2 Vacuum Phenomena in Quantum Electrodynamics (QED)

So far, we concentrated on scalar field theories, investigating the question as to whether quantum fluctuations can induce a non-trivial vacuum state. This state then lead to interesting physical features such as the spectrum of excitations and the realization of symmetries.

In QED, the theory of the interactions of light & matter, the situation is somewhat different. Fluctuations are typically not strong enough to induce a non trivial vacuum state. Still, quantum fluctuations are ubiquitous and therefore can influence the physical properties of the trivial vacuum.

In order to reveal the influence of fluctuations, it is interesting to expose the vacuum to further external influences and measure the response of the vacuum + fluctuations on these external parameters. One of the most widely considered "knobs" to probe the quatum vacuum is to expose it to strong external fields.

Before we do any serious quantum computations, let us try to formulate our expectations in terms of a classical field theory language. Let us start with the classical Maxwell Lagrangian of electrodynamics including a source term,

$$\mathcal{L}_{ED} = \mathcal{L}_M + \mathcal{L}_j = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \qquad (2.1)$$

where $\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

The general equation of motion for an action depending on $F^{\mu\nu}$ is

$$\frac{\delta S}{\delta A_\nu} = 0$$

$$\Rightarrow \quad 0 = \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -j^\nu - \partial_\mu \frac{\partial \mathcal{L}}{\partial F_{\alpha\beta}} \underbrace{\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}}_{= \delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha}$$

$$= -j^\nu - 2\partial_\mu \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \qquad (2.2)$$

Inserting the Maxwell action, we immediately get Maxwell's equation

$$\partial_\mu F^{\mu\nu} = j^\nu , \tag{2.3}$$

which yields a high-precision description of electromagnetic (EM) fields for a given current. One particular feature is the linearity in terms of the EM fields corresponding to the action being quadratic in the fields. As a consequence, the superposition principle holds in classical ED.

These considerations imply that the classical Maxwell action must be the limiting case of a full quantum effective action. Since QFT is the true theory of nature, the question arises: Why is the classical limit so simple?

General properties of $\mathcal{L}_{eff}$ are:

(1) Lorentz scalar

(2) gauge invariance

(3) mass dimension: $[\mathcal{L}] = 4$, such that $[S] = 0$

(4) CP invariance (no CP violation known in QED)

In the scalar examples, the assumption of homogeneity of the vacuum state has left us with very few invariants as building blocks for the effective potential / action: ϕ^2 or $\phi^a \phi^a$.

Assuming homogeneity of the field strength here, the question now is: How many invariants can we construct from $F_{\mu\nu}$? The answer is: 2. These are typically written as

$$\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (\vec{B}^2 - \vec{E}^2) \qquad (2.4a)$$

(Lorentz scalar)

and $$\mathcal{G} = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = - \vec{E} \cdot \vec{B} \qquad (2.4b)$$

(pseudo-scalar, CP odd)

where

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \qquad (2.5)$$

is the dual field strength tensor.

Any other algebraic expression can be reduced to these two invariants by means of the identities

$$F^{\mu\alpha} F^{\nu}{}_{\alpha} - \tilde{F}^{\mu\alpha} \tilde{F}^{\nu}{}_{\alpha} = 2 \mathcal{F} g^{\mu\nu}$$
$$F^{\mu\alpha} \tilde{F}^{\nu}{}_{\alpha} = \tilde{F}^{\mu\alpha} F^{\nu}{}_{\alpha} = \mathcal{G} g^{\mu\nu} \tag{2.6}$$

These identities render all Lorentz-contracted monomials of the field strength tensor reducible to $\mathcal{F}$ and $\mathcal{G}$.

In general, the number of invariants of a tensor can be determined from

invariants = # indep. components − # sym. generators + # generators of invariant subgroup

here:

$F_{\mu\nu}$: $2 = 6 - \underset{\substack{\uparrow \\ \text{3 boosts} \\ \text{+ 3 rotations}}}{6} + \underset{\substack{\uparrow \\ \text{1 boost + 1 rotation} \\ \text{in the system,} \\ \text{where } \vec{E} \parallel \vec{B}}}{2}$

(NB: Let us contrast this with a non abelian YM theory, say with gauge group SU(2)

$F^{a}_{\mu\nu}$: $9 = 6 \cdot 3 - (6+3) + 0$)

Now, the assumption of homogeneity has a different character here as in the scalar case. Whereas in the latter case, the assumption is reasonable as inhomogeneities typically "cost" more action, the field here will be an external parameter that can be chosen at will.

Hence, we expect that $\partial_\mu F_{\alpha\beta} \neq 0$ in general, such that ∂_μ is also a building block of the effective action. Somewhat schematically, we may expect the QED effective action to be of the form

$$\int d^4x \, \mathcal{L}_{\rm eff} = \int d^4x \Big[-\mathcal{F} + c_1 \mathcal{F}^2 + c_2 \mathcal{G}^2 + \mathcal{O}(\mathcal{F}^3, \mathcal{G}^2\mathcal{F})$$

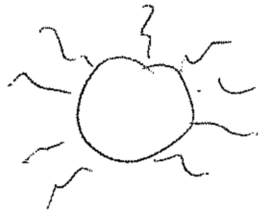
$$+ d_1 F_{\mu\nu} \partial^2 F^{\mu\nu} + \mathcal{O}(F^4 \partial^2, F^2 \partial^4) \Big], \qquad (2.7)$$

where terms which vanish on-shell (or are of higher-order) such as $\partial_\mu F^{\mu\nu}$ have already been dropped.

Eq. (2.7) constitutes a gradient expansion in terms of the field strength with coefficients c_1, c_2, d_1 of mass dimension

$$[d_1] = -2 \quad , \quad [c_{1,2}] = -4 \qquad (2.8)$$

As long as no other dimension-ful scale is available, the coefficients $c_{1,2}, d_1, \dots$ have to vanish in order for $[\mathcal{L}_{eff}] = 4$ to be satisfied. However, as soon as we allow for quantum fluctuations, the EM field can couple to electron-positron fluctuations



providing for another mass scale:

$$
\begin{aligned}
m_e &= 511 \text{ keV} \\
&\hat{=}\ 7.6 \cdot 10^{11} \text{ GHz} \ \hat{=}\ (3.8 \cdot 10^{-13} \text{m})^{-1} \equiv \frac{1}{\lambda_c} \\
m_e^2 &\hat{=}\ 1.3 \cdot 10^{9} \text{ Tesla}
\end{aligned}
\qquad (2.9)
$$

Hence, we expect $d_1 \sim \frac{1}{m_e^2} \sim \lambda_c^2$ and $c_{1,2} \sim \frac{1}{m_e^4}$

Now, if a field is "slowly varying" on length scales larger than the Compton wavelength λ_c, neglecting the derivative terms in (2.7) is a very good approximation. Almost any laboratory field is of this slowly varying type; then $\mathcal{L}_{eff} \simeq \mathcal{L}_{eff}(\mathcal{F}, \mathcal{G}^2)$

If in addition the fields are weak

$$|\vec{E}|, |\vec{B}| \ll E_{cr} = \frac{m_e^2}{e}, \qquad (2.10)$$

the nonlinear terms in (2.7) can be neglected, and we end up with $\mathcal{L}_{eff} \simeq -\mathcal{F} = \mathcal{L}_M$ which is precisely the classical limit. In other words, classical electrodynamics holds, because the electron is so heavy!

In (2.10), we have introduced the so-called critical field E_{cr} which corresponds to the field strength where an electron can aquire an electrostatic energy over a Compton wavelength which is equal to its rest energy: $m_e c^2 \stackrel{!}{=} e E_{cr} \lambda_c = \frac{e E_{cr}}{m_e}$.

2.1 QED effective action
($\hat{=}$ Heisenberg-Euler action)

W. Heisenberg & H. Euler, Z. Phys. 98, 714 (1936)

V. Weisskopf ('36),
J. Schwinger ('51)

We start from the "bare" action of QED

$$\mathcal{L} = \bar{\psi} i \gamma_\mu (\partial^\mu - ieA^\mu)\psi - m\bar{\psi}\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} . \quad (2.11)$$

Here ψ is a 4-component complex Dirac spinor (anti-commuting Grassmann-valued field), and the γ matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = -2 g_{\mu\nu}$.
An example for a useful representation is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \{\gamma^i\} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} . \quad (2.12)$$

Later on, we will need

$$\sigma^{\mu\nu} := \frac{i}{2}[\gamma^\mu, \gamma^\nu] \text{, and in particular } \underbrace{\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}}_{=\sigma^{12}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (2.13)$$

We want to evaluate the one-loop effection in an EM field background $A^\mu_{cl.}$, $\psi_{cl} = 0 = \bar\psi_{cl}$. For this, we need the fluctuation operator

$$S^{(2)}_{\bar\psi\psi} = \frac{\delta^2 S}{\delta\bar\psi\,\delta\psi} = -i\not{D} + m, \quad \not{D} = \gamma_\mu D^\mu = \gamma_\mu(\partial^\mu - ieA^\mu_{cl}) \tag{2.14}$$

$$S^{(2)}_{\psi A} \sim \bar\psi_{cl} \to 0 \qquad \text{similarly } S_{\bar\psi A}, \text{ etc.}$$

$$S^{(2)}_{AA} \sim \text{const} \quad \text{indep. of } A^\mu_{cl}.$$

The full effective action to one-loop order reads

$$\Gamma = \underset{\substack{\uparrow \\ \text{Maxwell}}}{S_M} + \Gamma^1 \tag{2.15}$$

where

$$\Gamma^1 = \underset{\substack{\uparrow \\ \text{fermionic} \\ \text{Grassman functional} \\ \text{integral}}}{-} \overset{\substack{\text{Minkowski space} \\ \swarrow}}{(i)} \ln\det(-i\not{D} + m) \tag{2.16}$$

In even dimensions, we can define a matrix "γ_5" that anticommutes with all other γ^μ's, $\{\gamma^\mu, \gamma_5\} = 0$, and that satisfies $\gamma_5^2 = \mathbb{1}$.

This implies that

$$\det(-i\not{D}+m) = \det\left[(-i\not{D}+m)\gamma_5^2\right] = \det\left[\gamma_5(i\not{D}+m)\gamma_5\right]$$

$$= \det(i\not{D}+m) \qquad (2.17)$$

$$\Rightarrow \Gamma^1 = -i \ln\det(-i\not{D}+m) = -\frac{i}{2}\left\{\ln\det(-i\not{D}+m) + \ln\det(i\not{D}+m)\right\}$$

$$= -\frac{i}{2}\ln\det(\not{D}^2+m^2) \qquad (2.18)$$

Here, we need

$$\not{D}^2 = D_\mu D_\nu \gamma^\mu\gamma^\nu = D_\mu D_\nu\left[\frac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{=-2g^{\mu\nu}} + \frac{1}{2}\underbrace{[\gamma^\mu,\gamma^\nu]}_{=-i\sigma^{\mu\nu}}\right]$$

$$= -D^2 - i\sigma^{\mu\nu}D_\mu D_\nu$$

$$= -D^2 - i\sigma^{\mu\nu}\,\frac{1}{2}\underbrace{[D_\mu, D_\nu]}_{=ieF_{\mu\nu}}$$

$$= -D^2 - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} \qquad (2.19)$$

Inserting (2.19) into (2.18) and using the proper time representation, we arrive at

$$\Gamma^1 = -\frac{i}{2} \mathrm{Tr} \ln \left(-D^2 + m^2 - \frac{e}{2} \sigma F \right)$$

$$= \frac{i}{2} \int_{1/\Lambda^2}^{\infty} \frac{dT}{T} \mathrm{Tr}\, e^{-(-D^2+m^2-\frac{e}{2}\sigma F)T} \qquad (2.20)$$

where we have again used a proper time regulator for the UV divergences. For performing the trace over the spectrum, we use a less general simplifying background field in terms of a constant homogeneous magnetic field (let us stress that (2.20) is valid for an arbitrary EM background.)

$$A_{cl.\mu} = \frac{1}{2} B \begin{pmatrix} 0 \\ -y \\ x \\ 0 \end{pmatrix} \Rightarrow \vec{B} = B\, \hat{e}_z \qquad (2.21)$$

$$F_{12} = -F_{21} = B$$

$$\Rightarrow -D^2 = -\left(-\partial_t^2 + \partial_z^2 + D_\perp^2[A] \right)$$

$$-\frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu} = -e\sigma^{12} F_{12} = -eB \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad (2.22)$$

The eigenvalues of $-D^2$ are a sum of plane wave eigenvalues $-p_t^2 + p_z^2$ (or after rotation to Euclidean time $p_t \to i p_\tau$), $p_\tau^2 + p_z^2$, and Landau-levels known from the QM problem of a particle in a magnetic field, $-D_\perp^2 \to eB(2n+1)$, $n = 0, 1, \ldots$

such that

$$\Gamma^{\Lambda} = \frac{i}{2}\int_{1/\Lambda^2}^{\infty}\frac{dT}{T}\, e^{-m^2T}\, \mathrm{Tr}\, e^{-(p_t^2+p_z^2)T}\, e^{-eB(2n+1)T}\, e^{\frac{e}{2}\sigma F T} \tag{2.23}$$

The trace reads

$$\mathrm{Tr} \rightarrow i\int_{-\infty}^{\infty}\frac{dp_t}{2\pi/L}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi/L}\sum_{n=0}^{\infty} g(n)\, \mathrm{tr}_\gamma \tag{2.24}$$

where the density of states $g(n)$ for the Landau levels can be worked out from the limit $eB \Rightarrow 0$ where the sum has to become $\rightarrow \int\frac{dp_\perp^2}{(2\pi/L)^2}$ again. We find

$$g(n) = L^2\,\frac{eB}{2\pi}\,. \tag{2.25}$$

The Dirac trace only affects the Pauli term $\sim \sigma F$:

$$\mathrm{tr}_\gamma\, e^{\frac{e}{2}\sigma F T} \underset{\substack{\uparrow\\ \text{magnetic}\\ \text{field}}}{=} 2\left(e^{eBT} + e^{-eBT}\right) = 4\cosh eBT \tag{2.26}$$

The sum over n yields

$$\sum_{n=0}^{\infty} e^{-eB(2n+1)T} = \frac{e^{-eBT}}{1-e^{-2eBT}} = \frac{1}{2\sinh eBT} \tag{2.27}$$

Combining all these findings, we obtain for the effective Lagrangian $\Gamma^1 = \int d^4x\, \mathcal{L}^1 \underset{\substack{\uparrow \\ \text{hom.} \\ \text{fields}}}{=} \Omega\, \mathcal{L}^1$

where $\Omega = L^4$:

$$\mathcal{L}^1(B) = -\frac{1}{8\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{dT}{T^3}\, e^{-m^2 T}\, eBT \coth eBT \tag{2.28}$$

(unrenormalized) Heisenberg-Euler Lagrangian.

Normalizing the effective action to $\mathcal{L}^1(B=0) = 0$ corresponds dividing the B-field dependent determinant by the $B \to 0$ limit. This is simply achieved by the replacement

$$eBT \coth eBT \to eBT \coth eBT - 1 \tag{2.29}$$

Still, there is a log-type divergence corresponding to the next Taylor coefficient of the integrand:

$$\begin{aligned}
\mathcal{L}^1(B) &= -\frac{1}{8\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{dT}{T^3}\, e^{-m^2T} \left(eBT \coth eBT - 1\right) \\
&= -\frac{1}{8\pi^2} \int_{1/\Lambda^2 \to 0}^{\infty} \frac{dT}{T^3}\, e^{-m^2T} \left(eBT \coth eBT - \frac{1}{3}(eBT)^2 - 1\right) \\
&\quad - \frac{1}{24\pi^2} (eB)^2 \underbrace{\int_{1/\Lambda^2}^{\infty} \frac{dT}{T}\, e^{-m^2T}}_{= \ln \frac{\Lambda^2}{m^2} + \text{const.} + \mathcal{O}\left(\frac{m^2}{\Lambda^2}\right)}
\end{aligned} \tag{2.30}$$

The divergent piece is proportional to $\sim (eB)^2 \sim \propto B^2$ and hence is of the same type as the Maxwell action. According to renormalization theory, we can merge this piece with the bare Maxwell action in order to define renormalized couplings and fields:

$$\mathcal{L}_{eff} = \underbrace{\mathcal{L}_M}_{= -\frac{1}{2}B^2} + \mathcal{L}^1 = -\frac{1}{2}\left(1 + \frac{1}{12\pi^2} e^2 \ln \frac{\Lambda^2}{\mu^2} + ..\right) B^2 - \frac{e^2}{24\pi^2} \ln \frac{\mu^2}{m^2} B^2$$

$$- \frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \left(eBT \coth eBT - \frac{1}{3}(eBT)^2 - 1 \right)$$

(2.31)

Introducing the wave function renormalization

$$Z_F^{-1} = \left(1 + \frac{e}{12\pi^2} \ln \frac{\Lambda^2}{\mu^2} + .. \right)$$

we can define the renormalized field and coupling

$$B_R^2 = B^2 Z_F^{-1} \quad , \quad e_R^2 = e^2 Z_F \qquad (2.32)$$

This renders $e_R, B_R = e_R(\mu), B_R(\mu)$ dependent on an RG scale, but leaves the combination $eB = e_R B_R$ RG invariant.

We end up with the renormalized Heisenberg - Euler Lagrangian

$$\mathcal{L}_R = -\frac{1}{2}\left(1+\frac{e_R^2}{12\pi^2}\ln\frac{\mu^2}{m^2}\right)B_R^2$$

$$-\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\, e^{-m^2T}\left(e_RB_RT\coth e_RB_RT-\frac{1}{3}(e_RB_RT)^2-1\right) \tag{2.33}$$

plus Terms of higher-loop order.

It is conventional to choose an "on-shell" renormalization condition at $\mu = m$, such that

$$\frac{e_R^2}{4\pi} = \alpha_R(\mu=m) \simeq \frac{1}{137} \tag{2.34}$$

The scale dependence of the coupling is described by the β-function

$$\beta_{e^2} := \mu\,\partial_\mu\, e_R^2(\mu) = \mu\,\partial_\mu\, e^2\left(1-\frac{e^2}{12\pi^2}\ln \ln\frac{\Lambda^2}{\mu^2}+\dots\right)$$

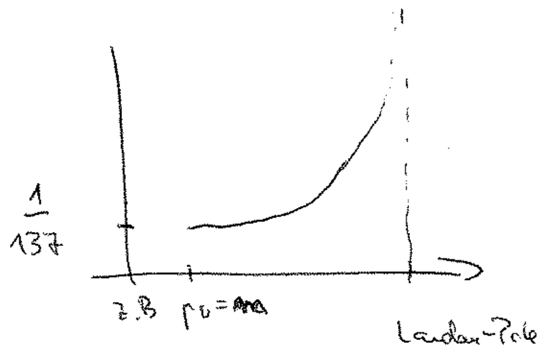
$$= \frac{1}{6\pi^2}\,e_R^4 + \mathcal{O}(e_R^6) \quad > 0 \tag{2.35}$$

Integrating the β_{e^2} function predicts a scale dependence of $e_R^2(\mu)$ of the type

$$e_R^2(\mu^2) = \frac{e_R^2(\mu_0^2)}{1 - \frac{e_R^2(\mu_0)}{12\pi^2} \ln \frac{\mu^2}{\mu_0^2}} \qquad (2.36)$$

yielding an increasing coupling with increasing momentum scale

The perturbative running yields a divergent coupling at the Landau pole



$$\mu_L = m^2 e^{\frac{12\pi^2}{e_R^2(m^2)}} \simeq m^2 e^{3\pi \cdot 137} \qquad (2.37)$$

which is far beyond any reasonable physics scale.
(NB: Still, also beyond perturbation theory there is evidence from Monte Carlo studies as well as functional methods that pure QED cannot be viewed as a fundamental quantum field theory.)

So far, we considered only the case of constant magnetic fields. The result can be generalized to

(a) constant electric fields

Boosting the present result to a different Lorentz-frame, yields

$$\mathcal{L}(B) \to \mathcal{L}(\sqrt{2\mathcal{F}}) \quad , \mathcal{F} = \frac{1}{2}(\vec{B}^2 - \vec{E}^2) \tag{2.38}$$

The case of a purely electric field can hence be obtained by analytic continuation:

$$\begin{aligned}\mathcal{L}^1(E) &= \mathcal{L}^1(B \to iE) \\ &= -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3} e^{-m^2T}\left(eET \cot eET + \frac{1}{3}(eET)^2 - 1\right)\end{aligned} \tag{2.39}$$

(b) For const. EM fields, the calculation has to be generalized to non-zero $\mathcal{F}$ and $\mathcal{G}$. The result is:

$$\mathcal{L}^1(\mathcal{F},\mathcal{G}) = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3} e^{-m^2T}\left\{(eT)^2|\mathcal{G}| \coth\left[eT\left(\sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F}\right)^{1/2}\right]\cot\left[eT\left(\sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F}\right)^{1/2}\right] - \frac{2}{3}(eT)^2\mathcal{F} - 1\right\} \tag{2.40}$$

Obviously, the action contains all orders of powers of $\mathcal{F}$ (and $\mathcal{G}^2$). This action hence encodes quantum corrections to class. ED in terms of nonlinearities of classical ED. This maps quantum phenomena onto a classical language. Expanding (2.40) to lowest orders in the fields yields

$$\mathcal{L}^1(\mathcal{F},\mathcal{G}) = \underbrace{\frac{8}{45}\frac{\alpha^2}{m^4}}_{=c_1}\mathcal{F}^2 + \underbrace{\frac{14}{45}\frac{\alpha^2}{m^4}}_{=c_2}\mathcal{G}^2 + \mathcal{O}\left(\frac{F^6}{m^8}\right) \tag{2.41}$$

as the weak-field limit of the Heisenberg-Euler action. The above-mention constants $c_{1,2}$ are hence fully predicted by QED. Diagrammatically, the expansion corresponds to (soft) photon scattering amplitudes

$\mathcal{L}^1 \sim$  $+ \; \ldots$

(NB: as these considerations are still restricted to homogeneous fields ignoring derivative terms, the effective action remains purely local. For general fields also non-local effects will appear.)