3 Vacuum of QCD

3.1 Introduction

The topic of this chapter is much too difficult to be comprehensively described on the pages to come. As Quantum Chromo Dynamics (QCD) is strongly interacting at low energies, the vacuum being the lowest energy state is most difficult. In fact, there is a plethora of ideas and models that attempt to understand aspects of the QCD vacuum each of them with their own pros and cons. In the following, we will therefore be much more modest: we will only be concerned with the question as to whether the ideas of the previous sections can be useful also in the case of QCD.

The answer will be twofold: taken at face value, these perturbative ideas are bound to fail spectacularly (even though the precise reasons why they fail is interesting to understand). Nevertheless, at a second glance, it is interesting to see that these ideas can serve as an inspiration for an extremely simple vacuum model that can explain a couple of characteristic features of low-energy QCD.

Let us start by summarizing a few essentials of QCD, the theory of quarks and gluons. From experiment, we know that quarks exist in $N_c=3$ "colors",

$$\psi^i = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix} \qquad (3.1)$$

A symmetry transformation in color space

$$\psi' = U\psi \qquad \text{where } U \in SU(N_c) \qquad (3.2)$$

in fundamental representation leaves physics observables invariant. The 3x3 matrices U can be parametrized by

$$U = e^{i\omega^a \tau^a} \qquad (3.3)$$

where the τ^a are the generators of the $su(N_c)$ Lie-algebra, satisfying

$$[\tau^a, \tau^b] = i f^{abc} \tau^c \qquad (3.4)$$

with structure constants f^{abc}. There are N_c^2-1 generators, such that $a,b,c = 1,\dots,N_c^2-1$.
(E.g. $\tau^a = \frac{\sigma^a}{2}$ (Pauli matrices) generate $SU(2)$, whereas $\tau^a = \frac{\lambda^a}{2}$ (Gell-Mann matrices) generate $SU(3)$.) We use the normalization

$$\mathrm{tr}\, \tau^a \tau^b = \frac{1}{2} \delta^{ab} \qquad (3.5)$$

The new aspect (compared to the bosonic $O(N)$-models discussed before) is that $SU(N_c)$ is a local symmetry, also called a gauge symmetry, of the theory,

$$\psi'(x) = U(x)\,\psi(x)$$
$$\bar{\psi}'(x) = \bar{\psi}(x)\,U^{-1}(x) \tag{3.6}$$

with $U^{-1} = U^{+}$ for $SU(N_c)$. Hence, a kinetic term requires a covariant derivative D_μ

$$\mathcal{L}_{kin} = \bar{\psi}\, i \not{D}\, \psi \quad \text{where} \quad D_\mu' = U(x)\, D_\mu\, U^{-1}(x) \tag{3.7}$$

Parametrizing D_μ with the aid of a "connection"

$$D_\mu = \partial_\mu - i g A_\mu \tag{3.8}$$

requires A_μ to be matrix valued and to transform as

$$A_\mu' = U A_\mu U^{-1} - \frac{i}{g}\,(\partial_\mu U)\, U^{-1} \tag{3.9}$$

The anti-hermiticity properties of D_μ require A_μ to be a hermitean $N_c \times N_c$ matrix which hence can be spanned by

$$A_\mu = A_\mu^a \tau^a . \tag{3.10}$$

(It is easy to check now, that $(\bar{\Psi} i \not{D} \Psi)' = \bar{\Psi} i \not{D} \Psi$.)

The requirement of local symmetry hence requires the existence of another field variable

$$A_\mu^a (x)$$

which we interpret as $N_c^2 - 1$ gluons – the non-abelian analogue of photons. Analogously to QED, we can read off the gluonic field strength $F_{\mu\nu} = F_{\mu\nu}^a \tau^a$ from the commutator of two covariant derivatives

$$-ig F_{\mu\nu} := [D_\mu , D_\nu] \quad \Rightarrow \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c . \tag{3.11}$$

From this definition, it is clear that $F_{\mu\nu}$ transforms covariantly

$$F_{\mu\nu}' = U F_{\mu\nu} U^{-1} \tag{3.12}$$

(i.e. the field strength is <u>not</u> gauge invariant!), but a gauge-invariant combination can be formed,

$$\mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} F_{\mu\nu}^a F^{a\mu\nu} \tag{3.13}$$

which serves to define a kinetic term for the gluons

$$\mathcal{L}_{gluon} \equiv \mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} , \qquad (3.14)$$

which itself defines "Yang-Mills theory" in analogy to Maxwell's theory. It is important to note that (3.14) already defines an interacting theory,

$$\mathcal{L} \sim \dots g (\partial_\mu A_\nu) A^\mu A^\nu + \dots g^2 (A_\mu A_\nu)^2 , \qquad (3.15)$$

i.e. gluons interact with each other classically (which is different from photons).

QCD can now be defined on the classical level completely analogous to QED:

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi} i \not{D} \psi - m \bar{\psi} \psi \qquad (3.16)$$

(Here, we ignored for the moment that there are more than just one kind of quarks with actually different masses m_q.)

From the action principle, we can derive a classical equation of motion for the gluon field:

$$D_\mu^{ab} F^{b\mu\nu} = \left(\partial_\mu \delta^{ab} + g f^{abc} A_\mu^c\right) F^{b\mu\nu} = j^{a\nu} \tag{3.17}$$

where we have defined the covariant derivative in adjoint representation

$$D_\mu^{ab} = \left(\partial_\mu \mathbb{1} - i g T^c A_\mu^c\right)^{ab} \quad , \quad (T^c)^{ab} = -i f^{abc} \quad , \tag{3.18}$$

since T^c also generates the $su(N_c)$ algebra

$[T^a, T^b] = i f^{abc} T^c$.

In (3.17), we have summarized the quark terms into a current $j^{a\nu}$. As an <u>example</u>, let us consider the "current" of a static quark-anti-quark pair

$$j^{a0} = \underset{\text{charge}}{Q} \quad \underset{\text{unit vector in adjoint color space}}{\hat{n}^a} \left(\delta^{(3)}(\vec{x} - \vec{x}_1) - \delta^{(3)}(\vec{x} - \vec{x}_2)\right) \tag{3.19}$$

$q, \bar{q}$ position in coordinate space

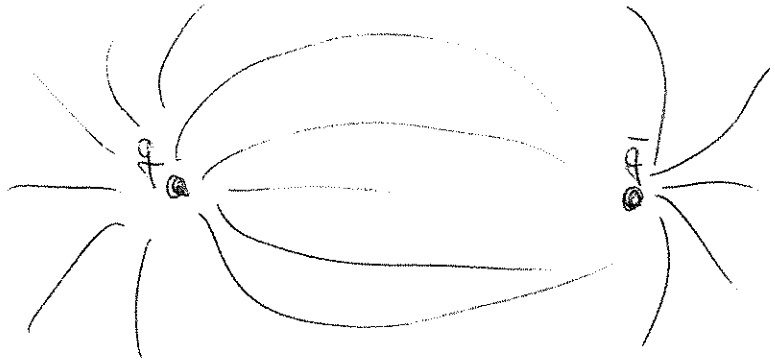
This equation of motion (3.17) can be solved by a pseudo-abelian field,

$$A_\mu^a = \hat{n}^a A_\mu \quad , \quad F_{\mu\nu}^a = \hat{n}^a F_{\mu\nu} \tag{3.20}$$

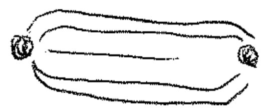
$f^{abc}\hat{m}_a\hat{m}_b = 0$

$$\Rightarrow \qquad \partial_\mu F^{\mu\nu} = j^{a\nu}\,\hat{m}^a \qquad (3.21)$$

This is identical to the corresponding problem of a dipole in classical electrodynamics. Hence, we find the same solution



yielding a static potential of Coulomb-type $V(r) \sim \frac{1}{r}$.

By contrast, measurements of the charmonium potential suggest $V_{q\bar{q}}(r) \sim r$, i.e. a linearly rising potential of a flux-type form 

This phenomenon inhibiting the liberation of single static color charges is summarized under the keyword "confinement"

(NB: a precise and useful definition of confinement is actually difficult ...)

It is important to note that classical self-interactions of the gluon field are not sufficient to explain confinement. In the spirit of this lecture course, we have to conclude that quantum effects modify the QCD vacuum structure not only quantitatively, but qualitatively.

3.2 Faddeev-Popov Quantization

For the perturbative construction of the functional integral, we face the problem of removing the singularities induced by a possible integration over gauge-equivalent field configurations. As in QED, this is done by gauge fixing, i.e. we require the fields to (predominantly) satisfy a certain gauge condition

$$F^a[A] = 0 \qquad (\text{e.g. } \partial_\mu A^a_\mu = 0) \qquad (3.22)$$

Here and in the following, we work in Euclidean space. However, in contrast to QED, we have to be more careful with implementing this constraint.

Assuming that we can make all gauge configurations A_μ to satisfy a certain gauge fixing condition

$$A_\mu \rightarrow A_\mu^{gf} \qquad \text{with } F[A_\mu^{gf}] = 0 \tag{3.23}$$

we would like to write down a measure

$$\int \mathcal{D} A^{gf} = \int \mathcal{D} A \; \delta[A - A^{gf}] \, .$$

However, in practice, we can only implement the gauge fixing condition in terms of $F[A]$ as the space of all inequivalent configurations A^{gf} is difficult to be parametrized. Now, implementing $F[A]=0$ with a δ-functional involves a Jacobian,

$$\int \mathcal{D} A \; \delta[F^a[A]] \; \Delta_{FP}[A] \tag{3.24}$$

where $\Delta_{FP}[A]$ is the Faddeev-Popov determinant

$$\Delta_{FP}[A] = \det \frac{\delta F^a[A^\omega]}{\delta \omega^b} \tag{3.25}$$

where $A^\omega = U A U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1}$, $U = e^{i\omega^a T^a}$. $\qquad$ (3.26)

We omit here the proof that (i) $\Delta_{FP}[A]$ is gauge invariant and that (ii) it facilitates to factorize the volume of the gauge group necessary to separate the integral over the

redundant field configurations.
(NB: we also ignore the fact that naive FP quantization is ill-defined for standard gauges due to the Gribov ambiguity as we will be satisfied with perturbation theory where these problems are not relevant.)

Furthermore, representing the δ functional by a gauge-fixing action, e.g.,

$$\delta[F^a[A]] \to e^{-\frac{1}{2\alpha}\int F^a F^a}\Big|_{\alpha\to 0} \equiv e^{-S_{gf}[A]}, \quad (3.27)$$

we arrive at a representation for the generating functional for Yang-Mills theory

$$Z[J] = \int \mathcal{D}A \; \Delta_{FP}[A] \; e^{-S_{YM} - S_{gf} + \int JA} . \quad (3.28)$$

The inclusion of quarks then is straightforward. In principle, the Faddeev-Popov quantization is also necessary in QED. However, for standard gauges $\Delta_{FP}[A]$ is actually independent of A and hence can (in many cases) be ignored.

3.3 Background field methods

For gauge theories, we face another problem: we wish to compute a gauge-invariant effective action $\Gamma[A] \equiv \Gamma[A^\omega]$, <u>but</u> need gauge fixing in order to define a functional integral.

This problem can be solved by the background-field method. Again, we will not detail this tool, but list just a few properties that are essential for our simple application below.

Let us decompose the quantum field into a

$$A = \bar{A} + Q \tag{3.29}$$

background field $\bar{A}$ and a fluctuation field Q. We also (artificially) split the gauge transformation of A into

$$\begin{aligned} \bar{A}'' &= U \bar{A} U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1} \\ Q'' &= U Q U^{-1} \end{aligned} \tag{3.30}$$

such that the $\bar{A}$ transform looks like a standard gauge transform. Eq. (3.30) is called a background field transformation (BFT).

A crucial point of the background field method is to choose a suitable gauge-fixing that fixes the Q-fluctuations, but leaves the artificial symmetry (3.30) intact. We use

$$\mathcal{F}^a[\bar{A}, Q] = D_\mu^{ab}[\bar{A}]\, Q_\mu^b \qquad (3.31)$$

such that

$$S_{gf}[\bar{A}, Q] = \frac{1}{2\alpha} \int (D_\mu^{ab}[\bar{A}]\, Q_\mu^b)^2 . \qquad (3.32)$$

Now (3.32) is invariant under the BFT (3.30), but fixes the standard gauge transformation

$$A' = U A' U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1} \qquad (3.33)$$

(but with $\bar{A}' = \bar{A}$ kept fixed).

For any given $\bar{A}$, the integration measure can be shifted

$$\mathcal{D}A \rightarrow \mathcal{D}Q \qquad (3.34)$$

without any non-trivial Jacobian. Having introduced the background field in the first place gives rise to the complication that the effective action now depends on two fields $\Gamma = \Gamma[Q, \bar{A}]$ (where here Q denotes the Legendre conjugate to the Q-source, i.e. the 'classical' field).

However, it is now possible to show that the desired effective action (the generator of 1PI correlation functions) can straightforwardly be obtained from the "bi-field" action:

$$\Gamma[A] \stackrel{!}{\equiv} \Gamma[Q=0, A=\bar{A}] \tag{3.35}$$

In this limit, $\Gamma[A]$ inherits the desired properties under gauge transformations from the auxiliary BFT. This is the "miracle" of the background field method.

It is now straightforward to work out the form of the Faddeev-Popov operator / determinant. For this, we note that the gauge transform of Q is

$$Q' = U(\bar{A}+Q)U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} - \underbrace{\bar{A}}_{\equiv \bar{A}'} \tag{3.36}$$

or infinitesimally with $U = e^{i\omega^a \tau^a} \simeq 1 + i\omega^a\tau^a + \mathcal{O}(\omega^2)$

$$\begin{aligned} Q'^a_\mu &= Q^a_\mu + f^{abc}\omega^b(Q^c_\mu + \bar{A}^c_\mu) - \frac{1}{g}\partial_\mu \omega^a \\ &= Q^a_\mu - \frac{1}{g} D^{ac}_\mu[\bar{A}+Q]\,\omega^c \, . \end{aligned} \tag{3.37}$$

From the gauge-fixing condition (3.31), we can compute the Faddeev-Popov operator

$$\frac{\delta F^a[\bar{A},Q]}{\delta \omega^b} = D_\mu^{ac}[\bar{A}]\,\frac{\delta Q^c}{\delta \omega^b} = -\frac{1}{g}\,D_\mu^{ac}[\bar{A}]\,D_\mu^{cb}[\bar{A}+Q], \tag{3.38}$$

yielding the Faddeev-Popov determinant

$$\Delta_{FP}[\bar{A},Q] = \det\left(-D_\mu^{ac}[\bar{A}]\,D_\mu^{cb}[\bar{A}+Q]\right) \tag{3.39}$$

(The factor $\frac{1}{g}$ can be absorbed into the normalization of the functional integral)

Since Δ_{FP} already contributes as a log-det to the effective action, i.e. has an inherent one-loop structure, the Q-dependence of Δ_{FP} contributes only to higher-loop order. (This becomes manifest upon introduction of "Faddeev-Popov-ghosts", i.e. auxiliary fields, that help writing Δ_{FP} as a local action contribution). Therefore, at one-loop-order we obtain a first contribution to the effective action

$$\sim -\ln \Delta_{FP} = -\ln\det\left(-D^2[\bar{A}]\right). \tag{3.40}$$

In addition to the Faddeev-Popov part, we need the generic contributions from the gluonic fluctuations. This is obtained from the second variation of the classical action S_{YM} read together with the gauge-fixing part. A straightforward computation gives

$$\left(S^{(2)}_{YM} + S^{(2)}_{gf}\right)^{ab}_{\mu\nu} =: M^{ab}_{\mu\nu} = -\bar{D}^{ac}_{\alpha}\bar{D}^{cb}_{\alpha}\,\delta_{\mu\nu} + 2g f^{abc}\,\bar{F}^{c}_{\mu\nu} + \left(1-\frac{1}{\alpha}\right)\bar{D}^{ac}_{\mu}\bar{D}^{cb}_{\nu}\,, \tag{3.41}$$

where we have uses $S_{YM}[A] = S_{YM}[\bar{A}+Q]$; $S^{(2)}$ denotes the second functional derivative WRT Q^a_μ and $\bar{D} \equiv D[\bar{A}]$. In addition to a Klein-Gordon type Laplacian (first term in (3.41)) and the gauge-fixing part (last term), we observe the term $\sim g\, f^{abc}\bar{F}^c_{\mu\nu} \equiv ig T^c \bar{F}^c_{\mu\nu}$. This is reminiscent to the spin-field coupling Pauli term $\sim \sigma_{\mu\nu}F_{\mu\nu}$ in the fermionic case. It has the meaning of a gluonic spin - color-field coupling and will play an interesting role later on. As in the case of fermions, this term can be interpreted as a "paramagnetic" contribution.

With this, we obtain the standard one-loop effective action for Yang-Mills theory

$$\Gamma^1_{YM}[\bar{A}] = S_{YM}[\bar{A}] + \frac{1}{2} \ln \det M^{ab}_{\mu\nu}[\bar{A}] - \ln \det \left((-\bar{D}^2)^{ab} \right) \tag{3.42}$$

In the case of QCD, we also have to include the quark loops. For each quark flavor, we get a contribution $\sim - \ln \det(-i\bar{\not{D}} + m)$ which is precisely of Heisenberg-Euler type.