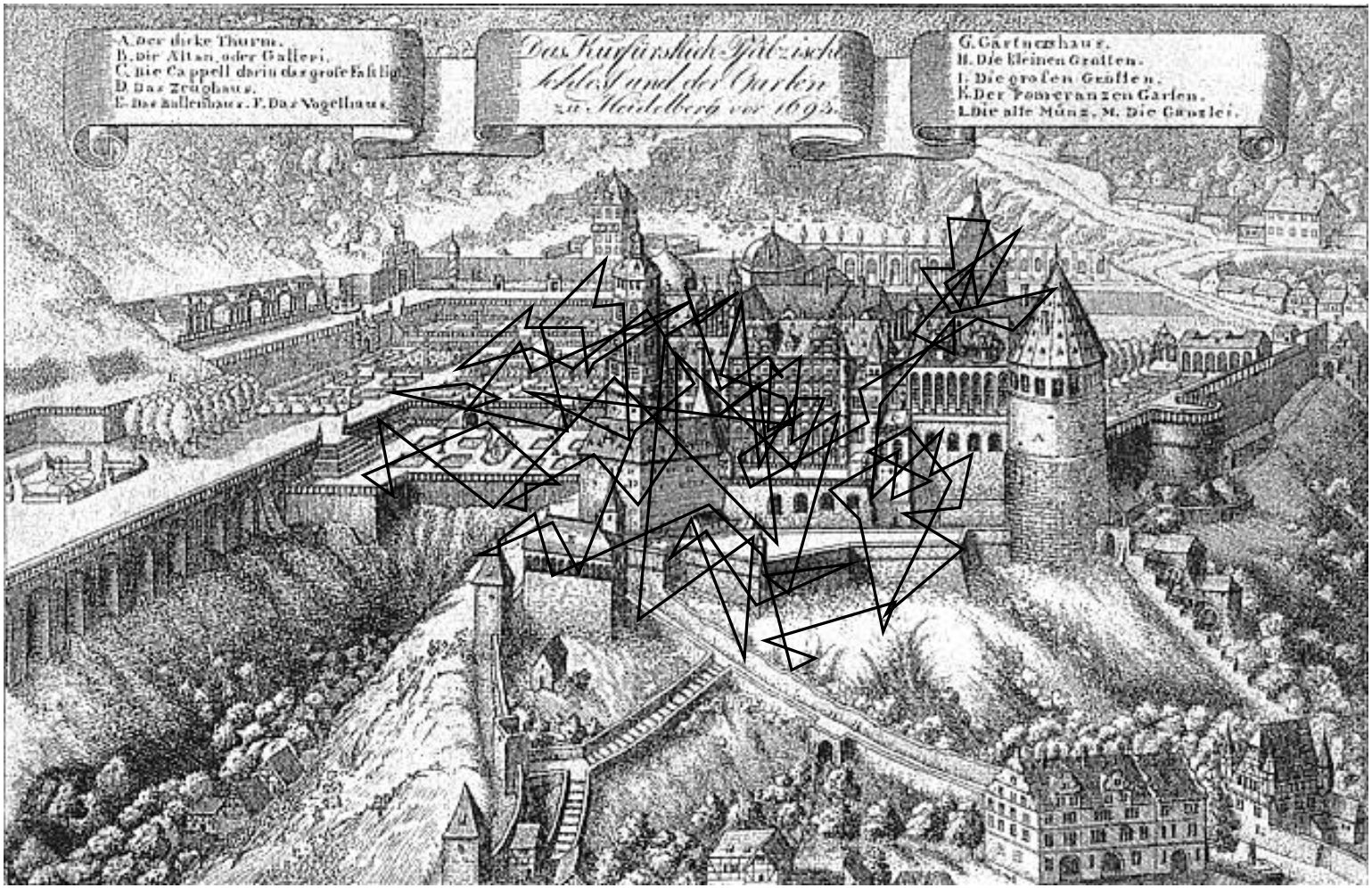

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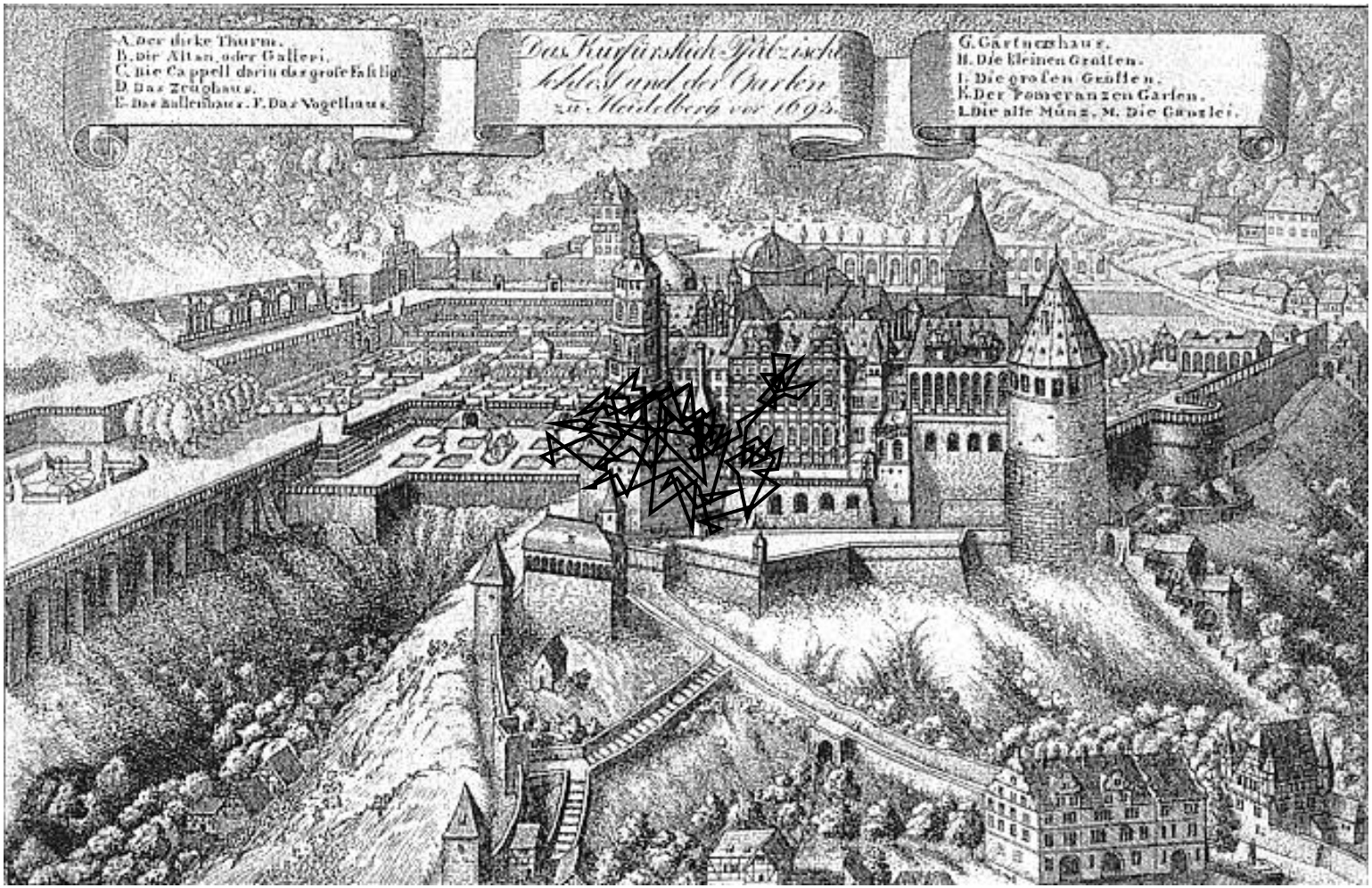

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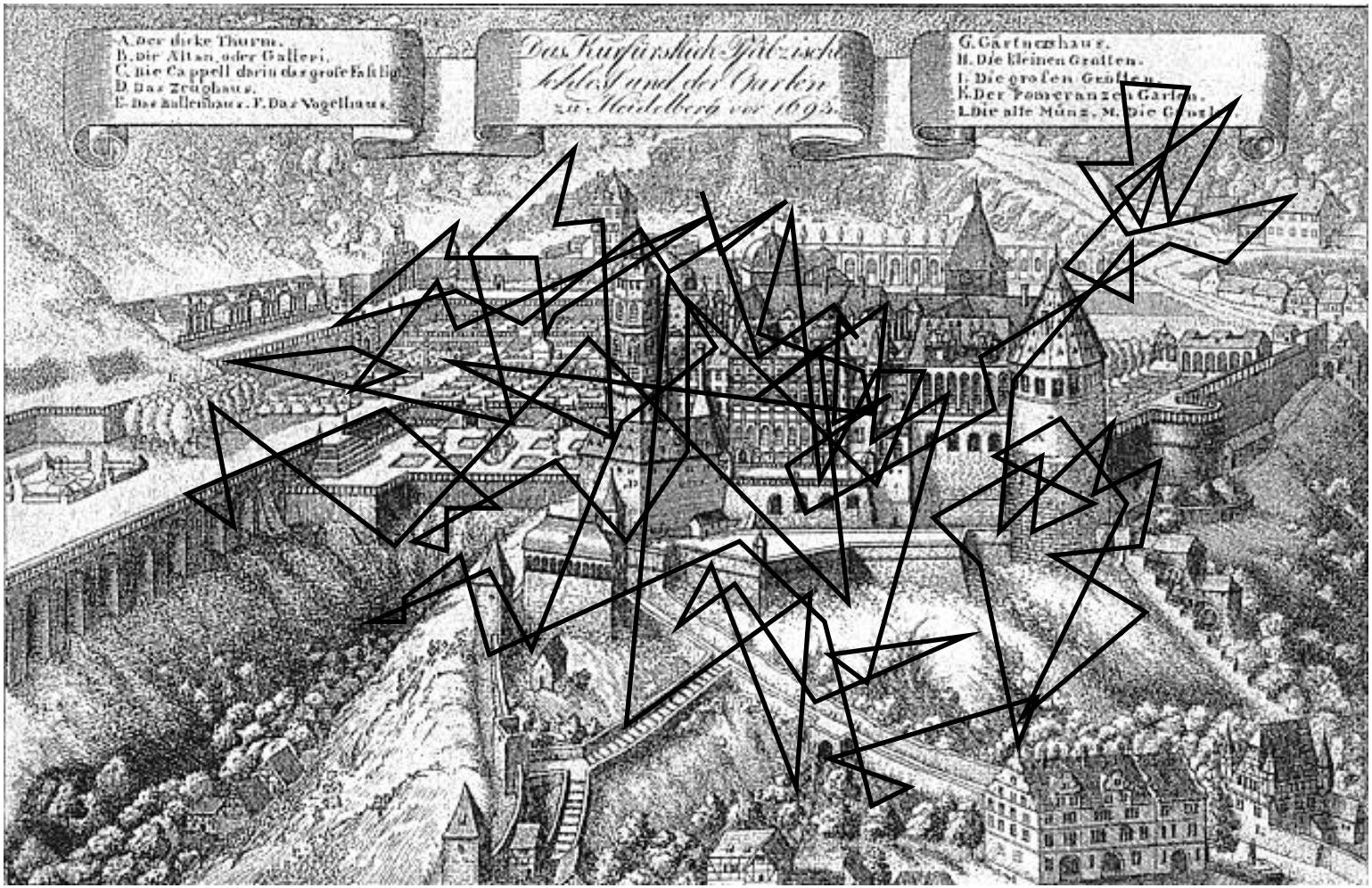
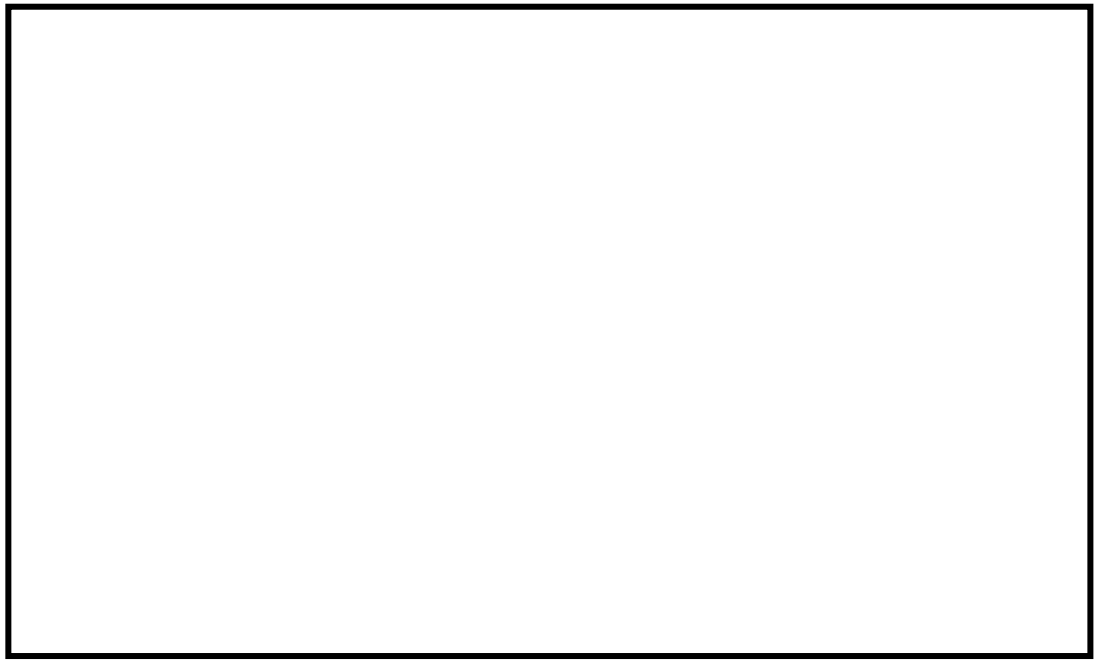
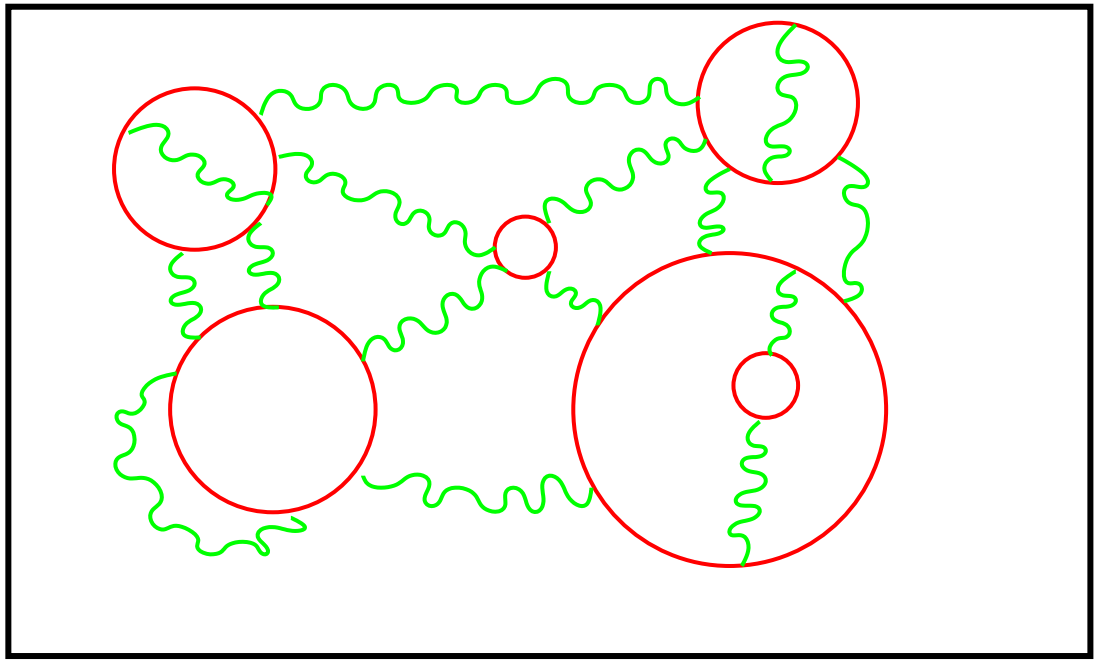


A view on the quantum vacuum.



- $\rho \longrightarrow 0$: “pneumatic vacuum”

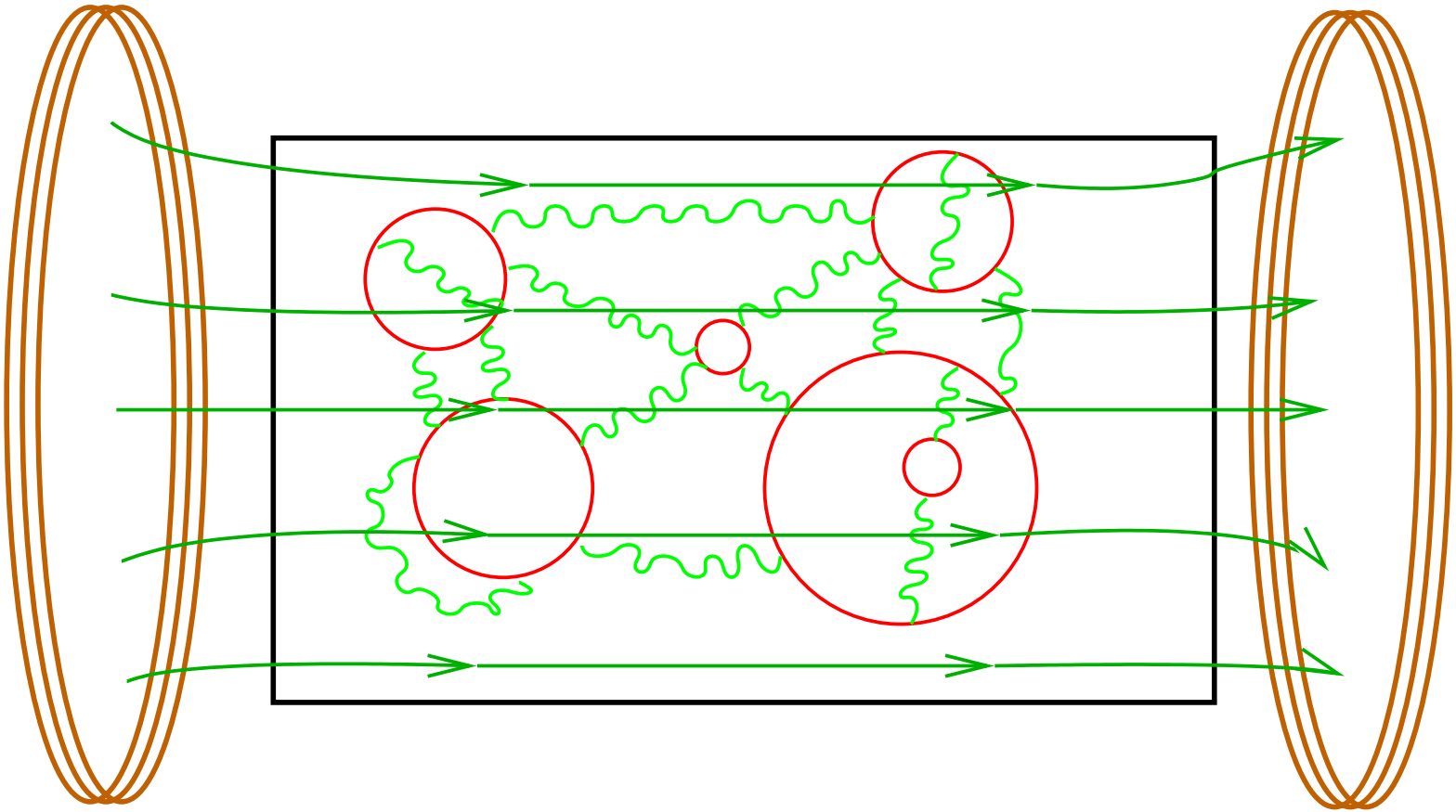
A view on the quantum vacuum.



- QFT: quantum fluctuations

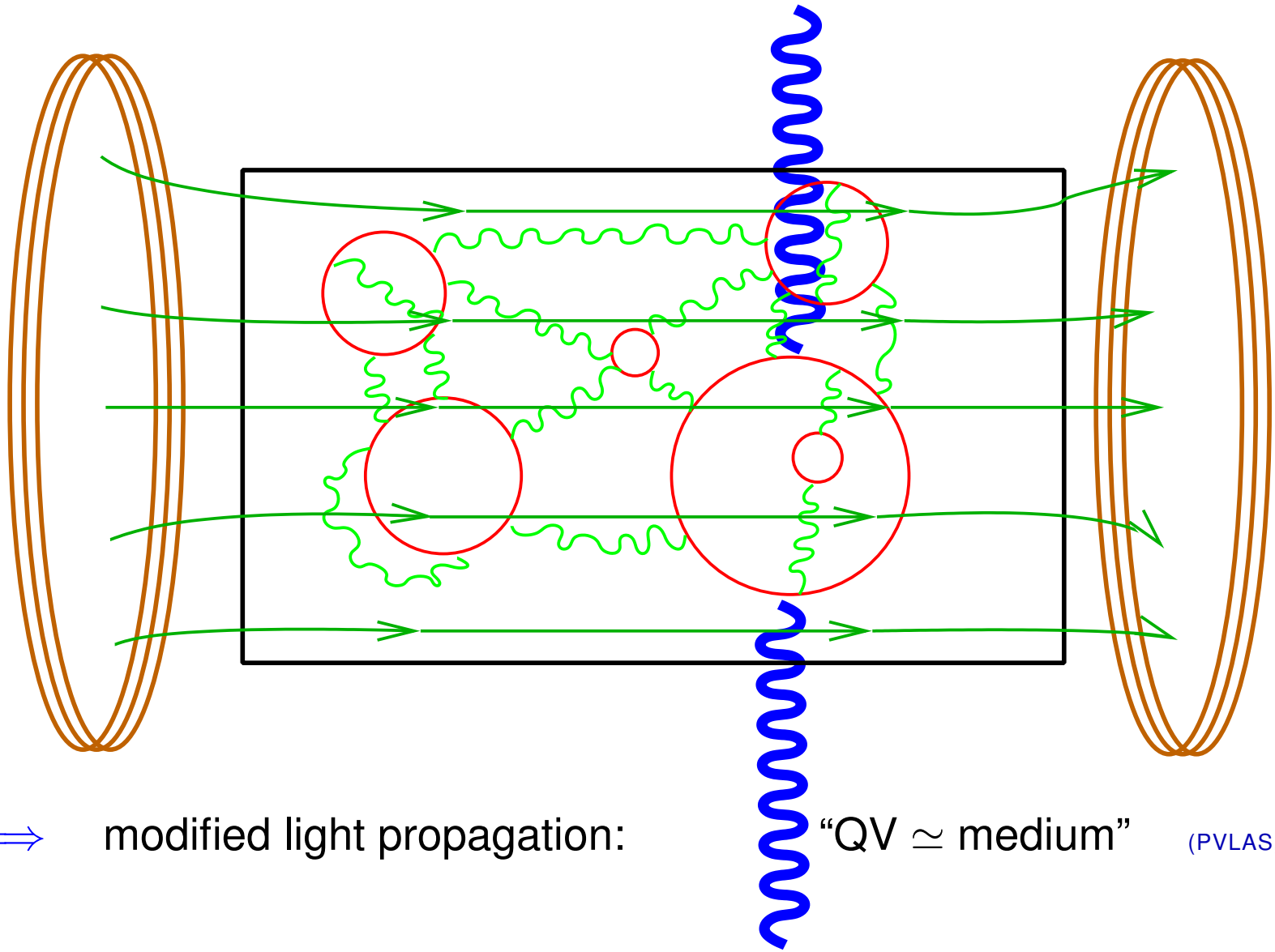
BUT: . . . just a picture !

A view on the quantum vacuum.



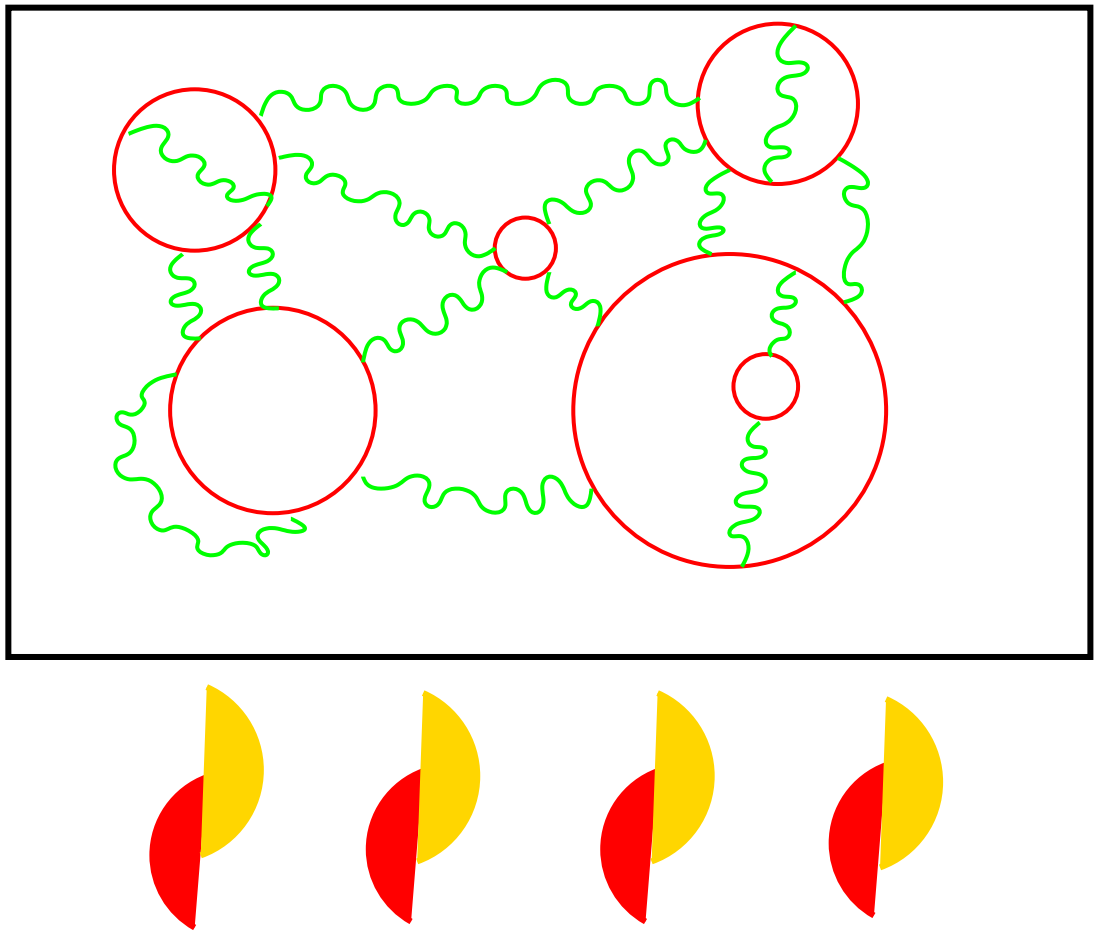
- Probing the quantum vacuum, e.g., by external fields:
  “modified quantum vacuum”

A view on the quantum vacuum.



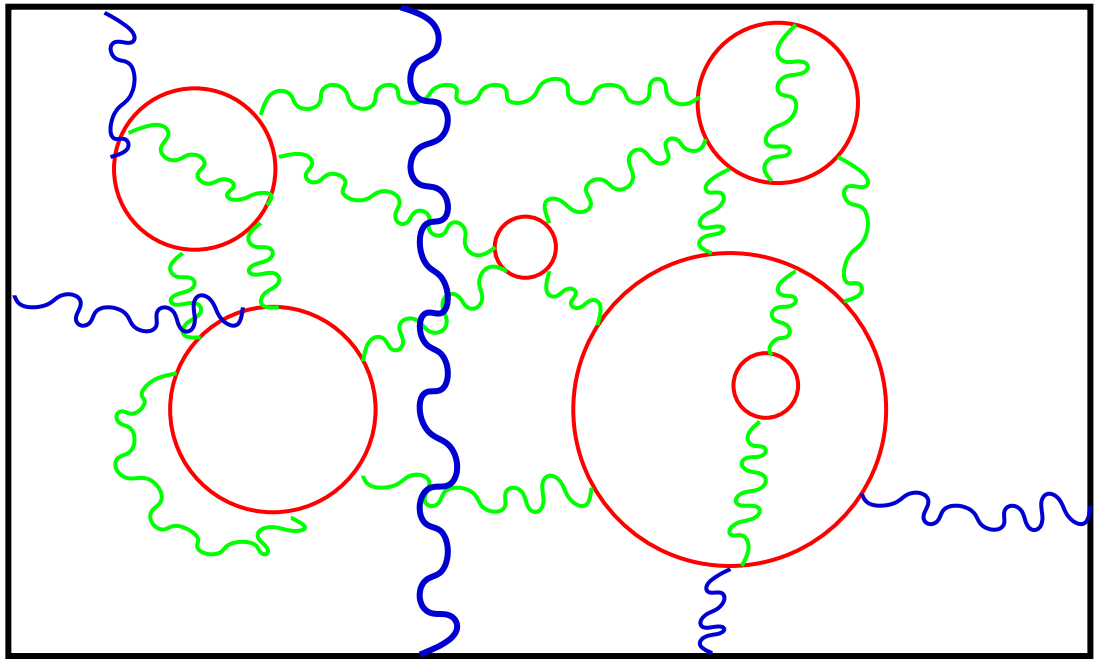
$\Longrightarrow$ modified light propagation: “QV $\simeq$ medium” (PVLAS,BMV,Q&A)

A view on the quantum vacuum.



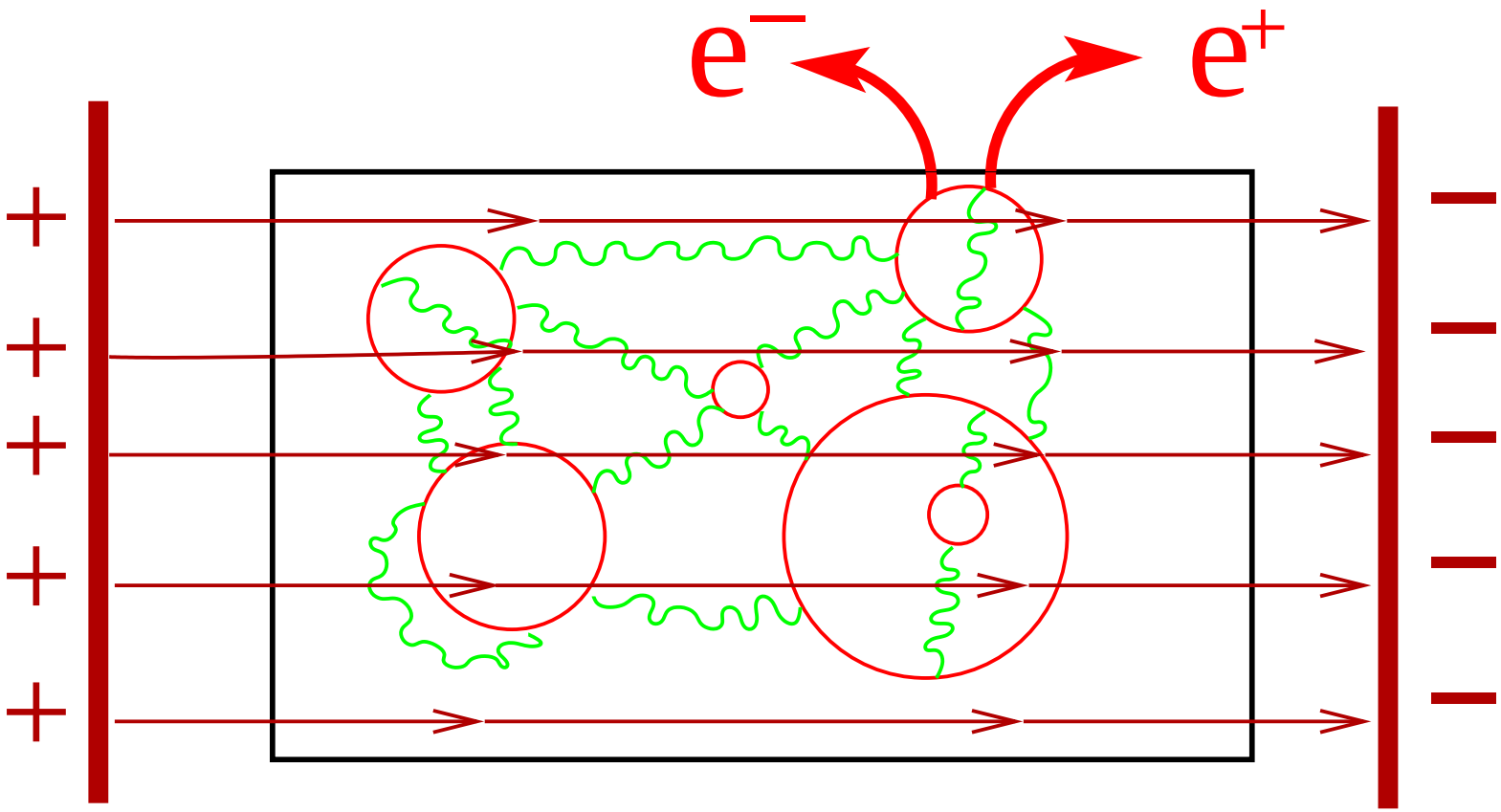
- Heat bath: quantum & thermal fluctuations

A view on the quantum vacuum.



- Boundary conditions: Casimir effect

A view on the quantum vacuum.



- electric fields: Schwinger pair production “vacuum decay”

Universal tool: effective action Γ.

Quantum vacuum with background A

$$\int \text{fluctuations} \quad \to \quad \Gamma[A]$$

$$\Gamma[A] \Longrightarrow \begin{cases} \frac{\delta\Gamma[A]}{\delta A} = 0, & \text{quantum Maxwell equations} \to \text{(light prop.)} \\ E_{\text{QV}} = \frac{\Gamma[A]}{T}, \quad F_{\text{Casimir}} = -\frac{\partial E_{\text{QV}}}{\partial A}, & \text{Casimir force} \\ \mathcal{W} = 2\,\text{Im}\,\frac{\Gamma[A]}{VT}, & \text{Schwinger pair production rate} \end{cases}$$

(HEISENBERG&EULER'36; WEISSKOPF'36; SCHWINGER'51)

(CASIMIR'48)

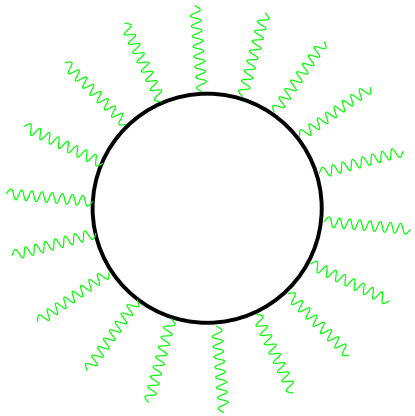
Universal tool: effective action Γ.

Quantum vacuum with background A, e.g., scalar QED

$$\int \text{fluctuations} \quad \rightarrow \quad \Gamma[A]$$

$$\Gamma[A] \quad = \quad -\ln \int \mathcal{D}\phi \, e^{-\int -|D(A)\phi|^2 + m^2|\phi|^2} \quad =$$

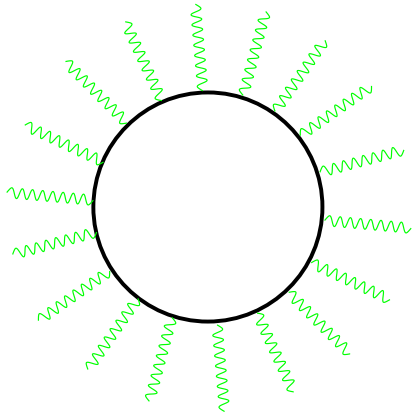
$$= \quad \sum_{\lambda} \ln\left(\lambda^2 + m^2\right)$$



- spectrum of quantum fluctuations: $-D(A)^2 \; \phi = \lambda^2 \; \phi$

Universal tool: effective action Γ.

$$\Gamma[A] \quad = \quad \sum_{\lambda} \ln\left(\lambda^2 + m^2\right) \quad =$$

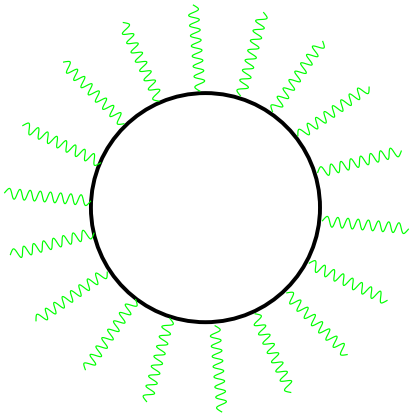
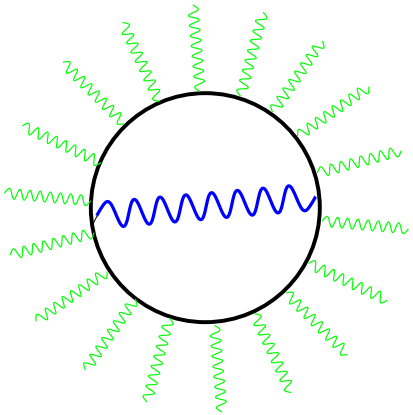


Problem solved, "in principle"

- find spectrum λ for a given background A
- sum over spectrum

Heisenberg-Euler effective action.

(Euler, Kockel'35; Heisenberg, Euler'36;Weisskopf'36; Schwinger'51; Ritus'76)

$$\Gamma =$$ $+$  $+$  $+\ldots$

1%

$$= -\int_x \mathcal{F} + \frac{1}{8\pi^2}\int_x \int \frac{ds}{s} e^{-im^2 s}\left[(es)^2|\mathcal{G}|\cot(es\sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F}})\right.$$
$$\left.\times \coth(es\sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F}})\right]\ldots$$

Conventions: $\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(B^2 - E^2)$, $\mathcal{G} = \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = -B\cdot E$

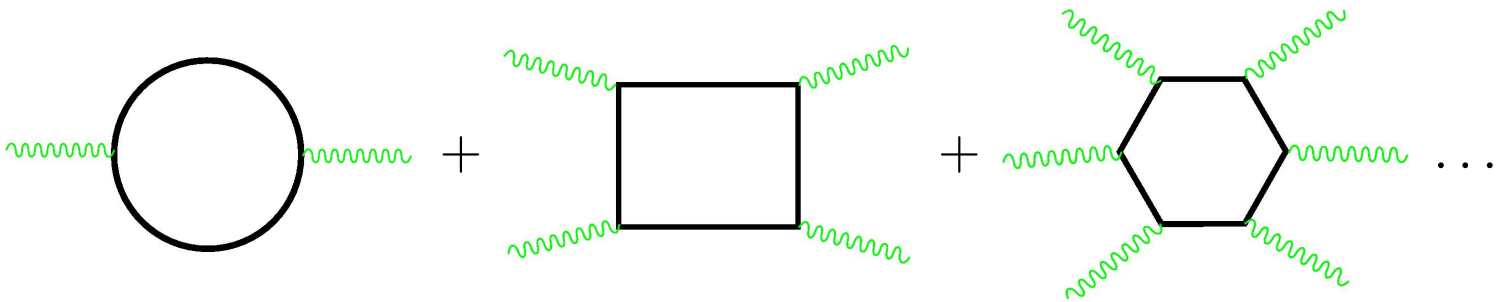
Heisenberg-Euler effective action.

(Euler, Kockel'35; Heisenberg, Euler'36;Weisskopf'36; Schwinger'51; Ritus'76)

▷ weak-field expansion

$$\Gamma = \int \Big\{ \quad -\mathcal{F} \quad + \frac{8}{45}\frac{\alpha^2}{m^4}\mathcal{F}^2 + \frac{14}{45}\frac{\alpha^2}{m^4}\mathcal{G}^2 + \quad \mathcal{O}(F^6) \quad \Big\}$$

$$= \quad + \quad + \quad \ldots$$



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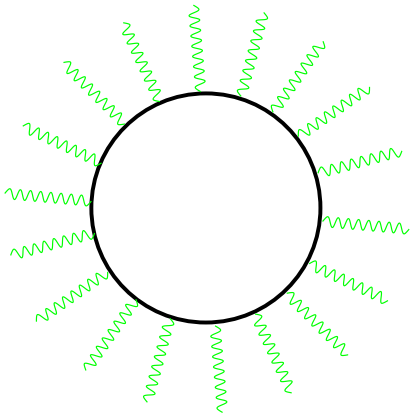
Universal tool: effective action Γ.

Remember ...

$$\Gamma[A] \quad = \quad \sum_{\lambda} \ln\left(\lambda^2 + m^2\right) \quad =$$



Problem solved, "in principle"

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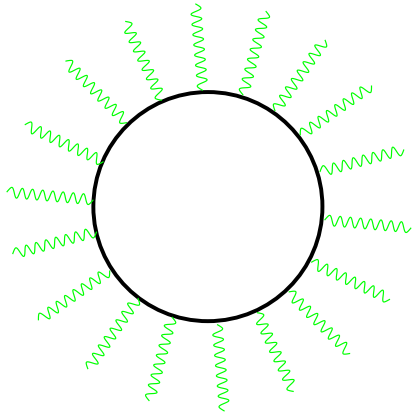
$$\Gamma[A] \quad = \quad \sum_{\lambda} \ln\left(\lambda^2 + m^2\right) \quad =$$



BUT:

In general practice:

- spectrum $\{\lambda\}$ not known analytically
- spectrum $\{\lambda\}$ not bounded
- $\sum_\lambda \to \infty$ (regularization)
- renormalization

Worldline representation of Γ.

- pedestrian approach

$$\Gamma[A] = \sum_{\lambda} \ln\left(\lambda^2 + m^2\right) = \mathrm{Tr}\,\ln\left[-(D(A))^2 + m^2\right]$$

$$= -\int_{1/\Lambda^2}^{\infty} \frac{dT}{T}\, e^{-m^2 T} \underbrace{\mathrm{Tr}\,\exp\left[D(A)^2\, T\right]}_{=\langle x|e^{iH(iT)}|x\rangle}$$

$$= -\int_{1/\Lambda^2}^{\infty} \frac{dT}{T}\, e^{-m^2 T}\, \mathcal{N} \int_{x(T)=x(0)} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ie\,\dot{x}\cdot A(x_\tau)\right)}$$

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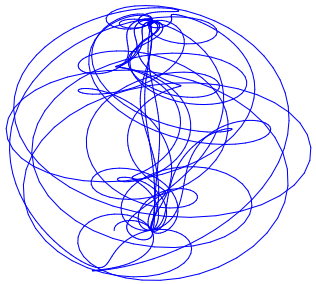
$x(T) \;=$ 

(Feynman'50)

⋮

(Halpern&Siegel'77)

(Polyakov'87)

⋮

(Bern&Kosower'92; Strassler'92)

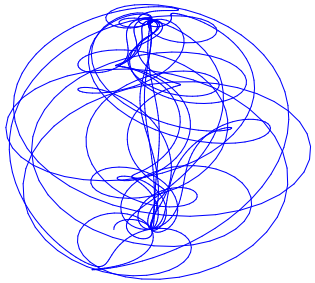
(Schmidt&Schubert'93)

(Kleinert'94)

Worldline representation of Γ.

$$\Gamma[A] \;=\; -\int_{1/\Lambda^2}^{\infty}\frac{dT}{T}\,e^{-m^2T}\;\mathcal{N}\int_{x(T)=x(0)}\mathcal{D}x(\tau)\,e^{-\int_0^T d\tau\left(\frac{\dot{x}^2}{4}+ie\,\dot{x}\cdot A(x\tau)\right)}$$

$$x(T) \;=$$



Worldline approach:

- effective action $\Gamma \sim \int$ closed worldlines $x(\tau)$
- worldline $\sim$ spacetime trajectory of ϕ fluctuations
- gauge-field interaction $\sim$ "Wegner-Wilson loop"
- finding $\{\lambda\}$ and $\sum_\lambda$ done in one finite (numerical) step (HG&LANGFELD'01)

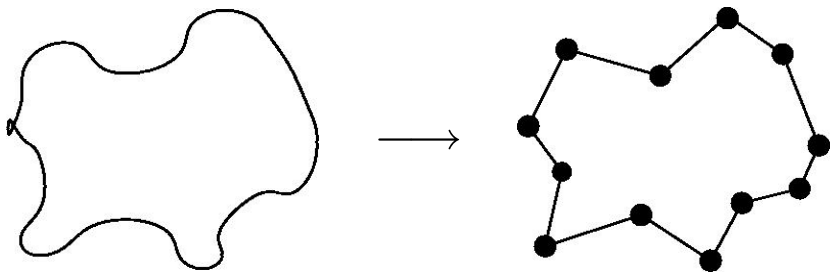
Worldline Numerics.

$$\int_{x(1)=x(0)} \mathcal{D}x(t) \quad \longrightarrow \quad \sum_{l=1}^{n_{\mathrm{L}}}, \qquad n_{\mathrm{L}} = \#\ \text{of worldlines}$$

$\longrightarrow$ statistical error

$$x(t) \quad \longrightarrow \quad x_i, \qquad i = 1, \ldots, N\ \text{(ppl)}$$

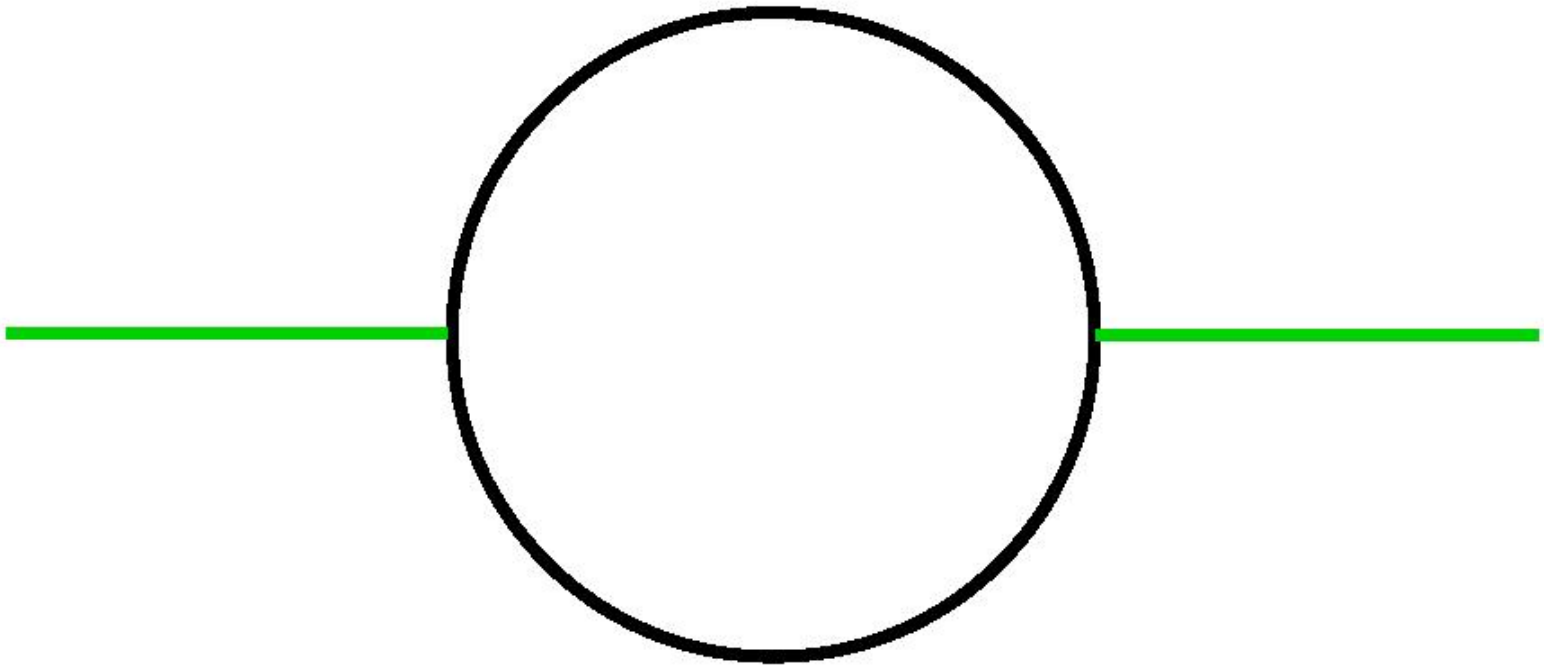
$\longrightarrow$ systematical error



$\longrightarrow$ spacetime remains continuous

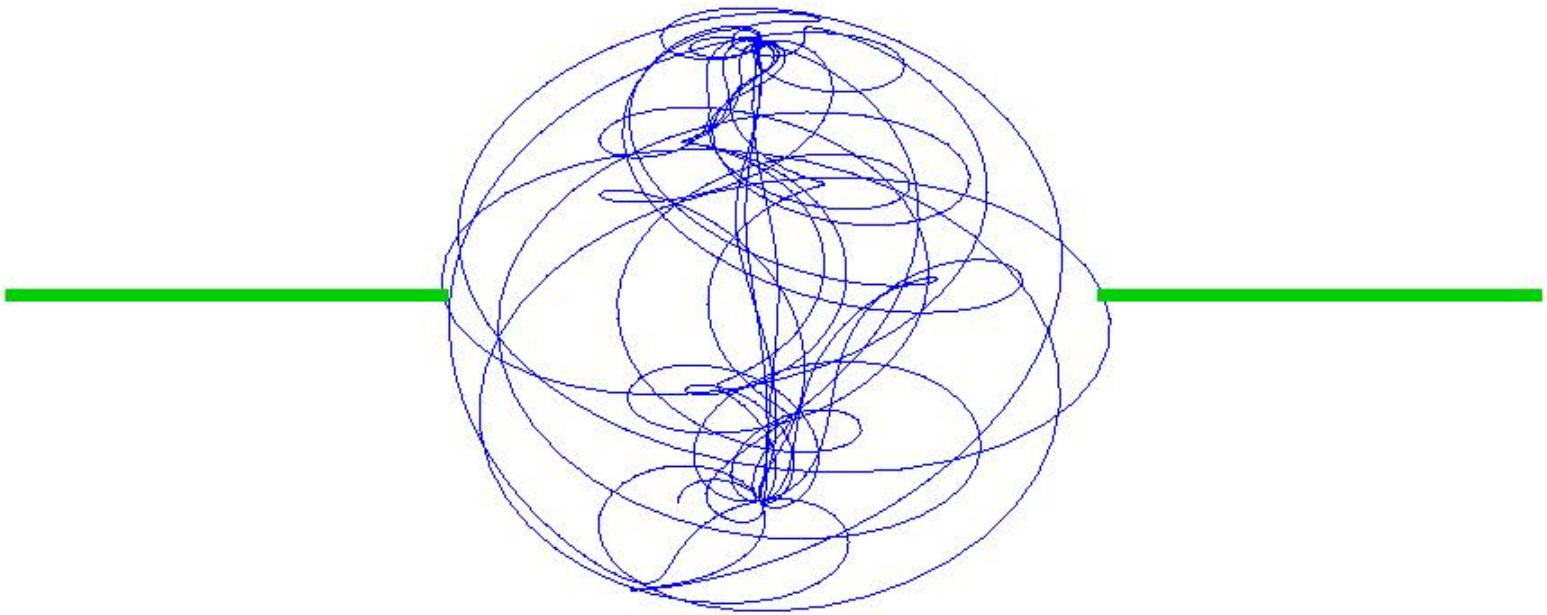
Trajectory of a Quantum Fluctuation.

- Feynman diagram (conventionally in momentum space)



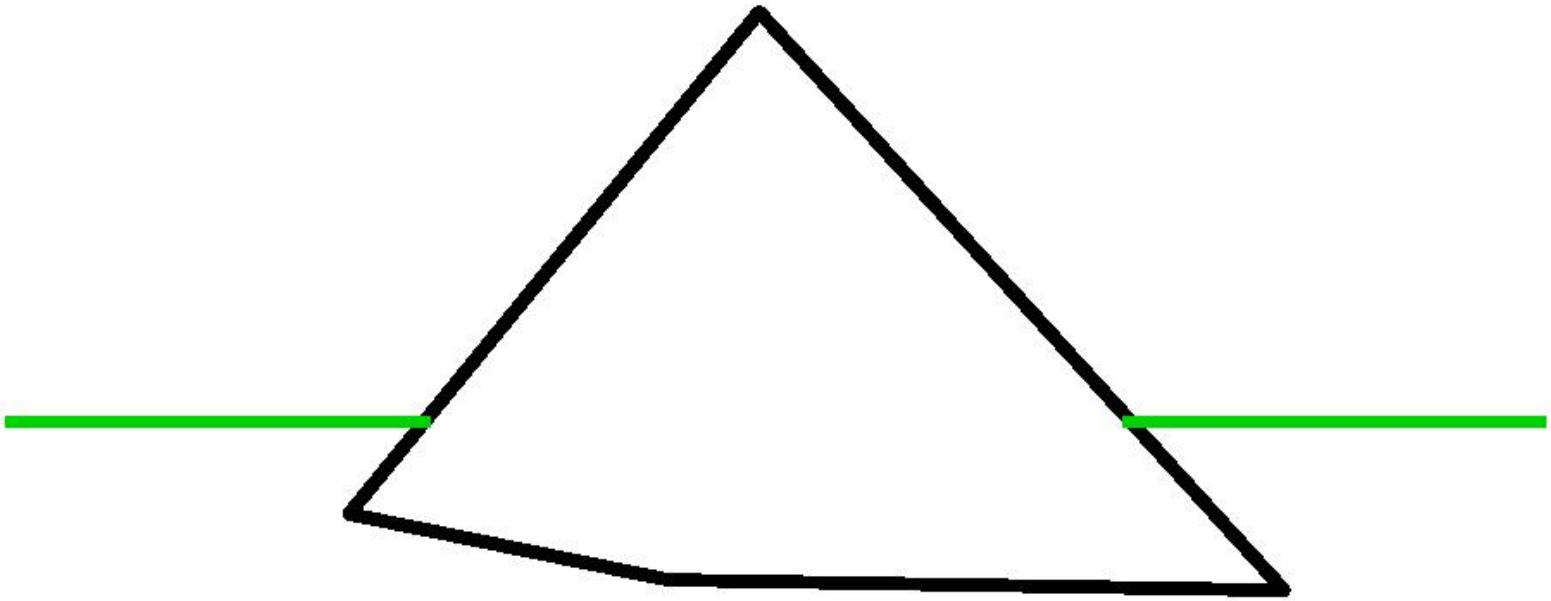
Trajectory of a Quantum Fluctuation.

- worldline (artist's view)



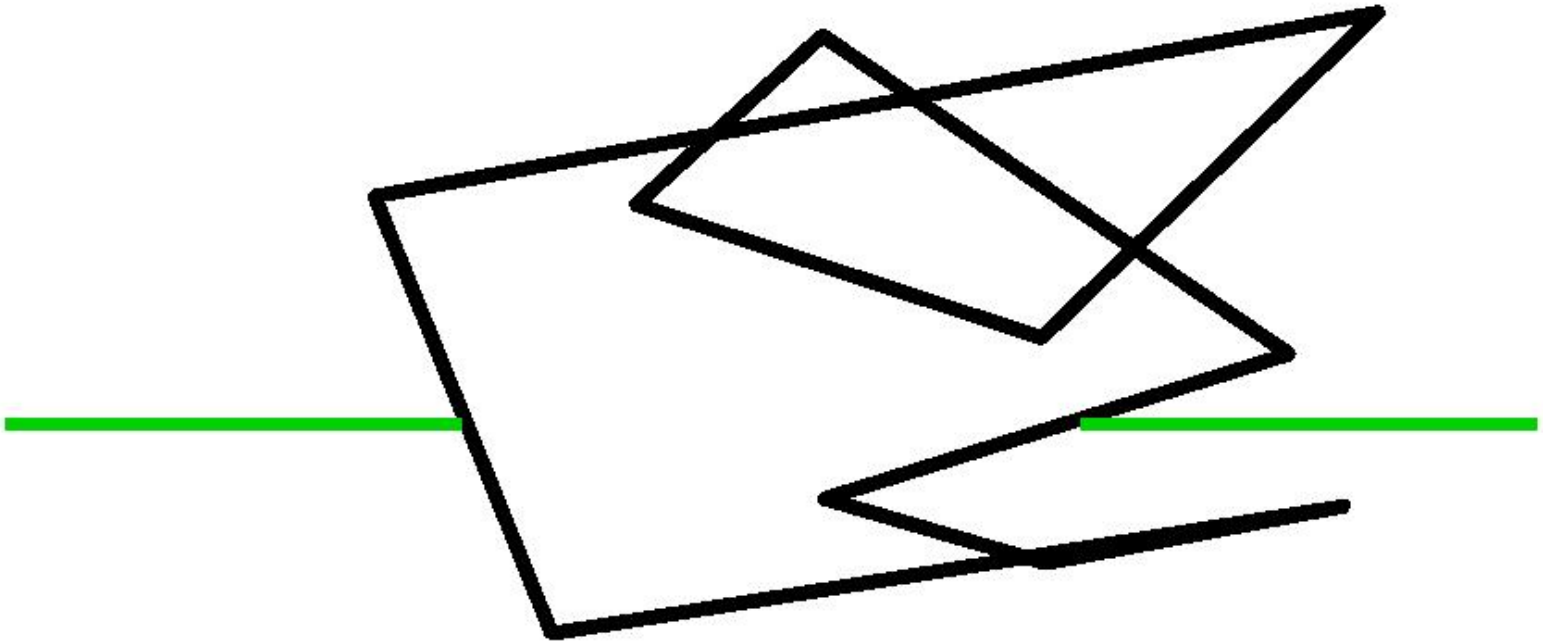
Trajectory of a Quantum Fluctuation.

- worldline numerics: $N = 4$ points per loop (ppl)



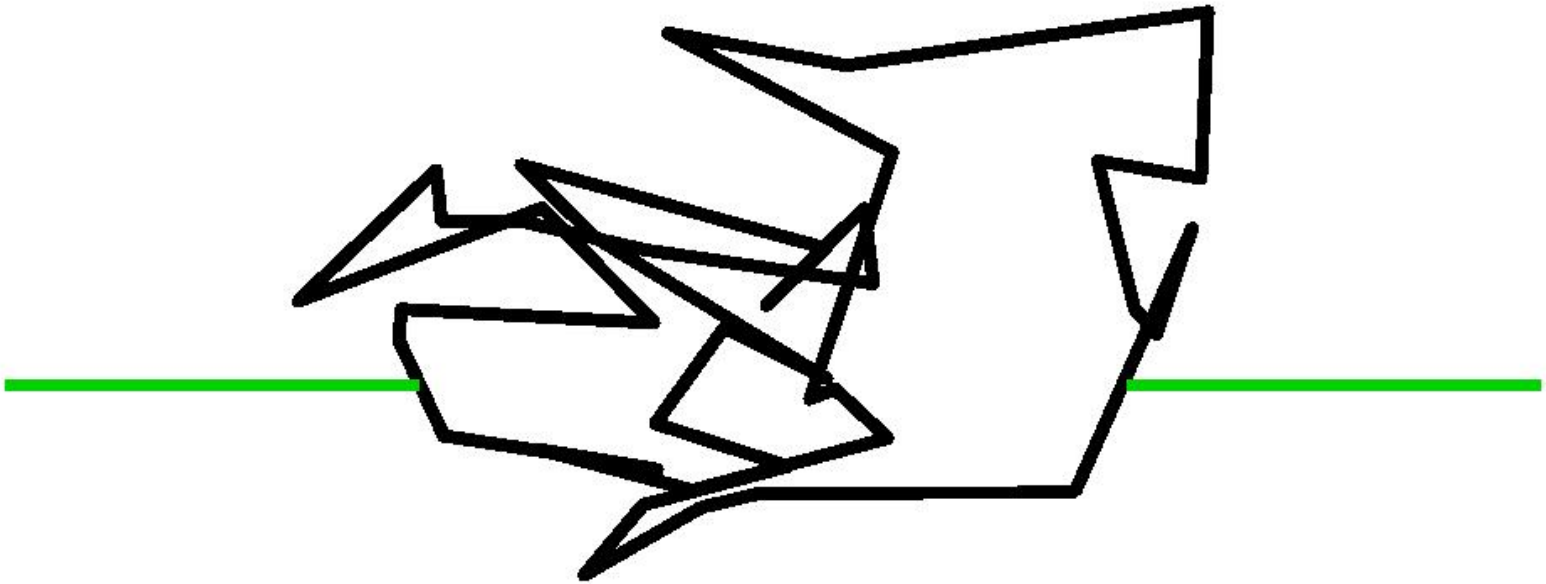
Trajectory of a Quantum Fluctuation.

- worldline numerics: $N = 10$ points per loop (ppl)



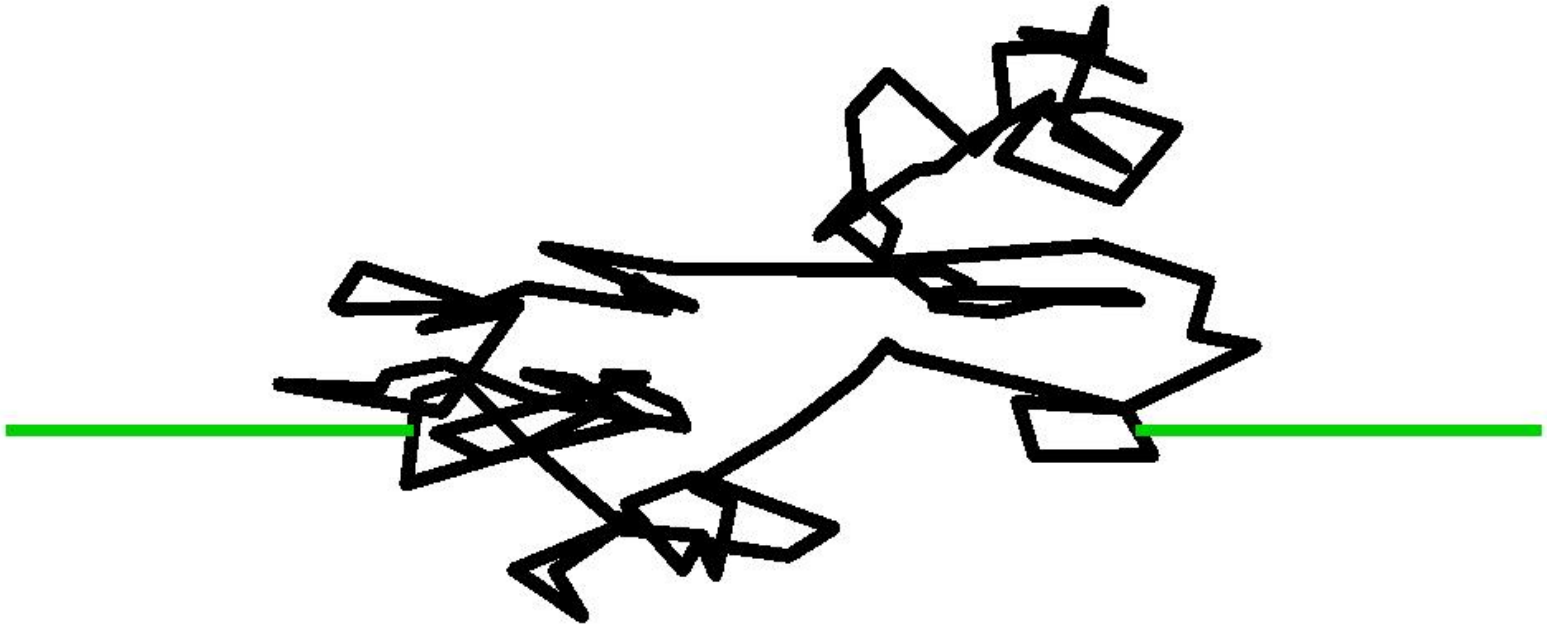
Trajectory of a Quantum Fluctuation.

- worldline numerics: $N = 40$ points per loop (ppl)



Trajectory of a Quantum Fluctuation.

- worldline numerics: $N = 100$ points per loop (ppl)

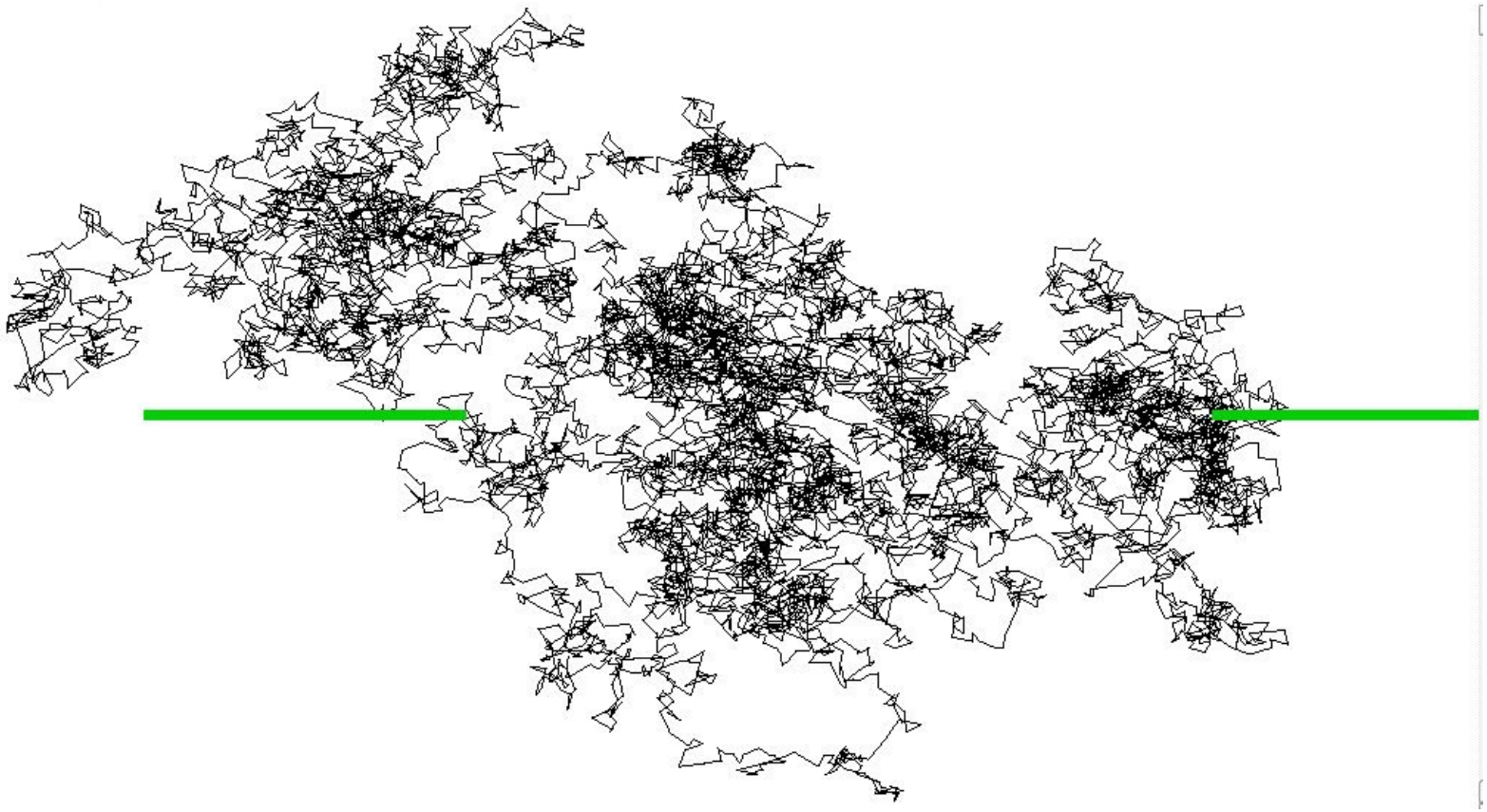


Trajectory of a Quantum Fluctuation.

- worldline numerics: $N = 1000$ points per loop (ppl)

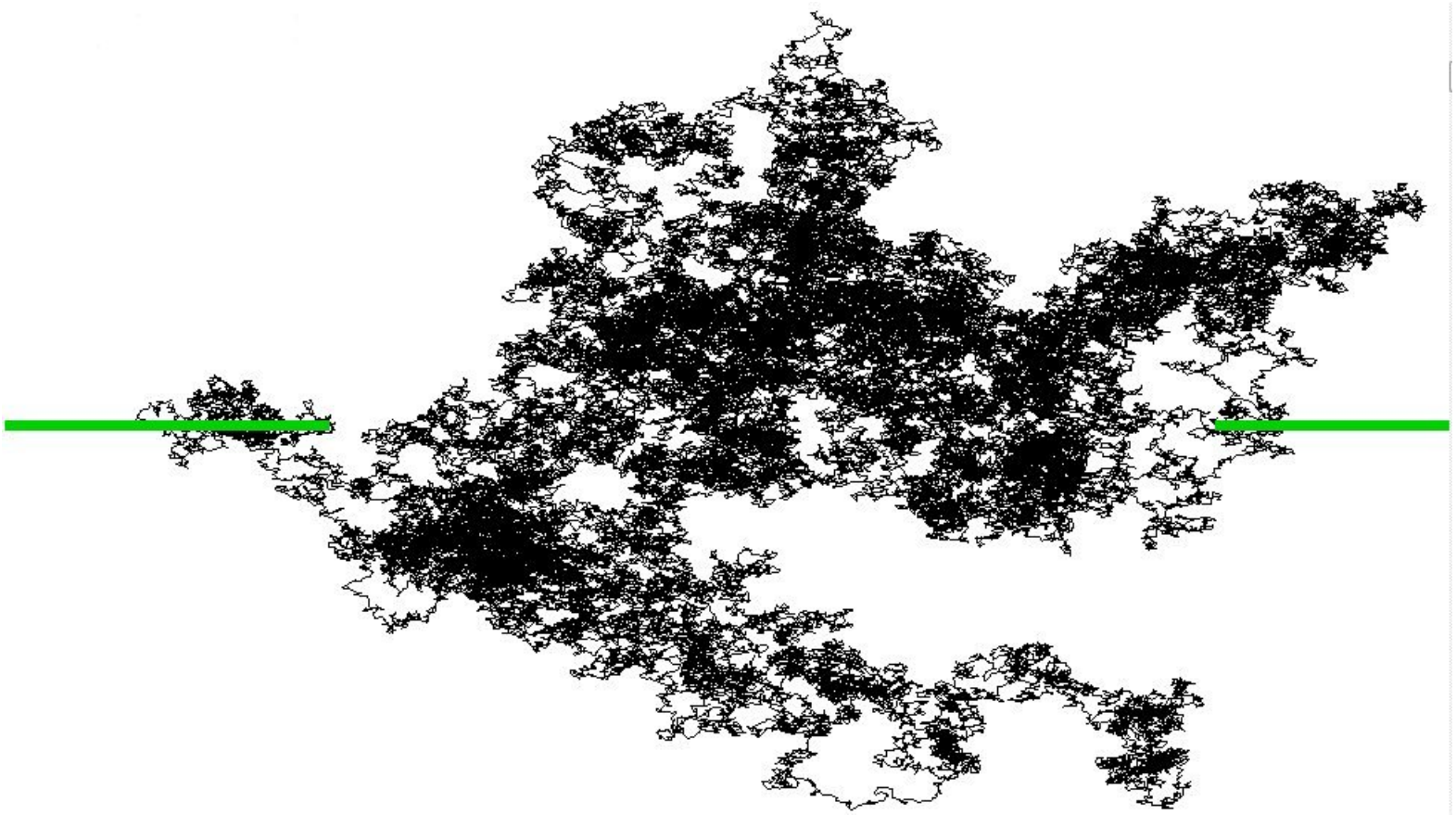
Trajectory of a Quantum Fluctuation.

- worldline numerics: $N = 10000$ points per loop (ppl)



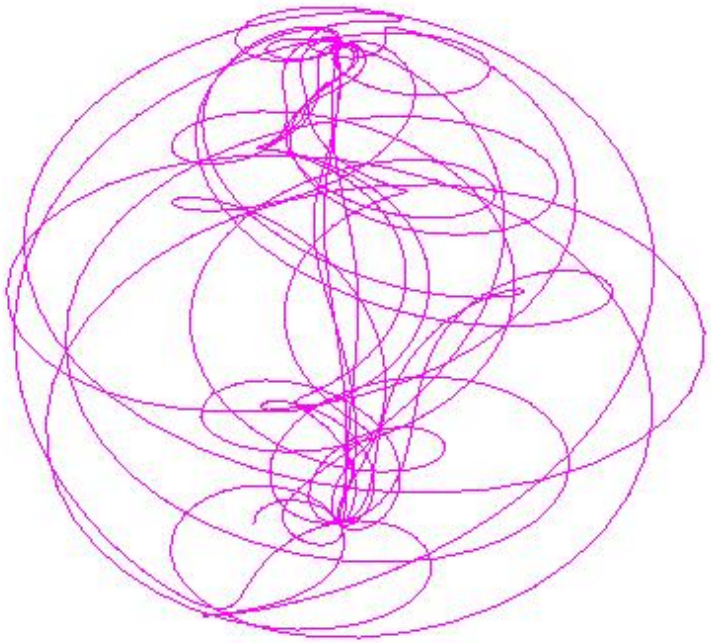
Trajectory of a Quantum Fluctuation.

- worldline numerics: $N = 100000$ points per loop (ppl)



Propertime T.

T

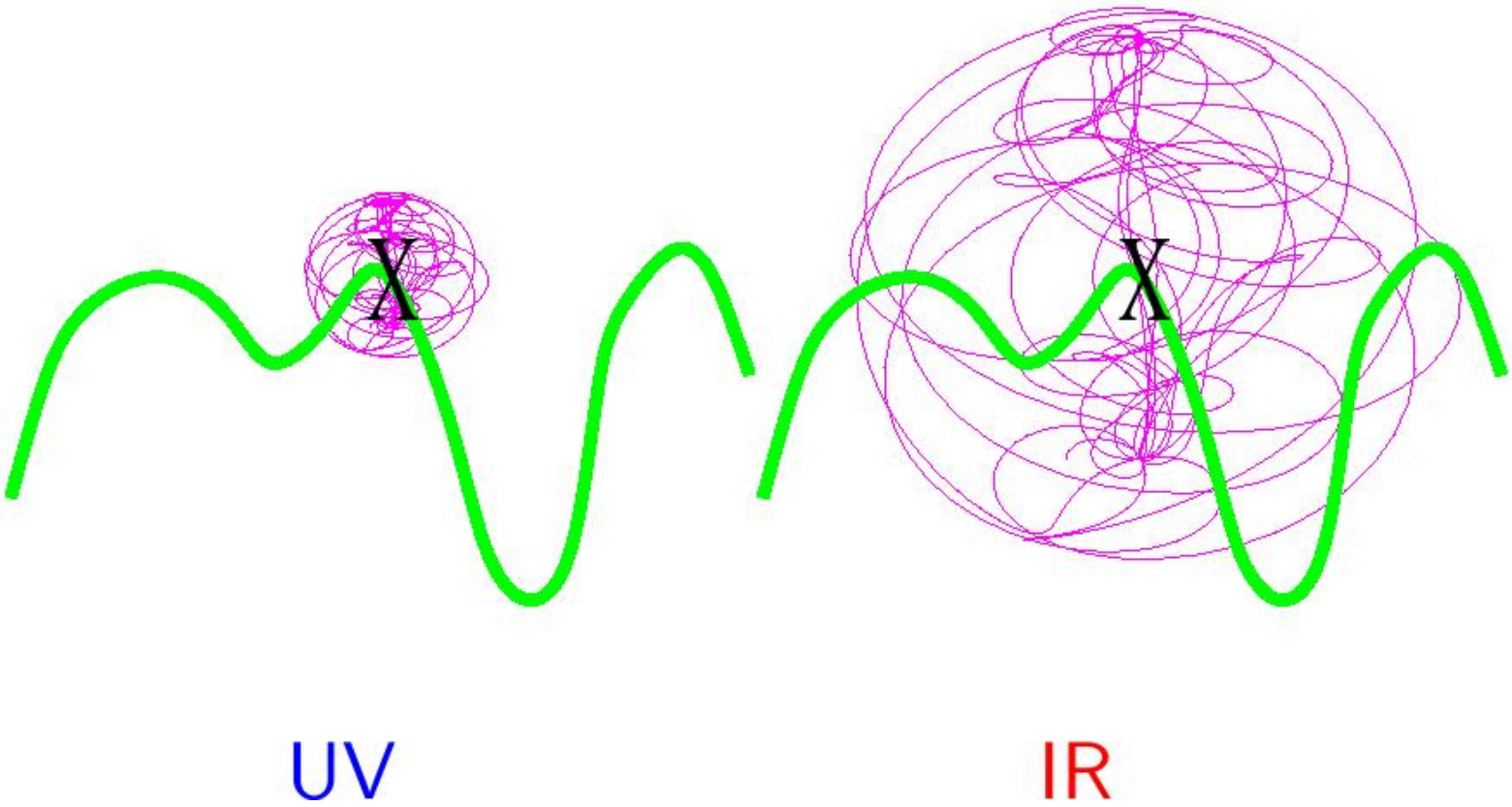


T

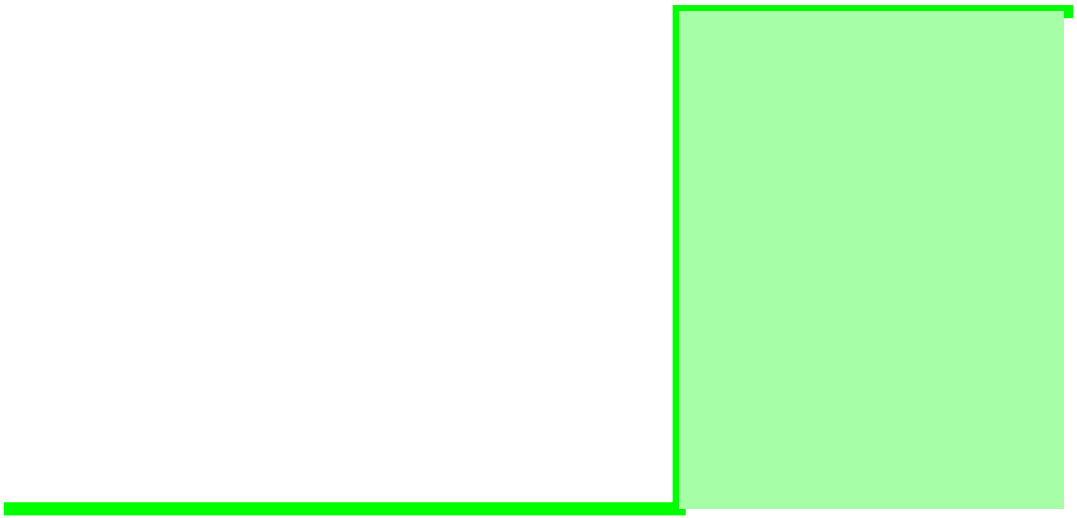
T $\sim$ regulator scale of smeared momentum shells

Propertime T.

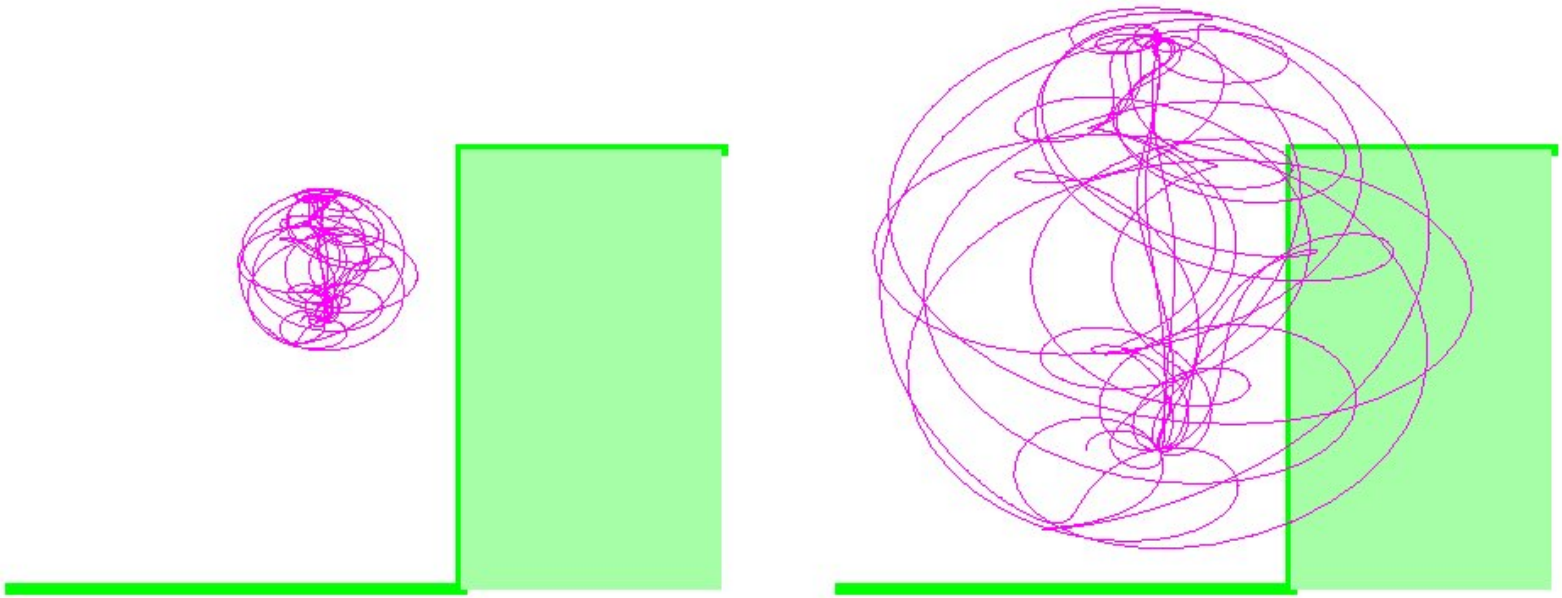
- "Measuring" the Wegner-Wilson loop $\exp\left(-ie\oint dx\cdot A\right)$



Magnetic Step.



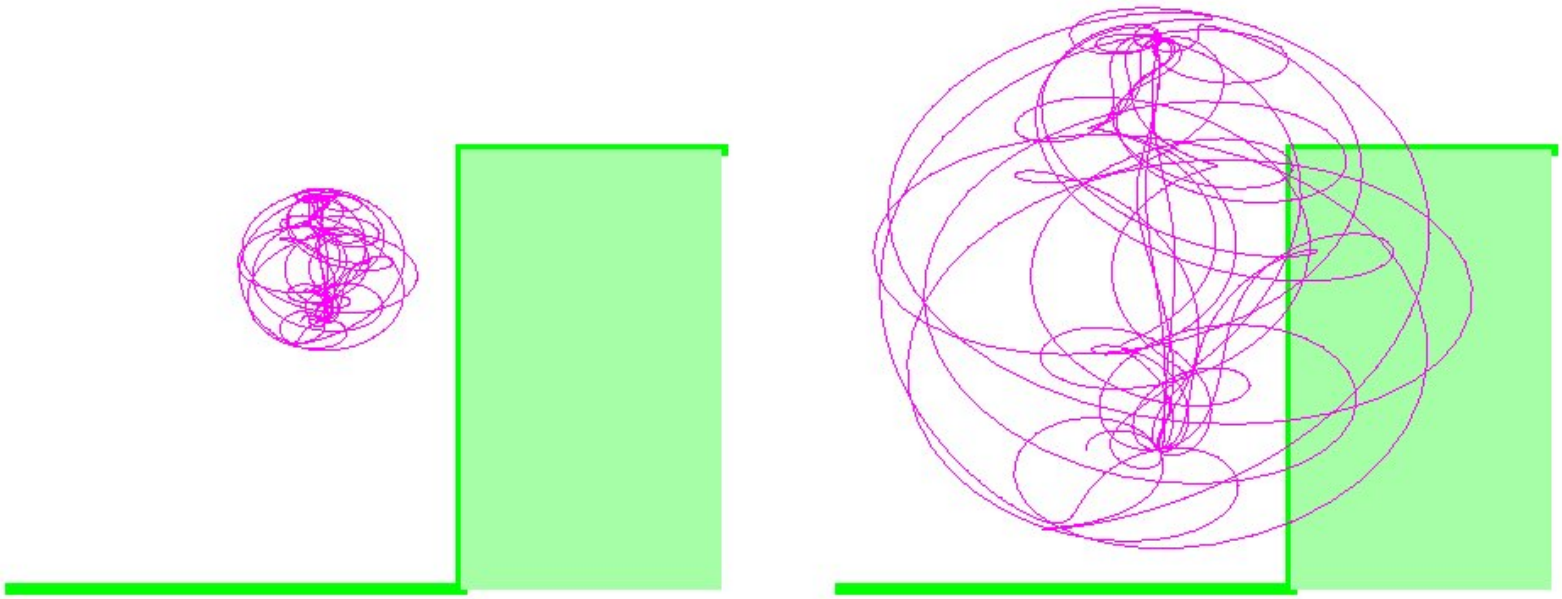
Magnetic Step.



UV, T

IR, T

Magnetic Step.



UV, T IR, T

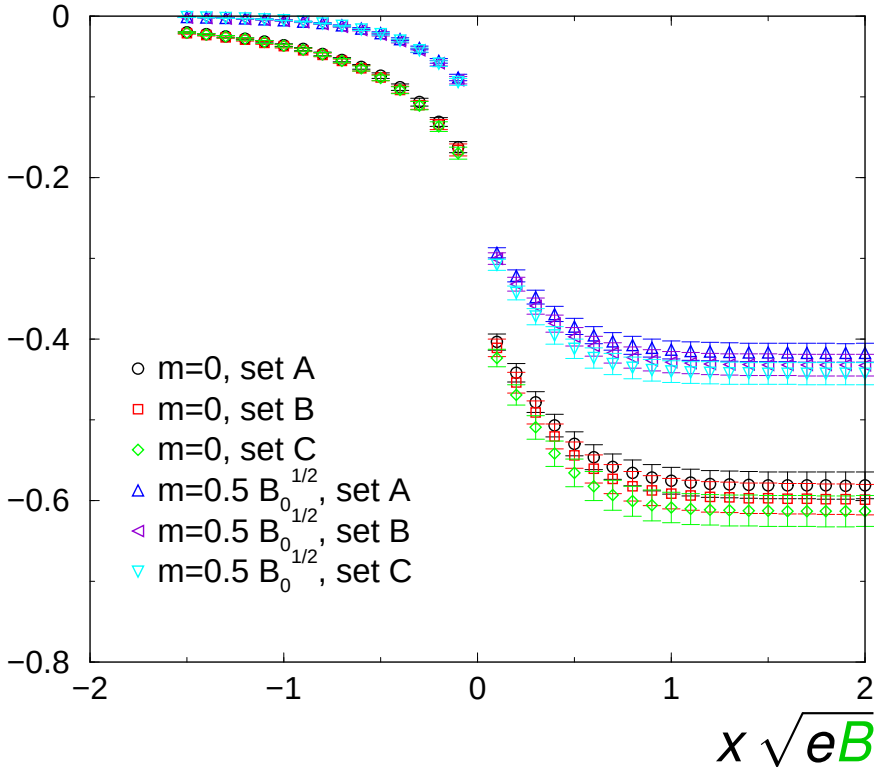
$\Longrightarrow$ “Quantum diffusion” of B field

Magnetic Step.

(HG,LANGFELD'01)

$D = 3$

$\left(\frac{4\pi}{eB}\right)^{3/2} \mathcal{L}_{\text{eff}}(x)$



HE limit

Magnetic Step.

(HG,LANGFELD'01)

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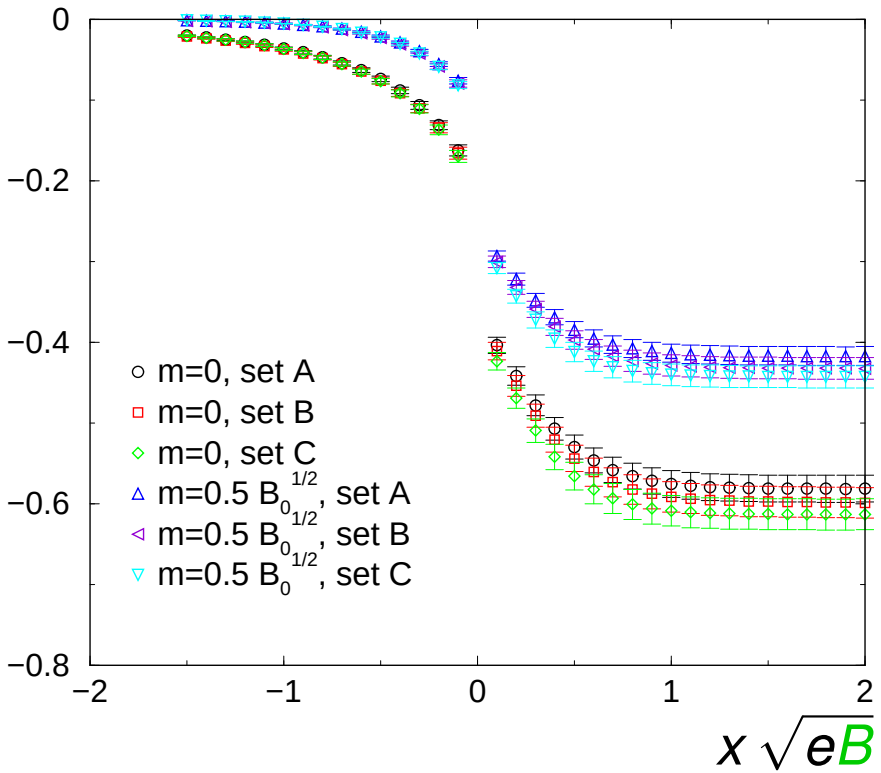
$\left(\frac{4\pi}{eB}\right)^{3/2} \mathcal{L}_{\text{eff}}(x)$



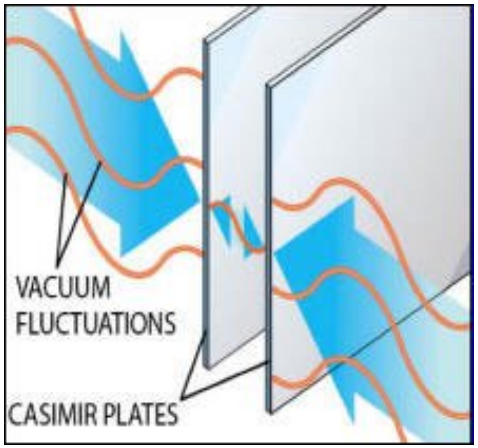
HE limit

▷ diffusion law: $\mathcal{L}^1_{\text{eff}}(x) \sim \exp\left(-3.255\ m\,x - 0.7627\ \sqrt{eB}\,x\right)$

Casimir Effect.

▷ Hendrik B.G. Casimir 1948:

$$\frac{F}{A} = -\frac{\pi^2}{240}\,\frac{\hbar c}{a^4}$$



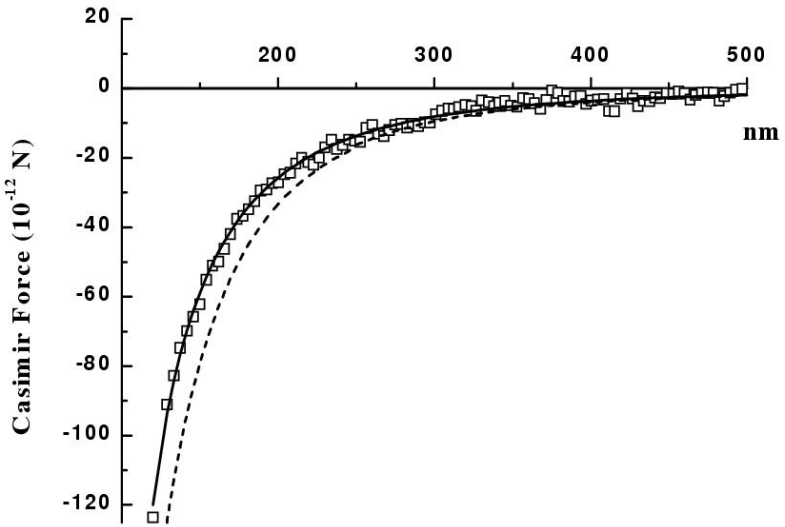
▷ precision measurements $\mathcal{O}(1\%)$

(LAMOREAUX'97)

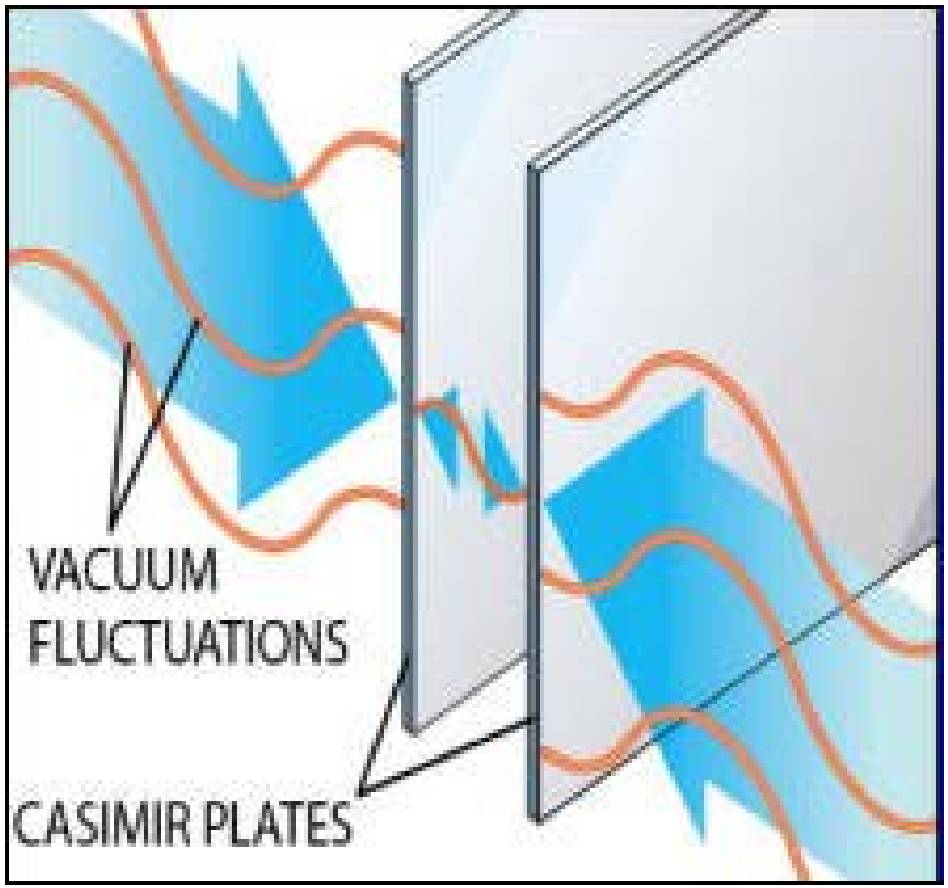
(MOHIDEEN ET AL.'98+)

(DECCA ET AL.'03+)

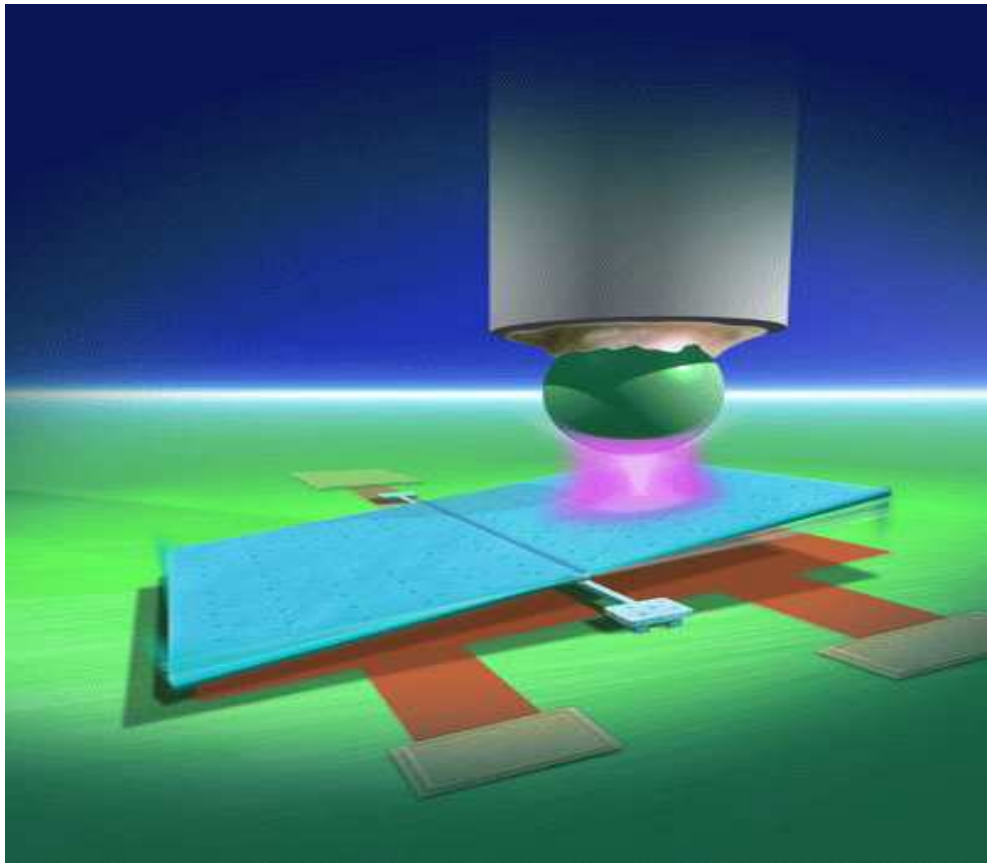
⋮



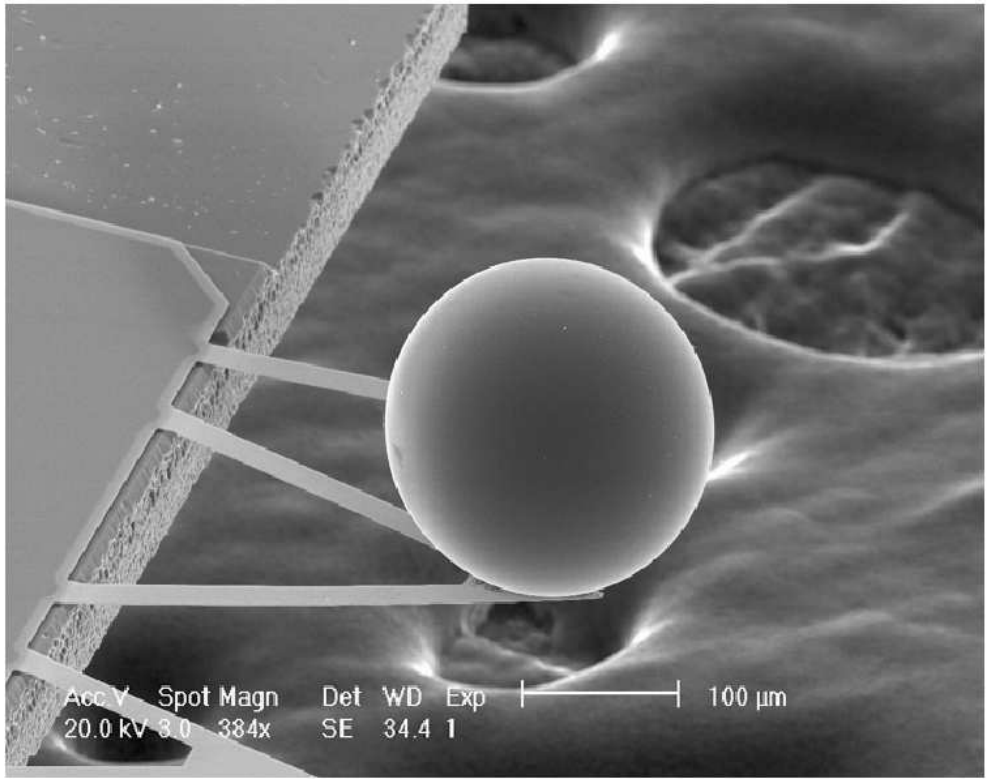
Casimir Effect.



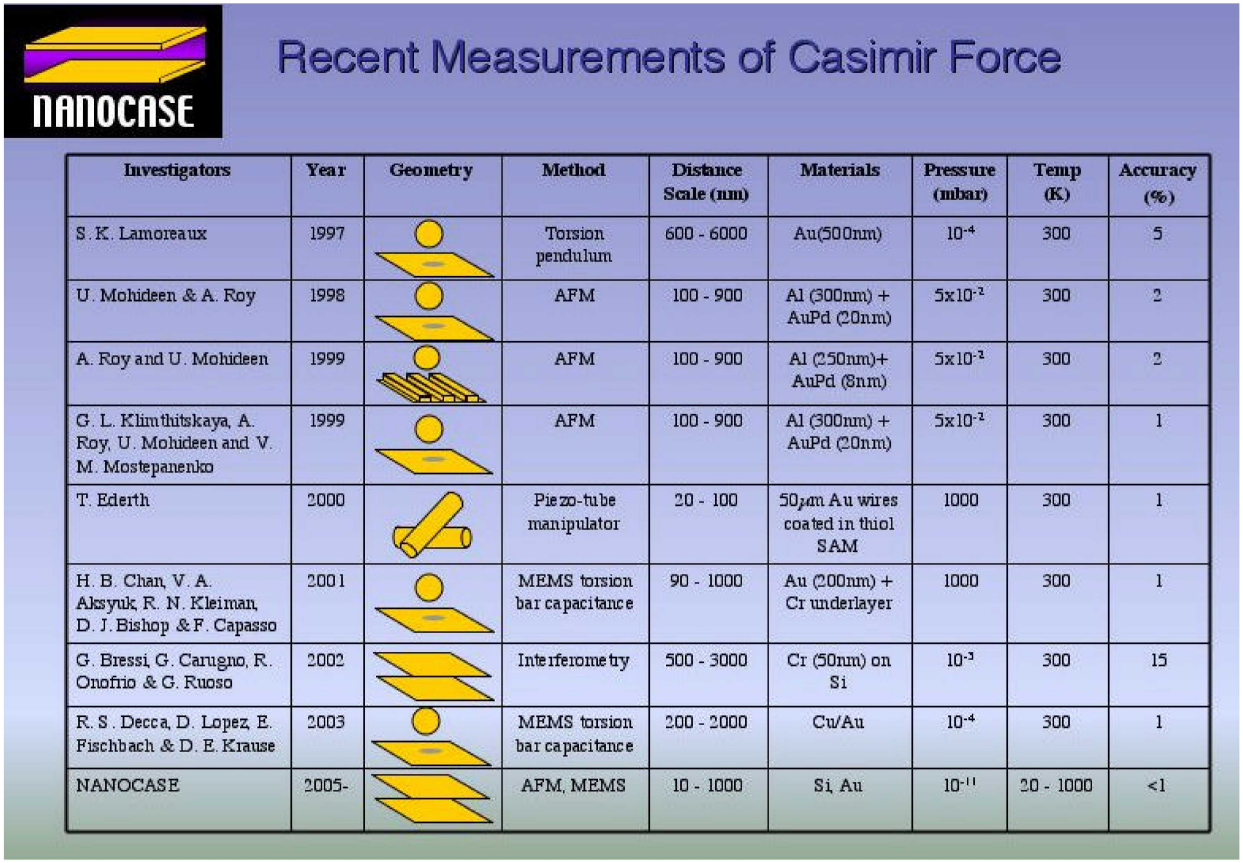
Casimir Effect.



Casimir Effect.



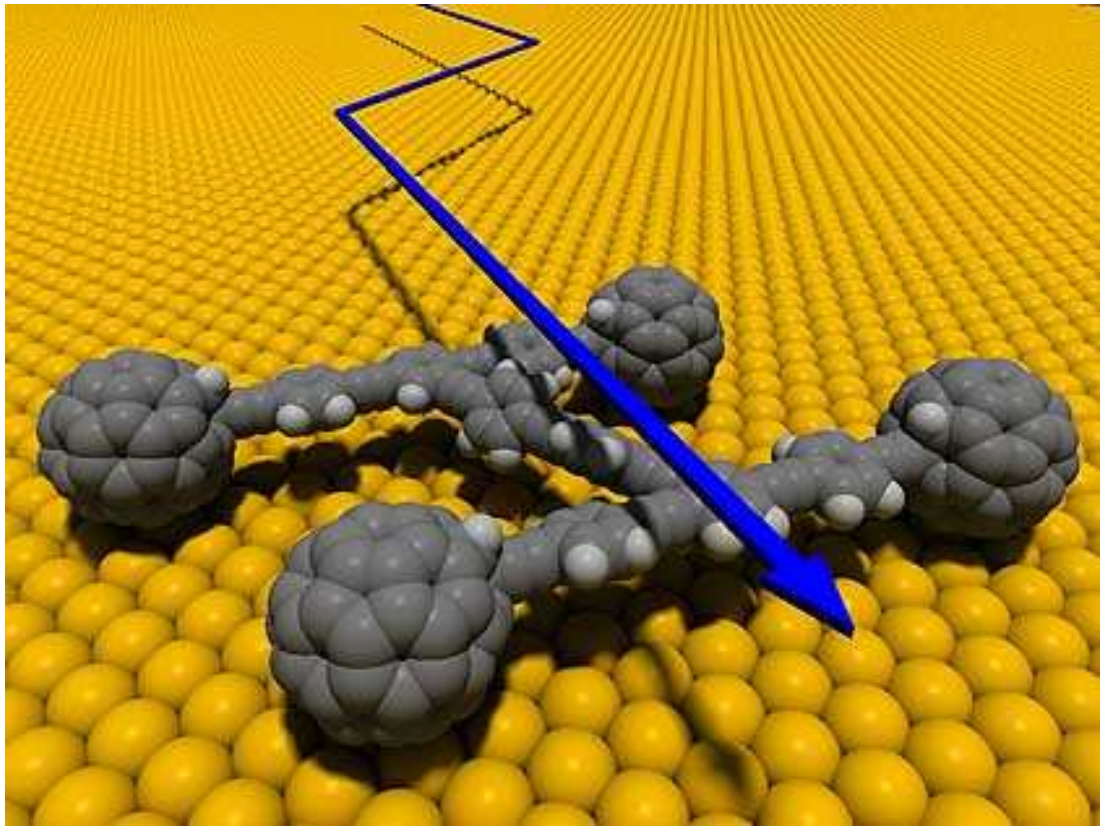
Casimir Effect.



Investigators	Year	Geometry	Method	Distance Scale (nm)	Materials	Pressure (mbar)	Temp (K)	Accuracy (%)
S. K. Lamoreaux	1997		Torsion pendulum	600 - 6000	Au(500nm)	10^{-4}	300	5
U. Mohideen & A. Roy	1998		AFM	100 - 900	Al (300nm) + AuPd (20nm)	$5x10^{-2}$	300	2
A. Roy and U. Mohideen	1999		AFM	100 - 900	Al (250nm)+ AuPd (8nm)	$5x10^{-2}$	300	2
G. L. Klimthitskaya, A. Roy, U. Mohideen and V. M. Mostepanenko	1999		AFM	100 - 900	Al (300nm) + AuPd (20nm)	$5x10^{-2}$	300	1
T. Ederth	2000		Piezo-tube manipulator	20 - 100	50μm Au wires coated in thiol SAM	1000	300	1
H. B. Chan, V. A. Aksyuk, R. N. Kleiman, D. J. Bishop & F. Capasso	2001		MEMS torsion bar capacitance	90 - 1000	Au (200nm) + Cr underlayer	1000	300	1
G. Bressi, G. Carugno, R. Onofrio & G. Ruoso	2002		Interferometry	500 - 3000	Cr (50nm) on Si	10^{-3}	300	15
R. S. Decca, D. Lopez, E. Fischbach & D. E. Krause	2003		MEMS torsion bar capacitance	200 - 2000	Cu/Au	10^{-4}	300	1
NANOCASE	2005-		AFM, MEMS	10 - 1000	Si, Au	10^{-11}	20 - 1000	<1

(BINNS'05)

Casimir Effect.

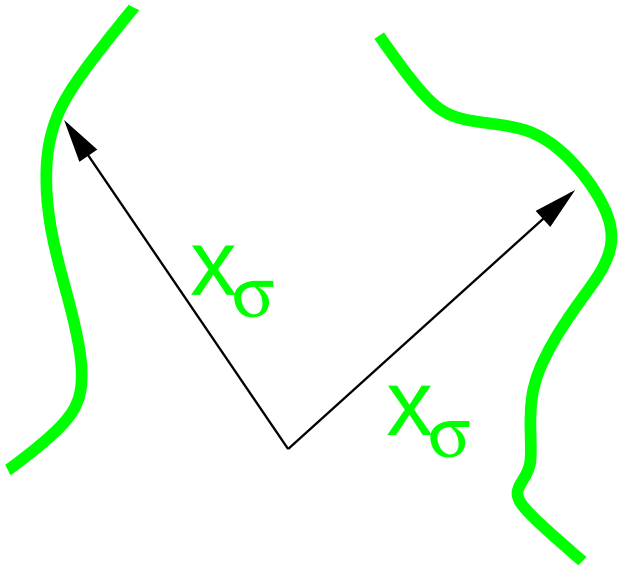


(TOUR&KELLY'05)

Morphology of the Casimir effect ?

Casimir Effect.

- Casimir effect $\hat{=}$ "strong-field QFT"

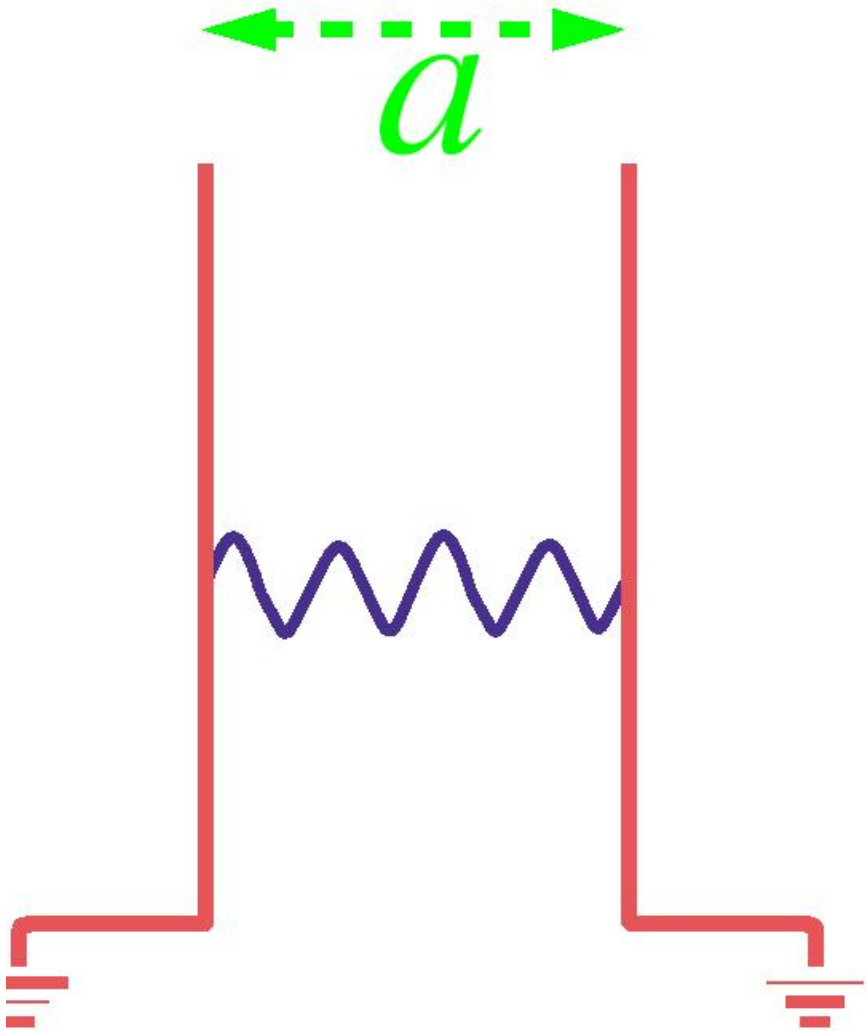


$$S = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + V\phi^2$$

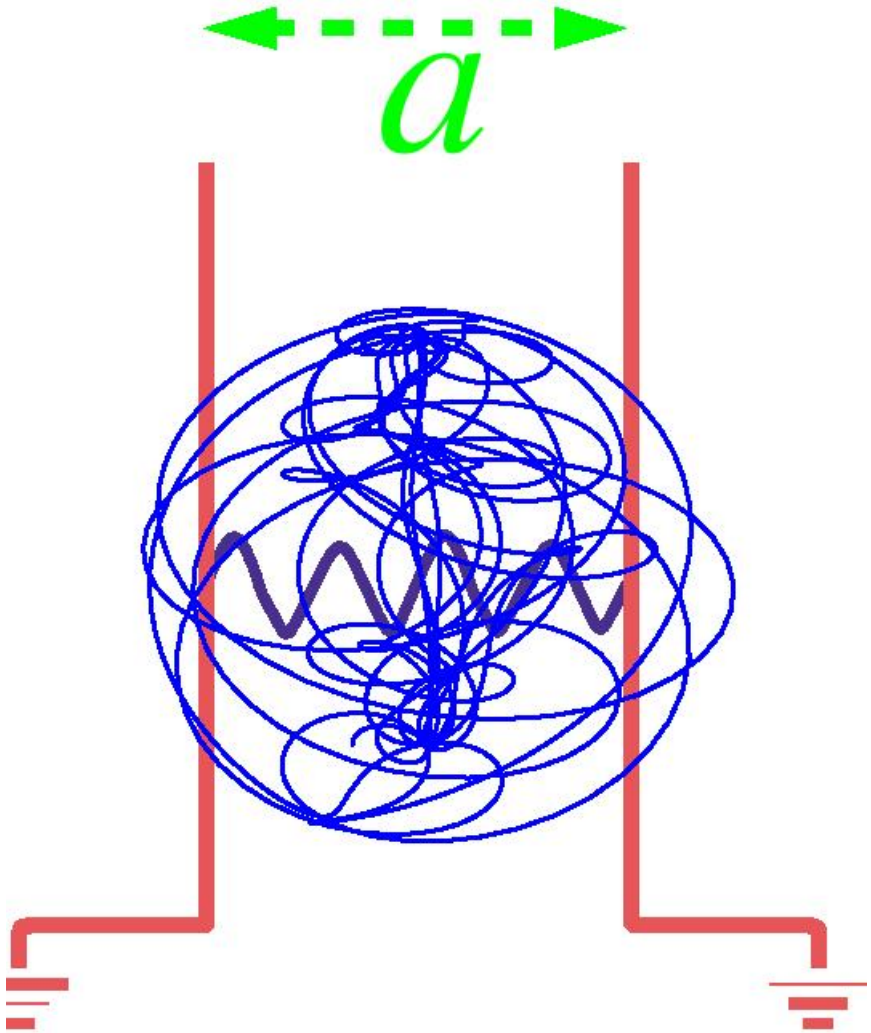
$$V(x) = \lambda \int_S d\sigma \left[\delta(x - x_\sigma) + \delta(x - x_\sigma)\right]$$

(BORDAG, HENNIG, ROBASCHIK'92; GRAHAM ET AL.'03)

Benchmark test: parallel plates

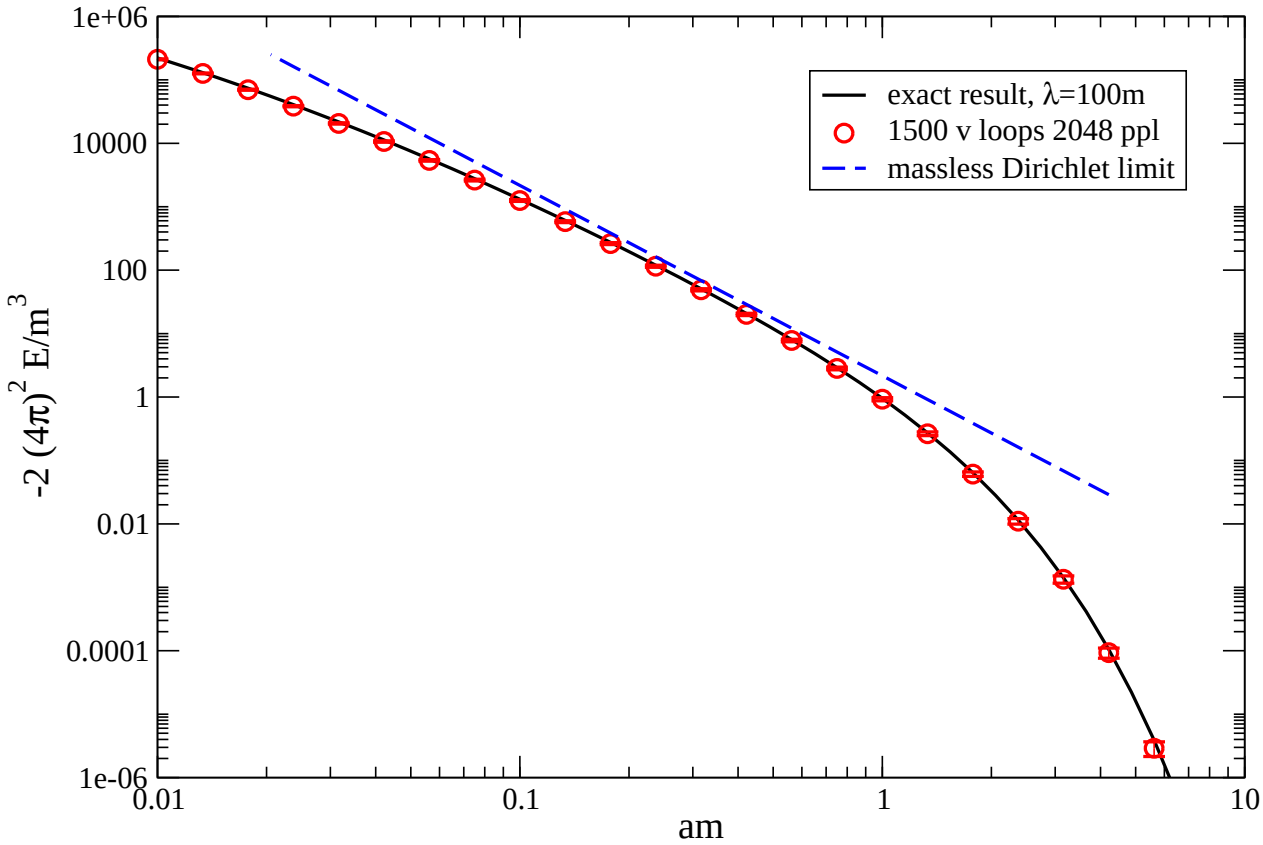


Benchmark test: parallel plates



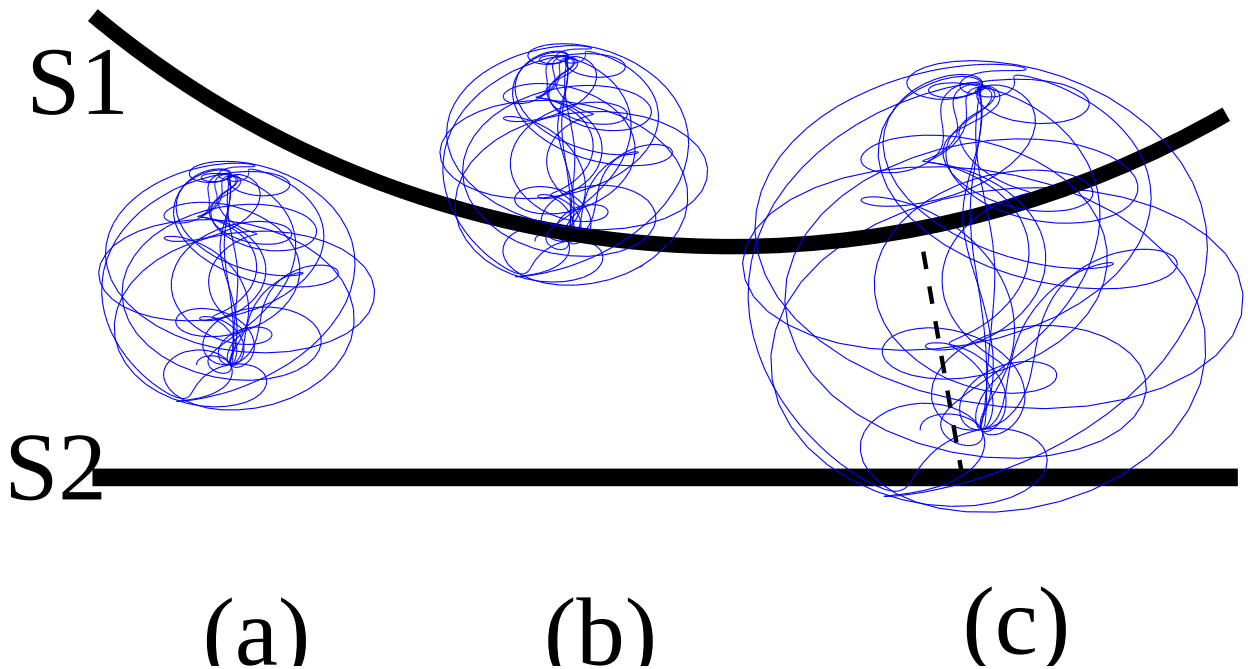
Benchmark test: parallel plates

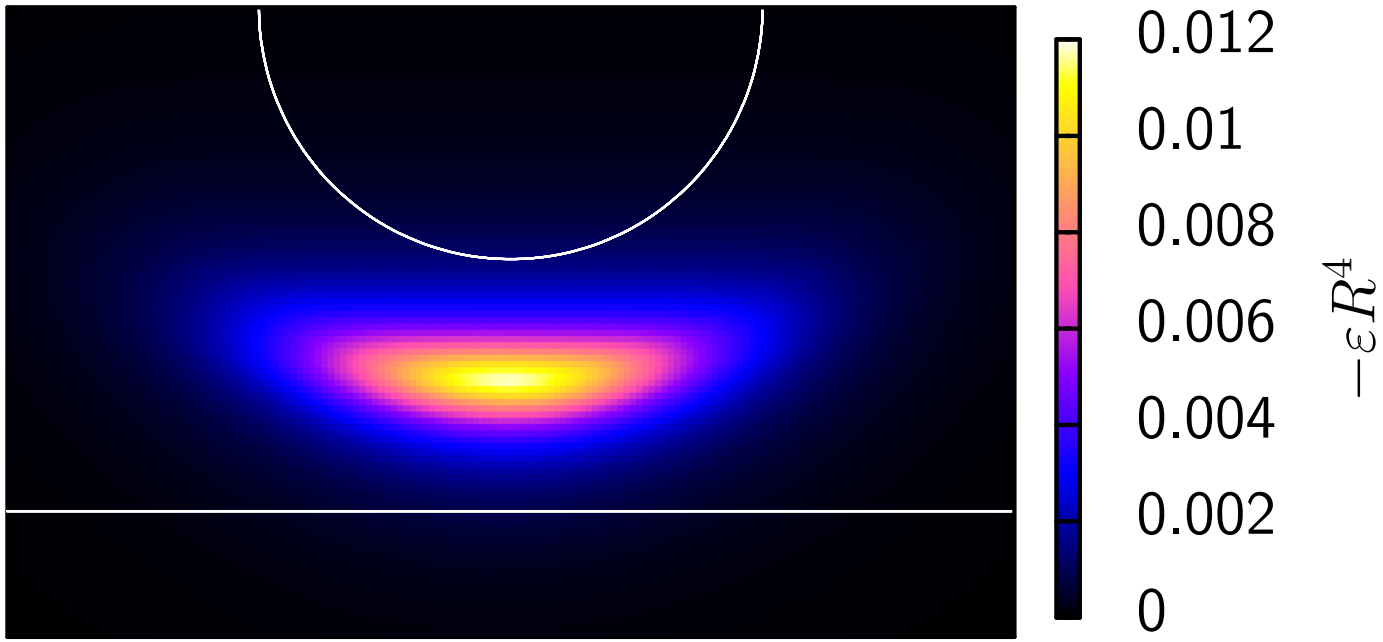
- for finite m, λ, a



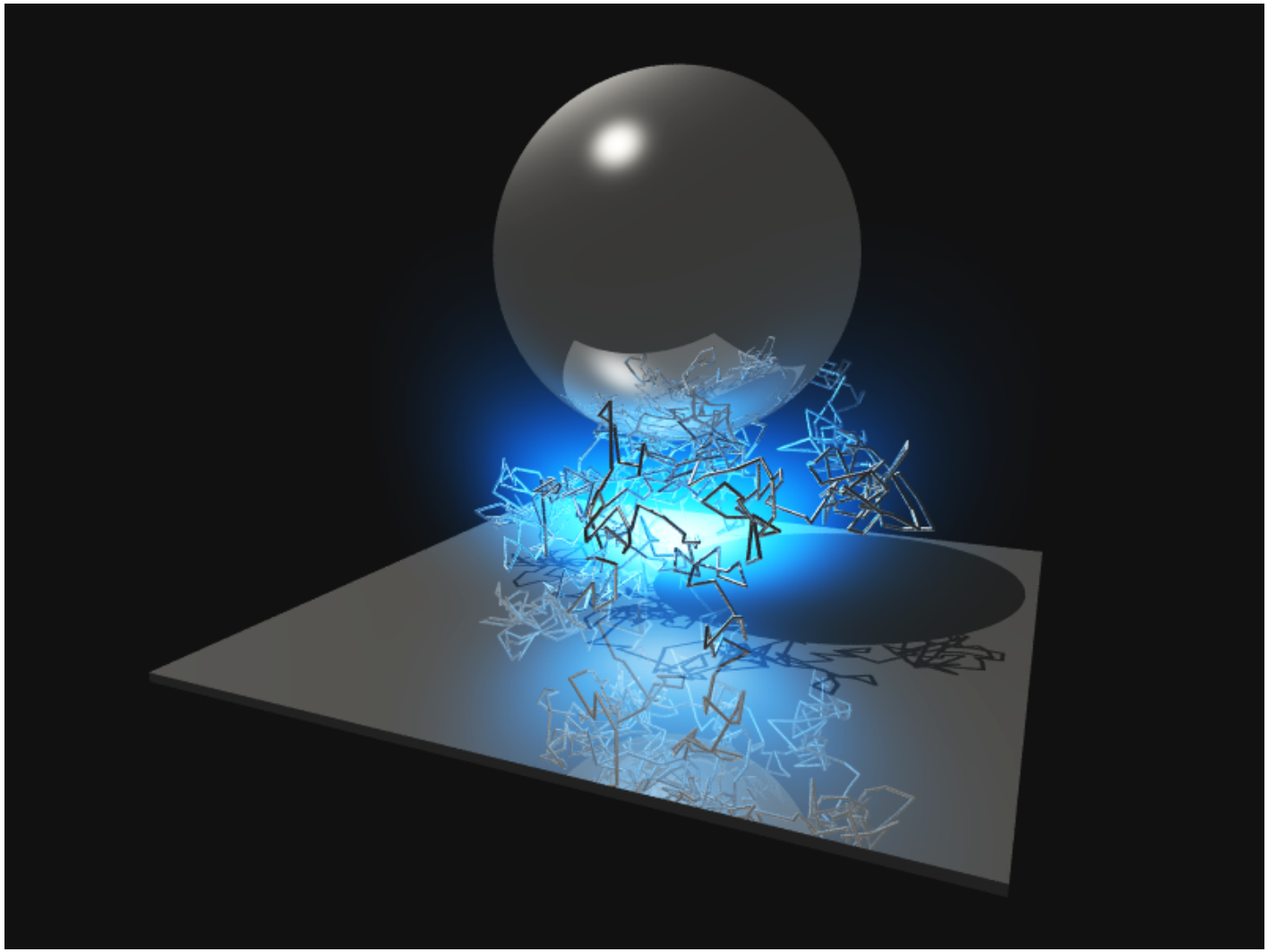
(BORDAG,HENNIG, ROBASCHIK '92)

(HG,LANGFELD,MOYAERTS '03)

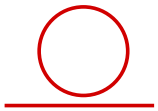

S1
S2
(a)
(b)
(c)

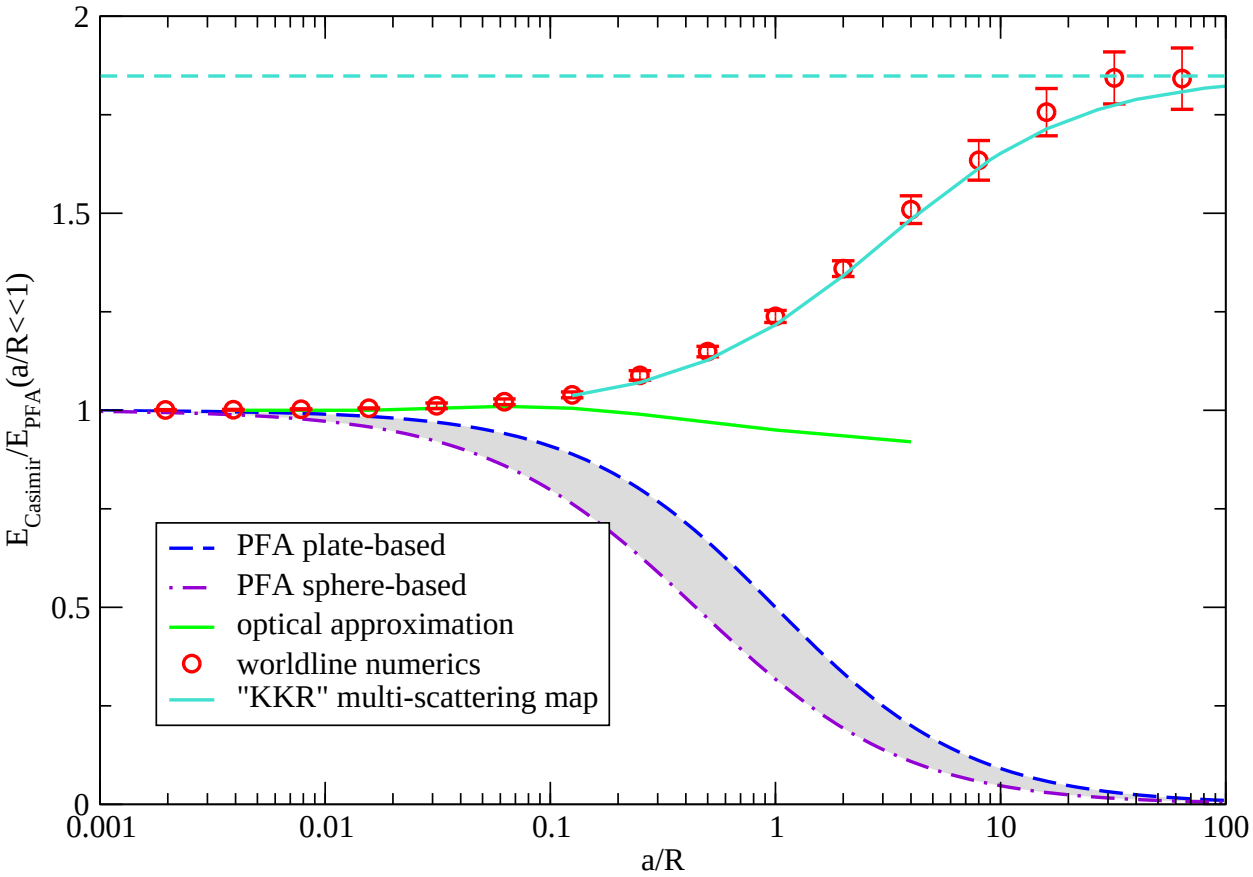

0.012
0.01
0.008
0.006
0.004
0.002
0
$-\varepsilon R^4$

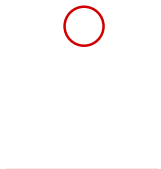
Casimir Effect: curvature effects on the worldline



Casimir Effect: sphere above plate.







(HG,LANGFELD,MOYAERTS'03; JAFFE,SCARDICCHIO'04; BULGAC,MAGIERSKI,WIRZBA'05; HG,KLINGMÜLLER'05)

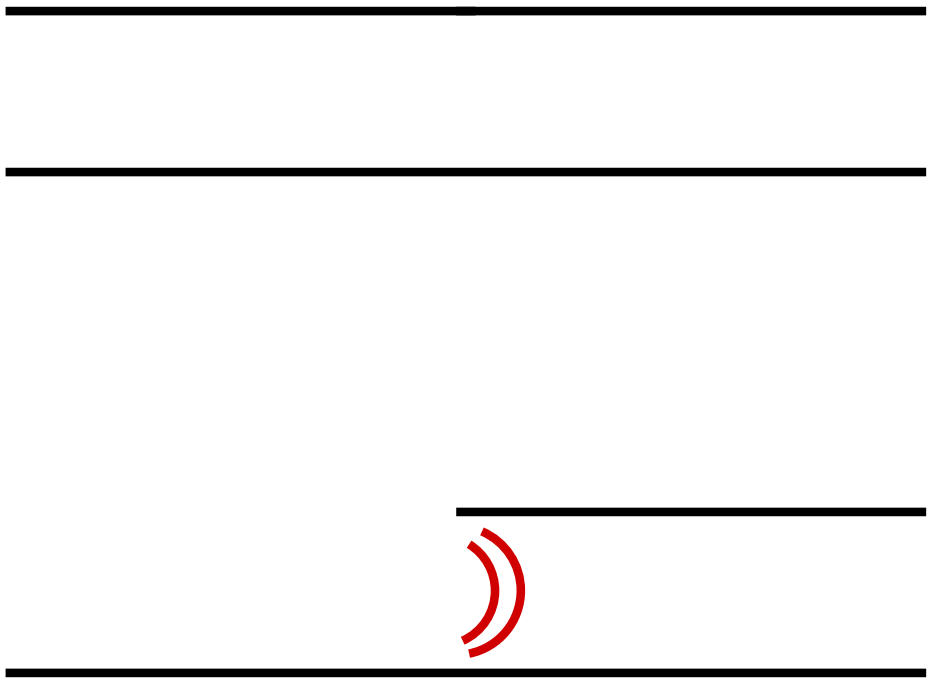
Casimir Edge Effects.

$$F_{\parallel} = -\gamma_{\parallel} \frac{\hbar c}{a^4} \cdot A$$

$$F_{1\text{si}} = ?$$

(CF. BRESSI,CARUGNO,ONOFRIO,RUOSO'02)

Casimir Edge Effects.



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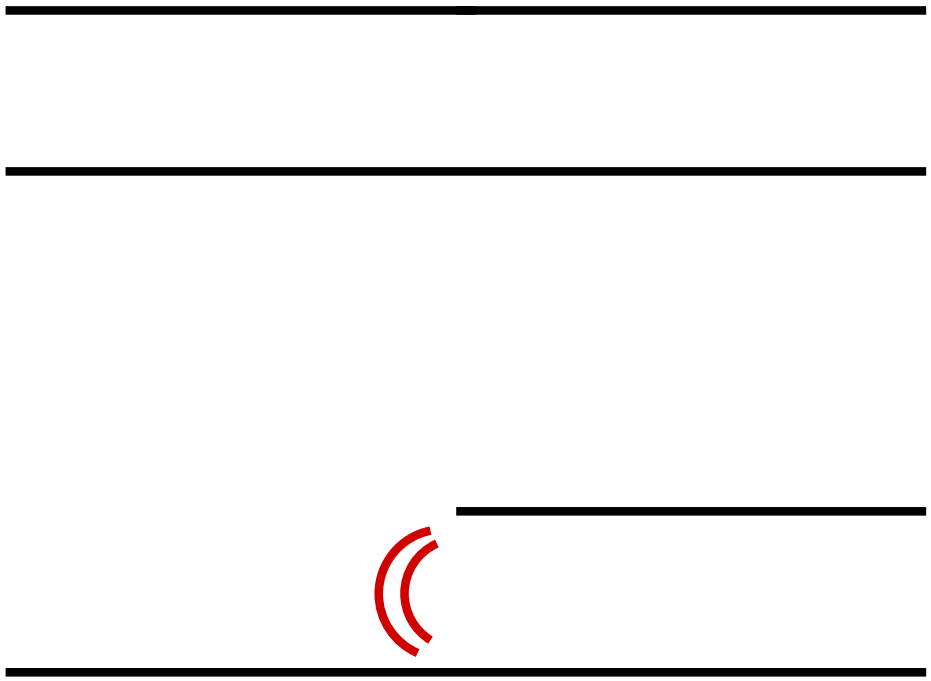

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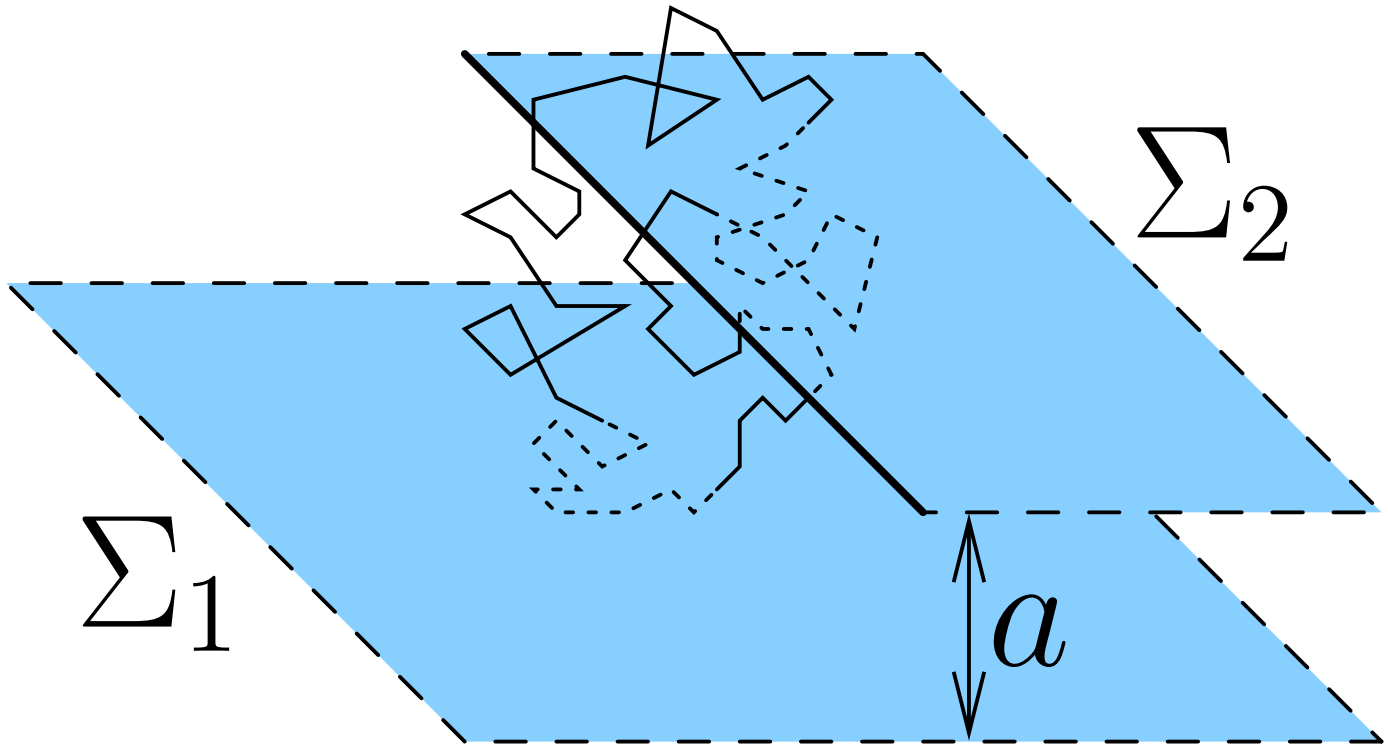
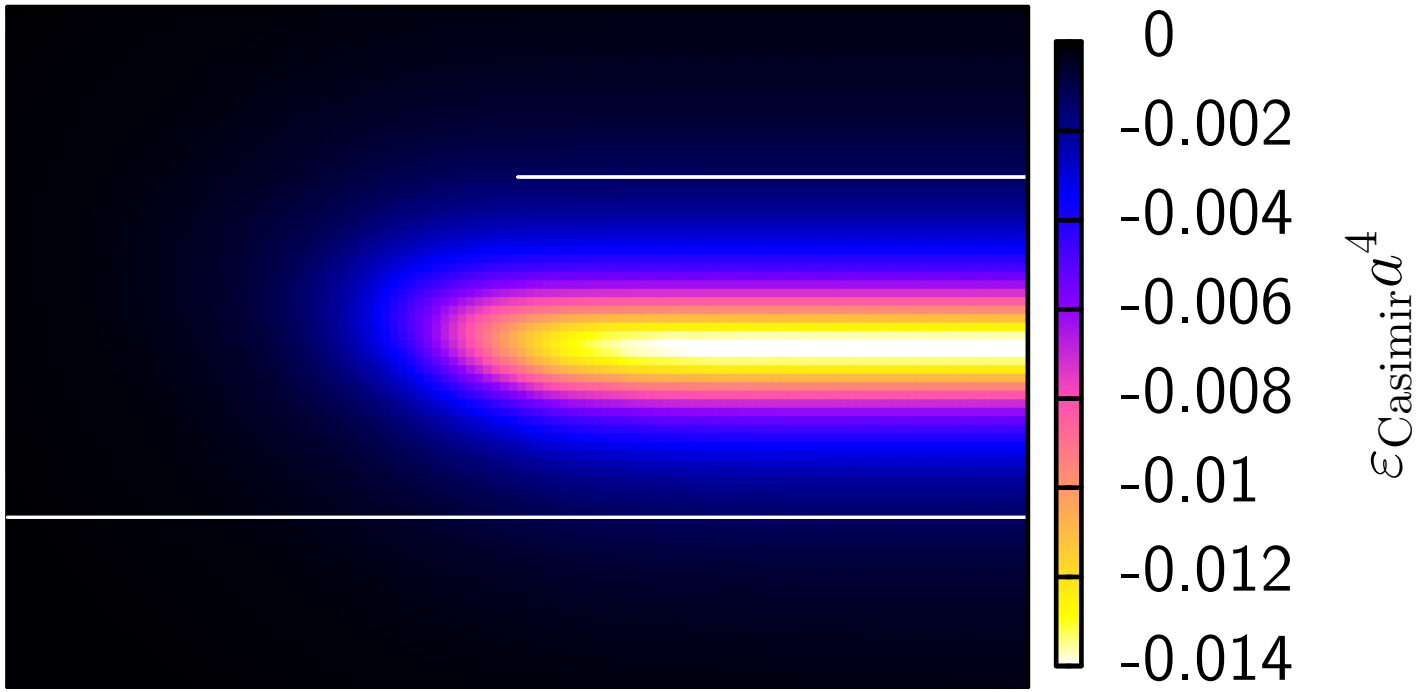
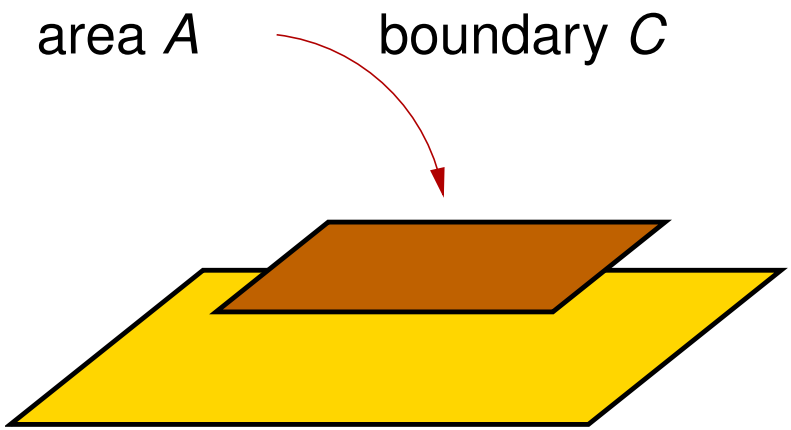


Casimir Edge Effects.



Casimir Edge Effects.

- effective description of a finite plate



$$F = -\gamma_{\parallel} \frac{\hbar c}{a^4} A_{\text{eff}},$$

- effective area: $A_{\text{eff}} \simeq A + \frac{\gamma_{1\text{si}}}{\gamma_{\parallel}} aC, \qquad \gamma_{1\text{si}} = 5.23(2) \times 10^{-3}$

(HG,KLINGMULLER'06)

Casimir Edge Effects.

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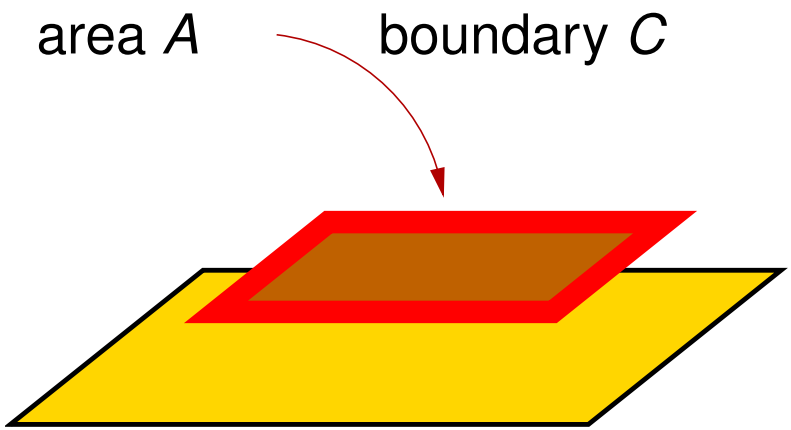

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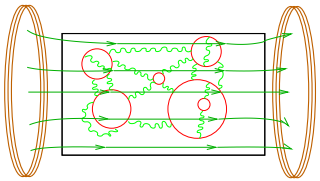
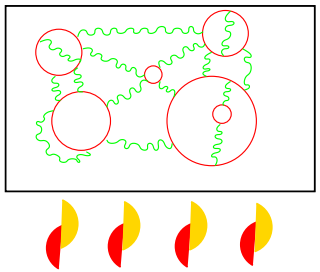
(HG,KLINGMULLER'06)

Further Worldline Applications.



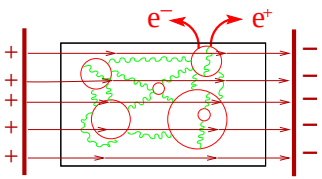
Heisenberg-Euler effective actions, spinor QED, flux tubes, quantum-induced vortex interactions

(HG,Langfeld'01; Langfeld,Moyaerts,HG'02)



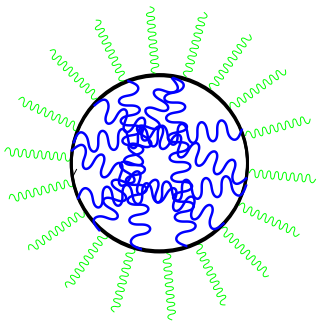
thermal fluctuations, free energies

(HG,Langfeld'02)



"spontaneous vacuum decay", Schwinger pair production in inhomogeneous electric fields

(HG,Klingmüller'05)

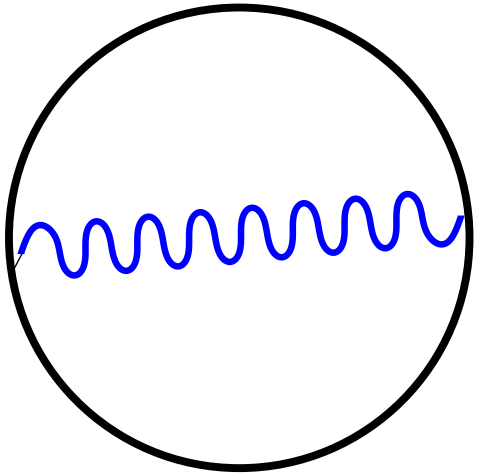


nonperturbative effective actions

(HG,Sánchez–Guillén,Vázquez'05)

Higher loops *per pedes*

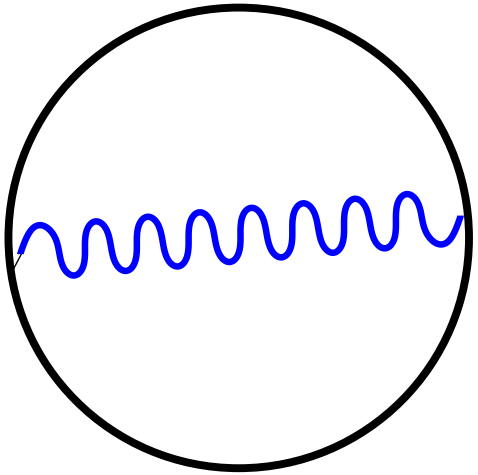
- Feynman diagrammar:



$$\sim \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \prod_i \Delta_i(q_i)$$

Higher loops *per pedes*

- Worldline:



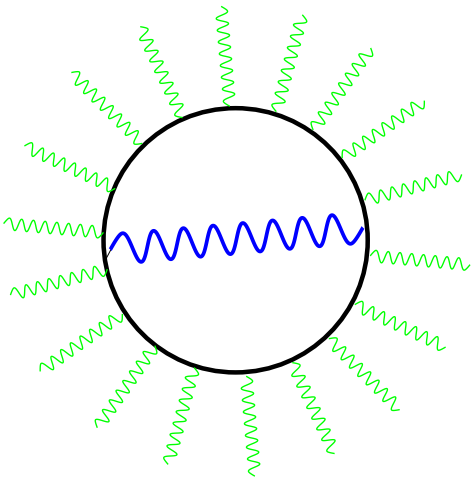
$$\sim \quad \int_T \left\langle \int d\tau_1 d\tau_2 \, \Delta(x(\tau_1), x(\tau_2)) \right\rangle_x$$

- photon propagator in coordinate space

$$\Delta(x_1, x_2) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} e^{ip(x_1 - x_2)} = \frac{\Gamma\left(\frac{D-2}{2}\right)}{4\pi^{D/2}} \frac{1}{|x_1 - x_2|^{D-2}}$$

Higher loops *per pedes*

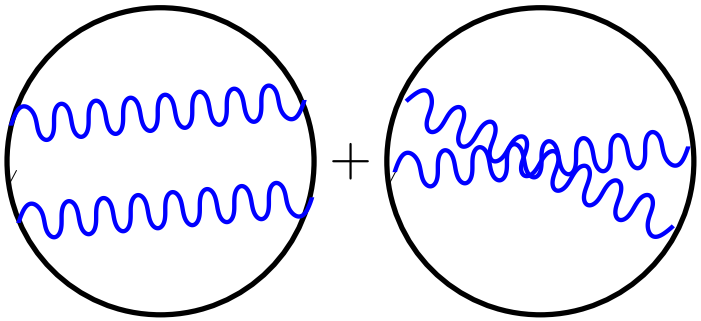
- Worldline:



$$\sim \int_T \left\langle e^{-ie\oint dx\cdot A(x)} \int d\tau_1 d\tau_2\, \Delta(x(\tau_1), x(\tau_2)) \right\rangle_x$$

Higher loops *per pedes*

- Feynman diagrammar:

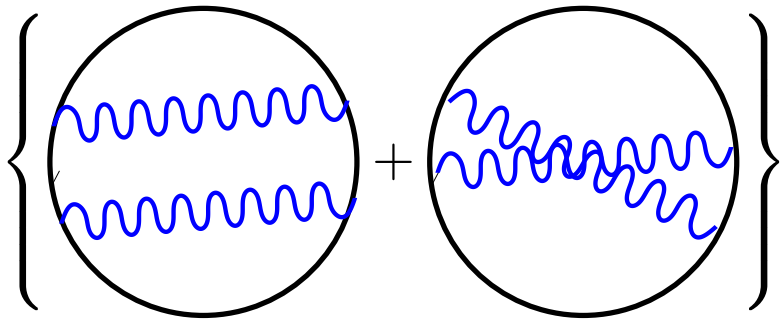


$$\sim \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \int \frac{d^D p_3}{(2\pi)^D} \prod_i \Delta_i(q_i)$$

$$+ \int \frac{d^D p_1}{(2\pi)^D} \int \frac{d^D p_2}{(2\pi)^D} \int \frac{d^D p_3}{(2\pi)^D} \prod_i \Delta_i(q_i)$$

Higher loops *per pedes*

- Worldline:

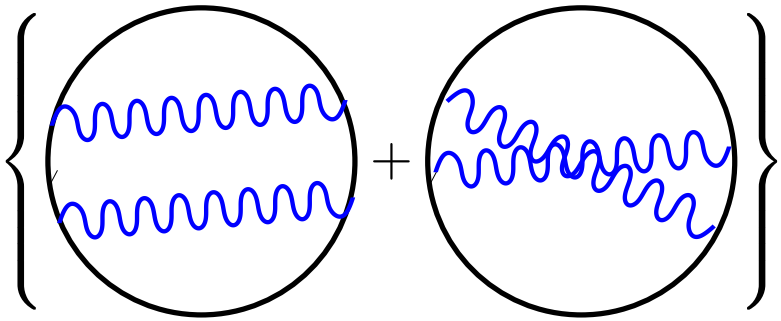


$$\sim \int_T \left\langle \int d\tau_1 d\tau_2 d\tau_3 d\tau_4 \, \Delta(x(\tau_1), x(\tau_2)) \Delta(x(\tau_3), x(\tau_4)) \right\rangle_x$$

- both diagrams in one expression

Higher loops *per pedes*

- Worldline:



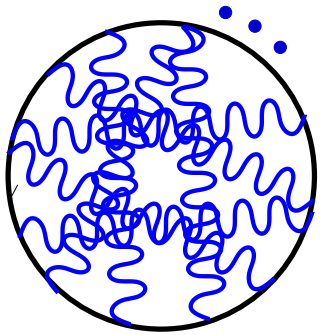
$$\sim \int_T \left\langle \left(\int d\tau_1 d\tau_2 \, \Delta(x(\tau_1), x(\tau_2)) \right)^2 \right\rangle_x$$

- both diagrams in one expression

Higher loops *per pedes*

▷ Worldline, all possible photon insertions:

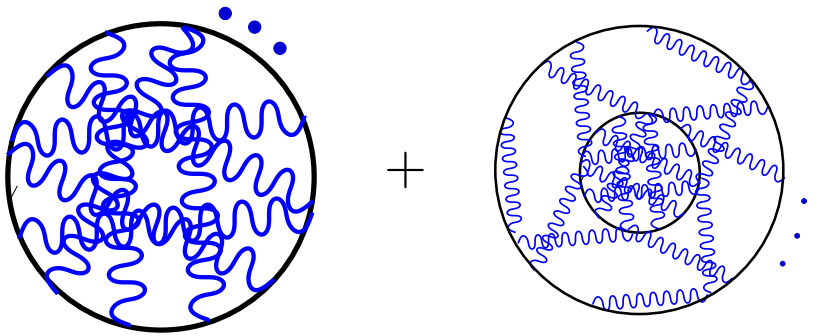
$$\sum \quad \sim \quad \int_T \left\langle \exp\left(-\frac{e^2}{2}\int d\tau_1 d\tau_2\, \Delta(x(\tau_1), x(\tau_2))\right) \right\rangle_x$$



$\Longrightarrow$ "quenched approximation" (further charged loops neglegted)

(FEYNMAN'50)

Systematics: small-N_f expansion



$$\sim \quad N_f \int_T \left\langle e^{-\frac{e^2}{2}\int\int \Delta} \right\rangle_x \quad + \quad N_f^2 \int_{T_1,T_2} \left\langle F_2\{x_1,x_2\} \right\rangle_{x_1,x_2}$$
$$+ \quad N_f^3 \int_{T_1,T_2,T_3} \left\langle F_3\{x_1,x_2,x_3\} \right\rangle_{x_1,x_2,x_3} \quad + \quad \dots$$

$\Longrightarrow$ "particle-$\hbar$ expansion" (HALPERN&SIEGEL'77)

$\Longrightarrow$ arbitrary g, "small" A (. . . but not perturbative in A)

A scalar model in quenched approximation

ϕ: "charged" matter field, A: "scalar" photon

$$\mathcal{L}(\phi, A) = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2}(\partial_\mu A)^2 - \frac{\mathrm{i}}{2}\, h\, A\, \phi^2 .$$

- well-defined perturbative expansion
- well-defined small-N_f expansion
- $\sim h\, A\, \phi^2$ superrenormalizable, $[h] = 1$, in $D = 4$
- imaginary interaction $\sim$ QED

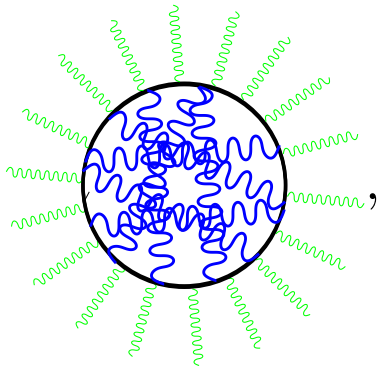
(. . . imaginary Wick-Cutkosky model)

Photon effective action

▷ quenched approximation

$$\Gamma_{\rm QA}[A] = \int_x \frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2(4\pi)^2}\int_0^\infty \frac{dT}{T^3}\, e^{-m^2 T}\left\langle e^{ih\int d\tau A}\, e^{-h^2\, V[x]}\right\rangle_x$$

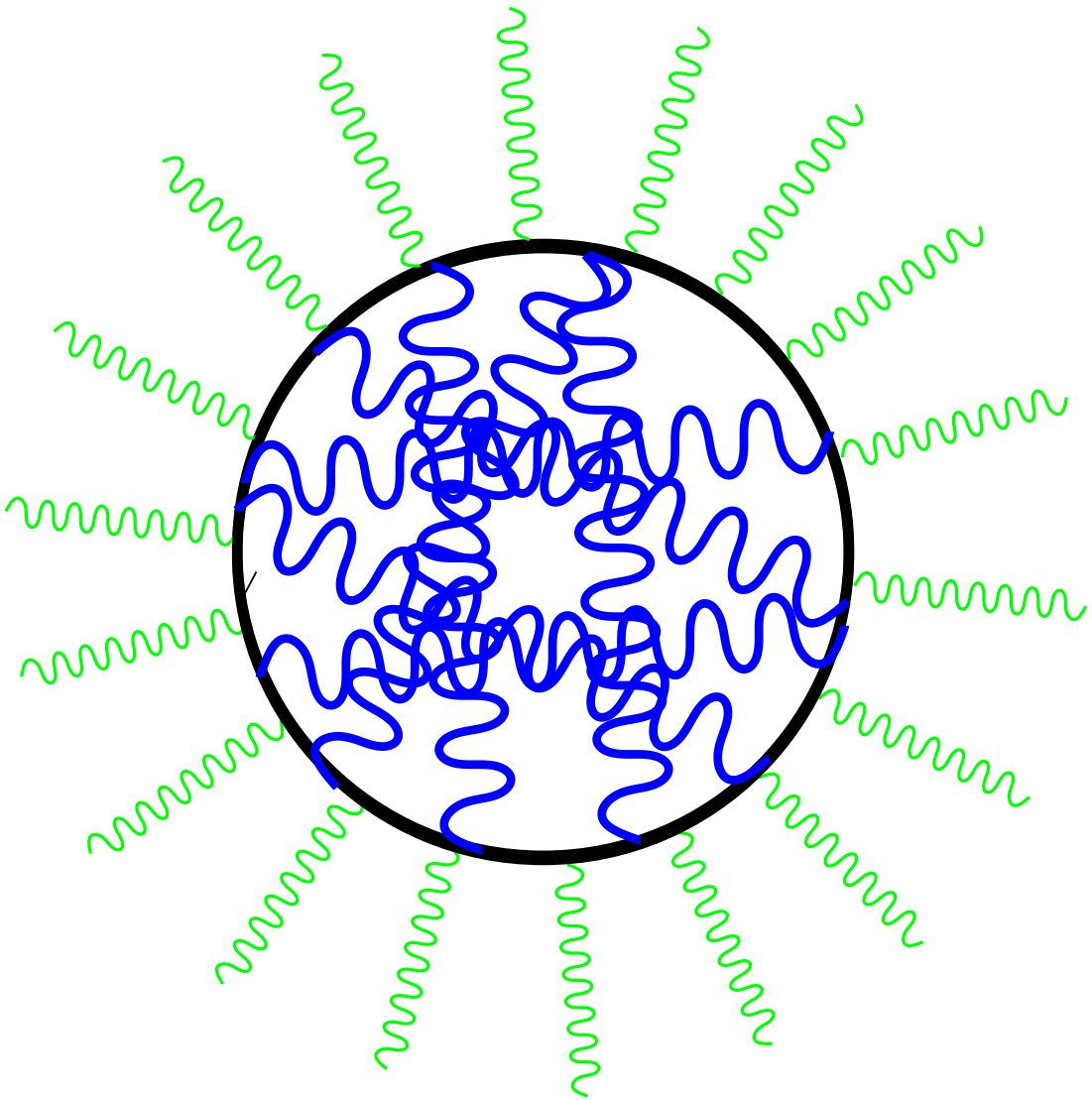
$$= \qquad , \tag{1}$$



▷ Worldline self-interaction potential

$$h^2\, V[x] := \frac{h^2}{8\pi^2}\int_0^T d\tau_1 d\tau_2\, \frac{1}{|x_1 - x_2|^2}$$

Self-interaction potential



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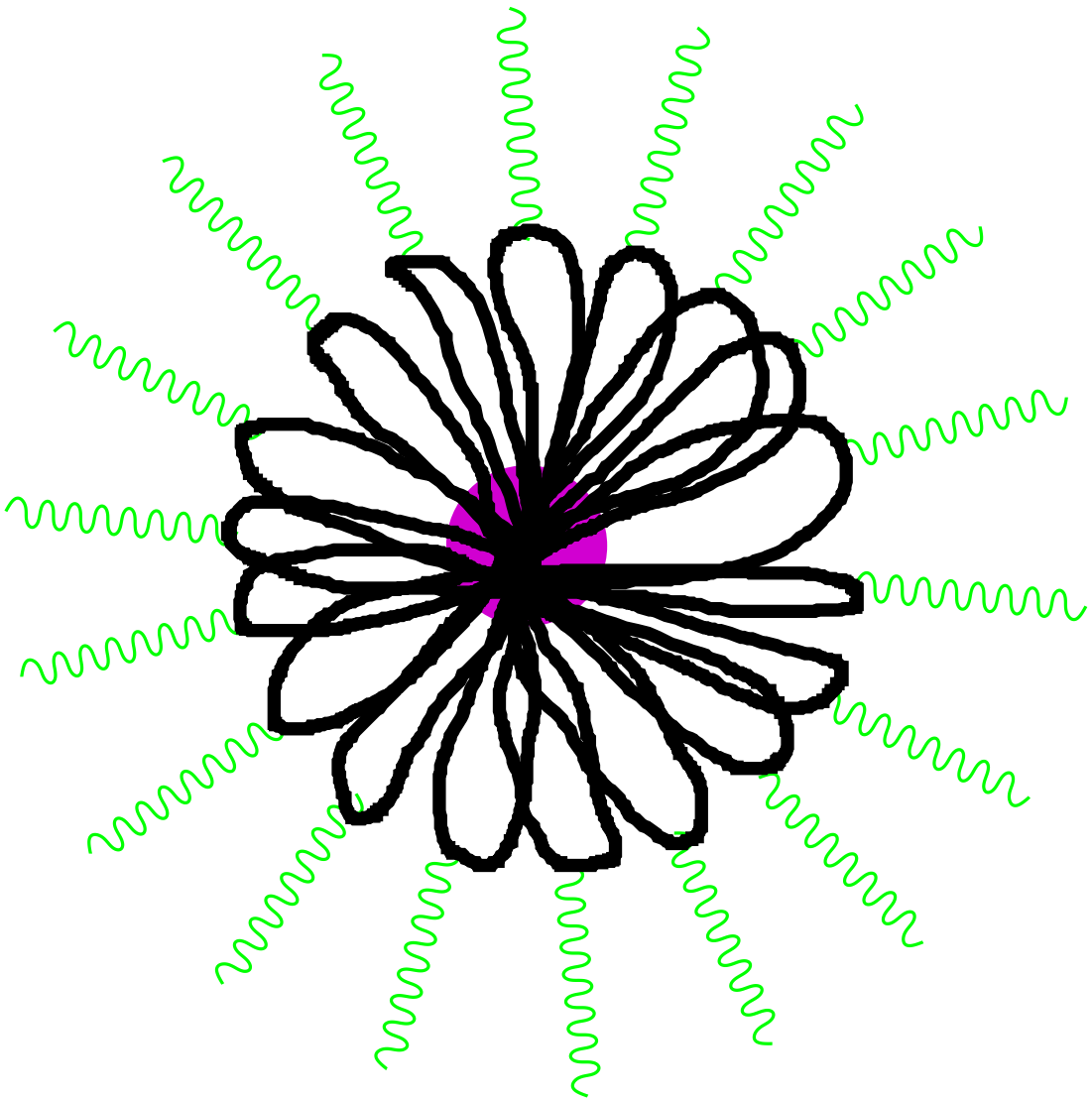
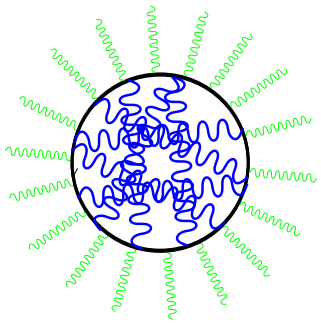


Quenched effective action

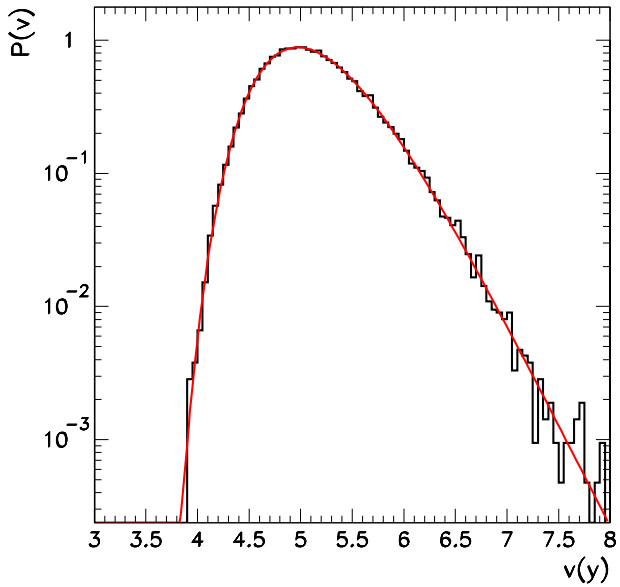
▷ soft-photon effective action, $A \simeq$ const. (. . . á la Heisenberg-Euler)

$$\Gamma_{\rm QA}[A] \;=\; -\frac{1}{2(4\pi)^{D/2}} \int_0^\infty \frac{dT}{T^{1+D/2}}\, e^{-m^2 T} e^{ihAT} \left\langle e^{-h^2\, V[x]} \right\rangle_x \;=$$



▷ PDF analysis

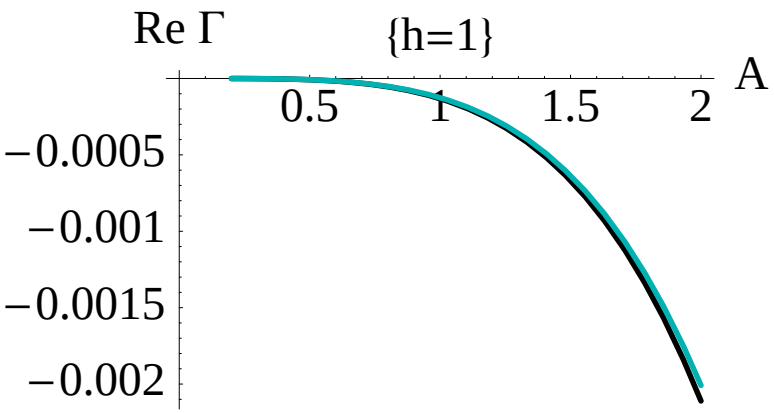
$$\left\langle e^{-h^2\, V[x]} \right\rangle_x = \int dV\, P_x(V)\, e^{-h^2 V}$$

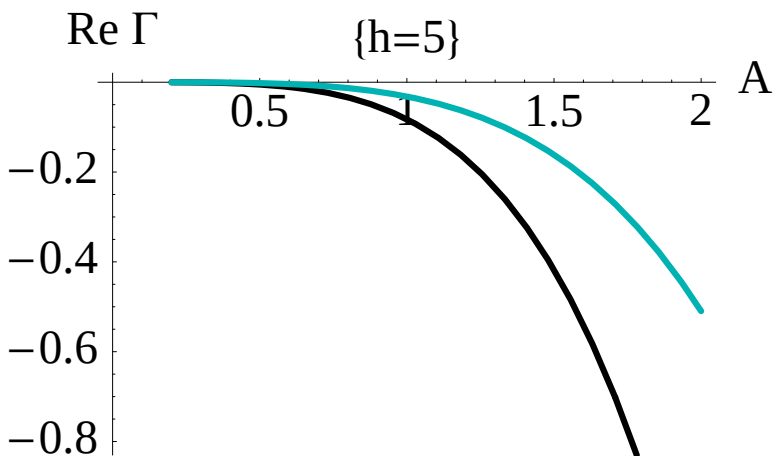


Renormalized effective action

(HG, Sanchez-Guillen, Vazquez'05)

$$\Gamma_{\mathrm{QA,R}}[A] = -\frac{1}{32\pi^2}\int d^4x \int_0^\infty \frac{dT}{T^3}\, e^{-m_R^2 T}\left(e^{\mathrm{i}hAT} - 1 - \mathrm{i}hAT + \frac{(hAT)^2}{2}\right) \times \left(\frac{\beta}{\beta + \frac{h^2}{8\pi^2}T}\right)^{1+\alpha}, \qquad \alpha \simeq 0.79, \quad \beta \simeq 13.2$$





Massless Limit?

▷ one-loop small-ϕ-mass limit: IR divergence

$$\Gamma_{\text{1-loop}}[A]\big|_{\frac{hA}{m_{\text{R}}^2}\gg 1} \simeq -\frac{1}{64\pi^2}\int d^4x\,(hA)^2\ln\frac{hA}{m_{\text{R}}^2}$$

▷ quenched small-ϕ-mass limit: finite

$$\Gamma_{\text{QA,R}}[A]|_{m_{\text{R}}=0} = -\frac{[-\Gamma(-2-\alpha)]\cos\frac{\pi}{2}\alpha}{2^{5-3\alpha}\pi^{2(1-\alpha)}\beta\alpha}\int d^4x\,(hA)^2\left(\frac{A}{h}\right)^{\alpha}[1+\mathcal{O}((A/h))]$$

$\Longrightarrow$ break-down of massless limit $\sim$ artifact of perturbation theory

. . . large log's summable

Conclusions.

- Probing the quantum vacuum by strong fields, Casimir boundaries, etc . . .

  . . . brings QFT to the desktop

  "quantum fields meet micro mechanics"

- Worldline numerics : { efficient tool ; intuitive picture }

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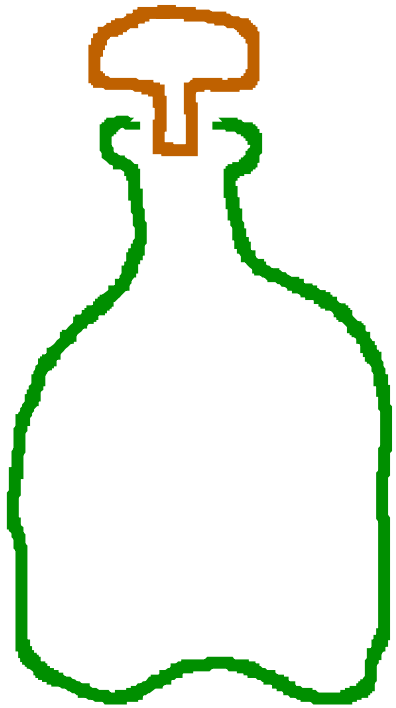

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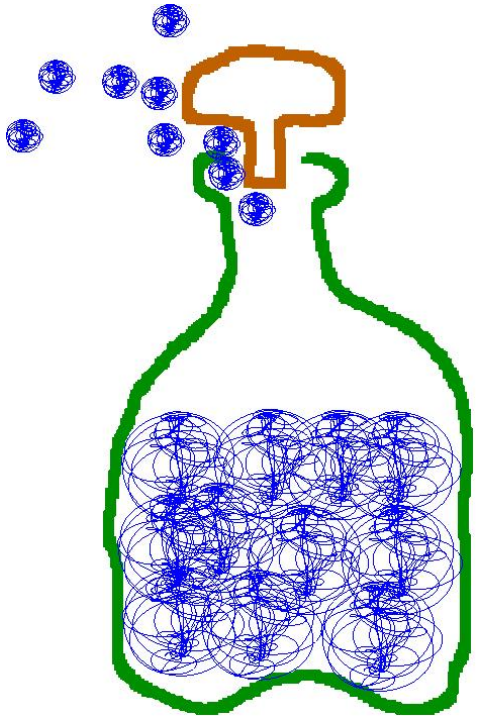


Fermions on the worldline I.

- Grassmann loops

$$\Gamma^1_{\text{spin}} = \ln\det\left[\gamma^\mu\partial_\mu + ie\gamma^\mu A_\mu + m\right]$$

$$= -\frac{1}{2(4\pi)^{D/2}}\int_{1/\Lambda^2}^{\infty}\frac{dT}{T^{1+D/2}}\,e^{-m^2T}\int_P\mathcal{D}x\int_A\mathcal{D}\psi\;e^{-\int_0^T d\tau L_{\text{spin}}}$$

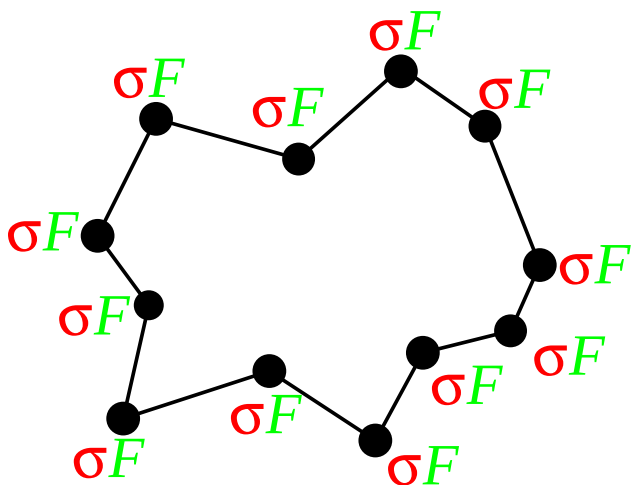
$$L_{\text{spin}} = \frac{1}{4}\dot{x}^2 + ie\dot{x}^\mu A_\mu + \frac{1}{2}\psi_\mu\dot{\psi}^\mu - ie\psi^\mu F_{\mu\nu}\psi^\nu$$

Fermions on the worldline II.

- spinor QED (parity-even part):

$$\Gamma = \frac{-\frac{1}{2}}{(4\pi)^{\frac{D}{2}}} \int d^D x_{\rm CM} \int_{1/\Lambda^2}^{\infty} \frac{dT}{T^{\frac{D}{2}+1}} \, {\rm e}^{-m^2 T} \left\langle W_{\rm spin}[A] \right\rangle_x$$

$$W_{\rm spin}[A] = W[A] \times P_T \exp\left(\frac{ie}{2} \int_0^T d\tau \, \sigma_{\mu\nu} F^{\mu\nu} \right)$$



Fermions on the worldline III.

▷ Spin factor

(STROMINGER'80, POLYAKOV'88)

$$\Gamma[A] \;=\; \frac{1}{2}\frac{1}{(4\pi)^{D/2}}\int_0^\infty \frac{dT}{T^{(1+D/2)}}\, e^{-m^2 T}\, \left\langle\, e^{-ie\oint dx A(x)}\; \Phi[x] \right\rangle_x$$

$$\Phi[x] := \mathrm{tr}_\gamma \mathcal{P} : e^{\frac{i}{2}\int_0^T d\tau\, \sigma\omega(\tau)} :$$

$$\omega_{\mu\nu}(\tau) = \frac{1}{4}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\; \ddot{x}_\mu(\tau+\frac{\rho}{2})\ddot{x}_\nu(\tau-\frac{\rho}{2})$$

(HG&HAMMERLING'05)