2.24 Pair production in electric fields

One of the most extensively studied phenomena described by the Heisenberg-Euler action is the phenomenon of spontaneous pair creation in electric fields. This has first been quantitatively discussed by F. Sauter (inspired by a remark of N. Bohr) and then first derived in QED by Heisenberg & Euler. Still, the phenomenon is often called "Schwinger pair production", as J. Schwinger in his profound work "On gauge invariance & vacuum polarization" in 1951 has put strong-field QED on solid QFT ground.

Let us start with the observation that

$$e^{i\int d^4x\, \mathcal{L}_{HE}[A]} = \langle 0|0\rangle_A \qquad (2.78)$$

can be interpreted as a vacuum persistence amplitude in the presence of an external field. So if $\mathcal{L}_{HE} \in \mathbb{R}$, then the probability for the vacuum to persist is

$$|\langle 0|0\rangle_A|^2 = 1 , \qquad (2.79)$$

and the vacuum is stable.

If however $\mathrm{Im}\, \mathcal{L}_{HE} > 0$, we obtain $0 < |\langle 0|0\rangle_A|^2 < 1$ such that

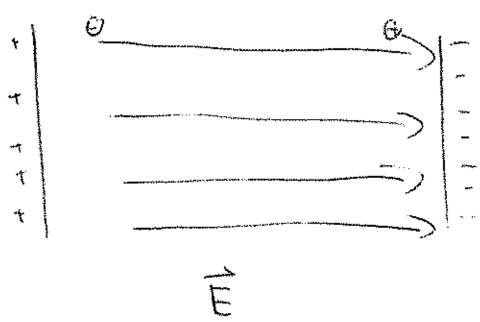
$$P_{[A]} = 1 - |\langle 0|0\rangle_A|^2 = 1 - e^{-2\int d^4x\, \mathrm{Im}\, \mathcal{L}_{HE}} \tag{2.80}$$

corresponds to the probability for the vacuum to decay in the presence of the external field A. The exponent

$$W = 2\, \mathrm{Im}\, \mathcal{L}_{HE}[A] \tag{2.81}$$

can be interpreted as a local decay rate. It is reminiscent to a tunnel rate.

As $\mathcal{L}_{HE}$ in the presence of a purely magnetic field is manifestly real, a magnetized vacuum is stable. However, in the presence of an electric field, it is plausible that the vacuum could be unstable towards a state with a non zero electron-positron number



This is because electric fields can transfer energy to charged particles. If it does so also to virtual fluctuations, a virtual e^+e^- pair can become real if the electrostatic energy acquired by a separated pair becomes of the same order as the rest mass:

$m = eEL$ where L is a separation scale between the charges. For electron-positron fluctuations, we expect L to be of the order of the Compton wavelength. This leads to value for a critical field strength where we expect pair production to set in

$$L \simeq \frac{1}{m} \quad \Rightarrow \quad E_c = \frac{m^2}{e} \simeq 1.3 \cdot 10^{18}\, V/m \quad (2.82)$$

(This is often called the Sauter or Schwinger field.)

Let us now take a look at the Heisenberg-Euler Lagrangian in an electric field $(T \to \frac{1}{eE} T)$

$$\mathcal{L}^1_{HE} = \frac{(eE)^2}{8\pi^2} \int_0^\infty \frac{dT}{T^3}\, e^{-\frac{m^2}{eE} T} \left(T \frac{\cos T}{\sin T} - 1 + \frac{1}{3} T^2 \right)$$

(2.83)

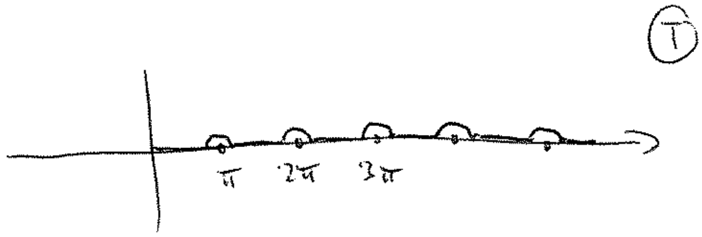
Here, we have rescaled to a dimensionless proper time variable.

Even though the integrand is real, the integral can acquire an imaginary part from the poles of the integrand at

$$T = n\pi \,, \quad n = 1,2,3,\dots \tag{2.84}$$

due to the sine in the denominator.

For consistency with the $m^2 \to m^2 - i\varepsilon$ causality prescription, we have to encircle these poles in the upper complex T half plane:



From these semi-circles, the integral acquires an imaginary part

$$i \,\mathrm{Im} \frac{1}{\sin T} \to i\pi \sum_{n=1}^{\infty} \delta(T - n\pi)\,(-1)^n \,, \tag{2.85}$$

yielding the famous Schwinger vacuum decay rate

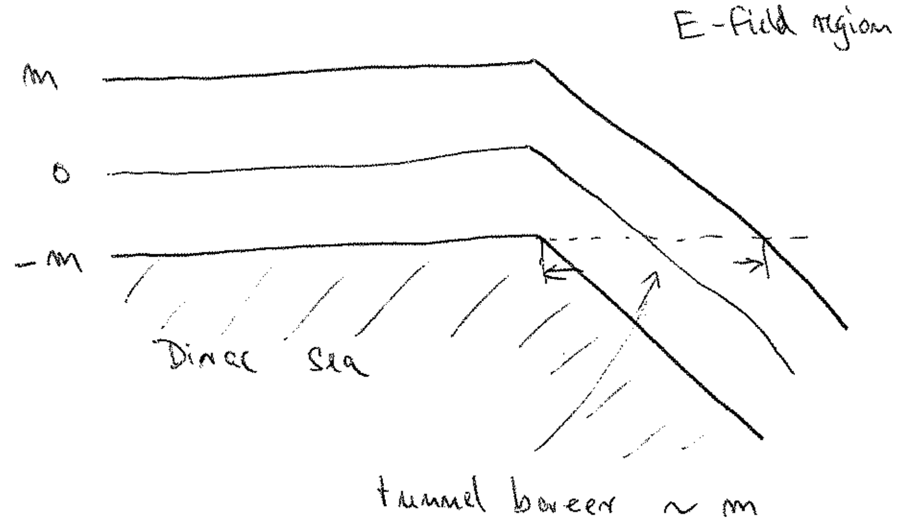
$$w = 2\,\mathrm{Im}\,\mathcal{L}_{HE} = \frac{(eE)^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{e^{-\frac{m^2}{eE} n\pi}}{n^2} \,. \tag{2.86}$$

In the weak-field limit, only the $n=1$ term is relevant

$$w\,(eE \ll m^2) \simeq \frac{(eE)^2}{4\pi^3}\, e^{-\pi \frac{m^2}{eE}} \,. \tag{2.87}$$

Several remarks are in order

- (2.86/87) indeed show a great similarity to quantum mechanical tunneling. Indeed, this analogy becomes transparent in a one-particle quantum mechanical picture involving the Dirac Sea:



- Pair production is obviously nonperturbative, as the coupling e occurs in the denominator of the exponent. However, it is important to note that the result is nonperturbative in (eE) rather than in α alone, i.e. one needs an infinite number of (zero-momentum) external legs. The result is still a one-loop result and higher-loop corrections can straightforwardly be calculated.

- There has been some discussion / confusion in the

literature w also is a "pair-production" rate. However, as is clear from the very beginning, it is a vacuum decay rate. A priori, w does not tell us anything about the state into which the vacuum decays (one pair, ..., 17 pairs ..).
In any case, in the weak-field limit, it is plausible that the decay rate of the vacuum should be dominated energetically by the decay into the one-pair state. Hence, the first term in the sum (2.87) estimates the one-pair production rate.

- Strictly speaking, the calculation presented here is conceptually misguided, as we have used the formalism of in-equilibrium field theory for an out-of-equilibrium decay process. Still, formalisms that account for the explicit time-dependence (real Minkowski time!) find the same result in the limit of asymptotic times, provided
   - interactions between the produced pairs are neglected (~ higher loop terms)
   - the density of the produced pairs is low
   - back-reactions from screening of the external field (→ plasma oscillations) are neglected

An experimental verification of Schwinger pair production is considered to be a crucial test of QED in a nonperturbative regime where no data is available yet. Obviously, standard macroscopic fields are too small, hence high-intensity laser systems are considered to provide for a suitable tool. The critical field strength corresponds to a critical intensity

$$I_c = E_c^2 \simeq 4 \cdot 10^{29}\ \mathrm{W/cm^2}$$

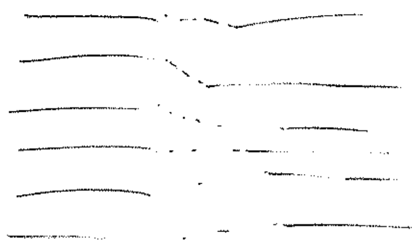
which is still a way to go for current systems.

2.4 Inhomogeneous fields: worldline formalism

So far, we have mainly considered the limit of constant fields in space and time which we have been able to treat analytically. In fact, only very few cases can be dealt with analytically. But what is even worse is that even numerical methods are not of naive immediate help. It is worthwhile to get a feel for the reason for this statement. The basic 1-loop formula for the effective action in the case of QED is

$$\Gamma = -i \operatorname{Tr} \ln(-i\not{D} + m) . \tag{2.88}$$

Taken at face value, we first have to compute all the eigenvalues of $i\not{D} + m$, and then trace over their logs. An obvious problem is that the spectra of such operators generically is not bounded, so at least a large number of eigenvalues has to be determined. This may be doable numerically with some precision. However, the more severe and less obvious problem is the trace operation. This trace is (actually needs to be) divergent as it not only encodes the vacuum energy to be subtracted but also information about renormalization. Therefore, what actually matters are not the eigenvalues, but the slight shift of the eigenvalues with respect to the vacuum eigenvalues modulo the shift that encodes renormalization



vacuum + external field

In order to isolate these physically relevant shifts, a high-precision computation seems to be necessary. Moreover, the tracing if done naively involves divergent expressions. Naive regularizations are typically bound to fail in practice.

Of course, one possibility always is to put the system onto a spacetime lattice. However, in this case it will be difficult to study problems with largely separated scales.

All these issues are solved within the (numerical) worldline formalism. For this, let us return to the exact expression (2.20)

$$\Gamma^{1}_{QED} = \frac{i}{2} \int_{1/\Lambda^2}^{\infty} \frac{dT}{T} \, \mathrm{Tr} \, e^{-(-D^2+m^2-\frac{e}{2}\sigma F)T} \, . \qquad (2.89)$$

Let us, for simplicity, ignore the spin degrees of freedom and turn to scalar QED

$$\Rightarrow \quad \frac{1}{2}\, e^{\frac{e}{2}\sigma F T} \quad \rightarrow \quad -1 \tag{2.90}$$

Spinor QED can also be dealt with (even though it involves some more technical challenges depending on the parameter regimes).

Furthermore, let us work directly in Euclidean space:

$$\int d^4x\, \mathcal{L}_{HE} = -\int_{1/\Lambda^2}^{\infty} \frac{dT}{T}\, e^{-m^2 T}\, \mathrm{Tr}\, e^{-(-D^2)T} \tag{2.91}$$

The (numerically) challenging problem is hidden in the trace. If we forget for the moment that T is an auxiliary integration variable, we may interpret $-D^2$ as a quantum mechanical Hamiltonian

$$H_W = -D^2 \tag{2.92}$$

for a single particle with time evolution in T:

$$U(T) = e^{-H_W T}. \tag{2.93}$$

$U(T)$ then is a time evolution operator in a Euclidean time T.

Now, we now from quantum mechanics that traces of time evolution operators can be represented as a path integral

$$\mathrm{Tr}\; e^{-(-\partial^2)T} = \mathcal{N} \int \mathcal{D}x \; e^{-\int_0^T d\tau \, L_W} \qquad (2.94)$$

where L_W is the corresponding Lagrangian, i.e. the Legendre transform of H_W. This Lagrange function is already familiar from classical electrodynamics:

$$L_W = \frac{1}{4} \dot{x}^2(\tau) + ie\, \dot{x}_\mu(\tau) A_\mu(x(\tau)) \qquad (2.95)$$

In (2.94), $\tau \in [0,T]$ is a parameter for the path $x_\mu(\tau)$ in spacetime. Since we represent a trace in $(2.94)_{LHS}$, the paths in $(2.94)_{RHS}$ are closed

$$x_\mu(0) = x_\mu(T) \qquad (2.96)$$

Therefore, the effective action can be represented as a worldline path integral

$$\int d^4x \, \mathcal{L}_{HE}^1 = - \int_{1/\Lambda^2}^{\infty} \frac{dT}{T} \, e^{-m^2 T} \, \mathcal{N} \int_{x(0)=x(T)} \mathcal{D}x \; e^{-\int_0^T \frac{\dot{x}^2}{4} d\tau} \; e^{-ie\oint dx_\mu A_\mu} . \tag{2.97}$$

This representation is extremely powerful and also leads to highly efficient calculation techniques in analytical calculations. This is, because a lot of "tricks" can be carried over from string-theory techniques (vertex operators, Green's function techniques, replacement rules, etc.), see the work by C. Schubert.

Here, we would like to emphasize the numerical advantages: as path integrals can be computed very efficiently with Monte Carlo techniques, Eq. (2.97) boils down to generating an ensemble of closed worldlines obeying a Gaußian velocity distribution $\sim e^{-\int_0^T \frac{\dot{x}^2}{4} d\tau}$ and then sampling the (gauge-invariant) Wegner-Wilson loop $e^{-ie\oint dx_\mu A_\mu}$

over this ensemble, schematically

$$\mathcal{N} \int \mathcal{D}x \; e^{-\int_0^T \frac{\dot{x}^2}{4} d\tau} \langle O[x] \rangle \sim \mathcal{N}' \sum_{\{x(\tau) = \}} \langle O[\;] \rangle \tag{2.98}$$

The important point is that this can be done for any $A_\mu(x)$, and that the result of the path integral is completely finite.

The remaining renormalization issues can be taken care of by the propertime integral. Typically, this can be done fully analytically.

Thereby, the worldline formalism combines the mutual strengths of the analytic & numerical approaches

(Concrete examples are shown as separate slides ...)