1.4.2 Effective potential for scalar QED

Though we learned a number of technicalities in the last section, we did not learn much about the physics of fluctuation-induced vacuum structures, as our approximations failed in the interesting regime.
We therefore turn to a system, where the loop expansion works. This is the case for scalar QED

$$\mathcal{L}_E(\varphi, A_\mu) = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* (D_\mu \phi) + m^2 \phi^* \phi + \frac{\lambda}{4!} 4 (\phi^* \phi)^2 \tag{1.83}$$

where $\phi = \frac{1}{\sqrt{2}} (\varphi_1 + i \varphi_2) \in \mathbb{C}$ is a complex charged scalar field (e.g. the charged pions). The gauge field A_μ occurs in the covariant derivative

$$D_\mu = \partial_\mu + ie A_\mu \tag{1.84}$$

and the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.85}$$

The Lagrangian is symmetric under local $U(1)$ transformations: (gauge transformations)

$$\phi(x) \to e^{-ie\Lambda(x)}\,\phi(x) \quad , \; e^{-ie\Lambda(x)} \in U(1)$$

$$A_\mu(x) \to A_\mu + \partial_\mu \Lambda(x) \qquad (1.86)$$

where $\Lambda(x)$ is an arbitrary smooth function of spacetime.

If $m^2 < 0$ (for positive λ), the scalar field acquires a vacuum expectation value ϕ_0

$$\phi_0^* \phi_0 = \frac{1}{2} v^2 \, . \qquad (1.87)$$

The gauge symmetry suggests a non linear parametrization of $\phi(x)$:

$$\phi(x) = \frac{1}{\sqrt{2}}\, e^{i\xi(x)/v}\,\big(v + \sigma(x)\big) = \frac{1}{\sqrt{2}}\big(v + \sigma(x) + i\xi(x) + \ldots\big) \qquad (1.88)$$

For a given field configuration $\phi(x)$, $A_\mu(x)$, we can perform a special gauge transformation

$$\Lambda(x) = \frac{\xi(x)}{ev} \qquad (1.89)$$

such that

$$\phi'(x) = \frac{1}{\sqrt{2}}\,(v + \sigma(x))$$

$$A'_\mu(x) = A_\mu(x) + \frac{1}{ev}\,\partial_\mu \xi(x) \qquad (1.90)$$

In terms of the new fields $\sigma(x), \xi(x), A'_\mu(x)$, the Lagrangian then reads

$$\mathcal{L}(\sigma, \xi, A'_\mu) = \frac{1}{4}(F'_{\mu\nu})^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2} e^2 v^2 A'_\mu A'_\mu$$

$$+ \frac{1}{2} e^2 (A'_\mu)^2 \sigma (2v + \sigma) + \frac{1}{2} \sigma^2 (3\lambda v^2 + m^2)$$

$$+ \mathcal{O}(\sigma^3, \sigma^4) \qquad (1.91)$$

Here we observe that σ occurs as a massive scalar and A'_μ as a massive photon. Most surprisingly, $\xi(x)$ has vanished completely. The latter observation is compatible with the counting of degrees of freedom: in the initial formulation (implicitly assuming $m^2 > 0$), we had 2 real scalar DoFs and 2 photon polarization modes (2 transversal modes). Now, we find 1 real scalar and 3 photon polarization modes (2 transv. and 1 longitudinal). The would-be Nambu-Goldstone boson ξ has been "eaten up" by the photon. This is known as the Higgs (Anderson, Brout, Englert, Kibble, Hagen, Guralnik) mechanism. The model is also called the abelian Higgs-model and plays a prominent role in 3d in the theory of superconductors.

We finally emphasize that the above analysis involved a special gauge choice (fixed by hand). Though the Higgs mechanism is sometimes referred to as the spontaneous breaking of gauge symmetry, this is a somewhat unfortunate nomenclature. By merely choosing a convenient gauge as we did above, the gauge symmetry is not broken (in fact it can't be broken according to Elitzur's theorem). The point is that particular gauges are convenient to identify the perturbative field content.

A particular gauge choice is also useful for performing the field quantization. Here, we choose a different gauge, the Landau-Lorenz gauge, classically satisfying the constraint

$$\partial_\mu A_\mu = 0 \tag{1.92}$$

We implement this constraint by a δ-functional

$$\delta(\partial_\mu A_\mu) = \lim_{\alpha \to 0} \mathcal{N}\, e^{-\frac{1}{2\alpha}\int (\partial_\mu A_\mu)^2} \tag{1.93}$$

to be inserted into the Schwinger functional,

$$Z[j_A, j_\varphi] = \int \mathcal{D}\phi \mathcal{D}A \; e^{-S[\varphi, A] + \int j_A A + \int j_\varphi \varphi - \frac{1}{2\alpha}\int (\partial_\mu A_\mu)^2} \tag{1.94}$$

where the limit $\alpha \to 0$ is implicitly understood, and the normalization of (1.93) has been absorbed into the functional measure.

Let us compute the one-loop effective potential, assuming a homogeneous background scalar field in the "real" direction $\phi_0 = \frac{1}{\sqrt{2}}(\varphi_{1,0} + i\varphi_{2,0})$, $\varphi_{1,0} \in \mathbb{R}$, $\varphi_{2,0} = 0$, $A_\mu = 0$

$$\Rightarrow \quad V(\phi) = m^2 \phi^* \phi + \frac{\lambda}{4!} 4 (\phi^* \phi)^2 = \frac{1}{2}(\varphi_1^2 + \varphi_2^2) + \frac{\lambda}{4!}(\varphi_1^2 + \varphi_2^2)^2$$

$$(D_\mu \phi)^* (D_\mu \phi) = \frac{1}{2}(\partial_\mu \varphi_1 - e A_\mu \varphi_2)^2 + \frac{1}{2}(\partial_\mu \varphi_2 + e A_\mu \varphi_1)^2 \tag{1.95}$$

In field space, the fluctuation operator reads

$$S^{(2)}_{\varphi_1\varphi_1}[\varphi_{1,0}] = -\partial^2 + m^2 + \frac{\lambda}{2}\varphi_{1,0}^2$$

$$S^{(2)}_{\varphi_2\varphi_2}[\varphi_{1,0}] = -\partial^2 + m^2 + \frac{1}{3}\frac{\lambda}{2}\varphi_{1,0}^2$$

$$S^{(2)}_{\varphi_1\varphi_2}[\varphi_{1,0}] \sim \varphi_{1,0}\varphi_{2,0}\Big|_{\varphi_{2,0}=0} = 0 \tag{1.96}$$

$$S^{(2)}_{A_\mu A_\nu}[\varphi_{1,0}] = \left(-\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu - \frac{1}{\alpha}\partial_\mu\partial_\nu\right) + e^2 \varphi_{1,0}^2 \, \delta_{\mu\nu}$$

$$S^{(2)}_{\varphi A_\mu} \sim \partial_\mu \varphi = 0 \quad \text{for } \varphi_{1,0} = \text{const}$$

Hence $S^{(2)}$ is block-diagonal in the components $\varphi_1, \varphi_2, A_\mu$. The calculation of the scalar trace-logs proceeds as in the pure scalar sector. Let us therefore take a closer look at the photon-loop. For this, we introduce the transversal and longitudinal projectors

$$P_{T\,\mu\nu} = \delta_{\mu\nu} - \frac{-\partial_\mu \partial_\nu}{-\partial^2} \quad , \; P_{L\,\mu\nu} = \frac{-\partial_\mu \partial_\nu}{-\partial^2} \tag{1.97}$$

$$\Rightarrow \; P_{T,L}^2 = P_{T,L} \quad , \; P_T + P_L = \mathbb{1} \quad , \; P_T P_L = 0$$

and write

$$S^{(2)}_{A_\mu A_\nu}[\varphi_{1,0}] = P_{T\,\mu\nu}\left(-\partial^2 + e^2 \varphi_{1,0}^2\right) + P_{L\,\mu\nu}\left(\frac{1}{\alpha}(-\partial^2) + e^2 \varphi_{1,0}^2\right).$$

This implies for the trace-log in the photon sector:

$$\mathrm{Tr} \ln \frac{S^{(2)}_{AA}[\varphi_{1,0}]}{S^{(2)}_{AA}[0]} = \underbrace{\mathrm{Tr}}_{=\mathrm{tr}_x \mathrm{tr}_L} P_T \ln \frac{(-\partial^2 + e^2 \varphi_{1,0}^2)}{-\partial^2} + \mathrm{Tr}\, P_L \underbrace{\ln \frac{\frac{1}{\alpha}(-\partial^2) + e^2 \varphi_{1,0}^2}{\frac{1}{\alpha}(-\partial^2)}}_{\to 0 \text{ for } \alpha \to 0}$$

$$= \underbrace{\mathrm{tr}_L P_T}_{= P_{T\,\mu\mu} = 3} \mathrm{tr}_x \ln \frac{(-\partial^2 + e^2 \varphi_{1,0}^2)}{-\partial^2} = 3\, \mathrm{tr}_x \ln \frac{(-\partial^2 + e^2 \varphi_{1,0}^2)}{-\partial^2} \tag{1.98}$$

which also boils down to a calculation similar to that for a real scalar field. Writing $\varphi_{1,0} = \varphi$, we find (for $m_R^2 = 0$)

$$V_{eff}(\varphi) = \frac{\lambda_R}{4!}\varphi^4 + \underline{\underline{\frac{C}{64\pi^2}\varphi^4\left(\ln\frac{\varphi^2}{\mu^2} - \frac{25}{6}\right)}} \qquad (1.99)$$

where $$\underline{\underline{C = \left(\left(\frac{\lambda_R}{2}\right)^2 + \frac{1}{9}\left(\frac{\lambda_R}{2}\right)^2 + 3e^4\right)}}$$

This is the celebrated Coleman-Weinberg effective potential.

Since the renormalization point μ is arbitrary, let us choose $\mu^2 = v^2$ (vacuum expectation value). This choice simultaneously fixes the couplings λ_R and e to each other. From

$$V_{eff}(\varphi) = \frac{\lambda_R}{4!}\varphi^4 + \frac{C}{64\pi^2}\varphi^4\left(\ln\frac{\varphi^2}{v^2} - \frac{25}{6}\right), \qquad (1.100)$$

we obtain the minimum condition

$$0 = V'_{eff}(\varphi)\Big|_{\varphi=v} = \left(\frac{\lambda_R}{6} - \frac{11}{3}\frac{C}{16\pi^2}\right)v^3 \qquad (1.101)$$

For $v \neq 0$, this implies

$$\lambda_R = \frac{11}{8\pi^2}C \quad , \qquad (1.102)$$

which to lowest-order in the couplings boils down to

$$\lambda_R = \frac{33}{8\pi^2} e^4 . \tag{1.103}$$

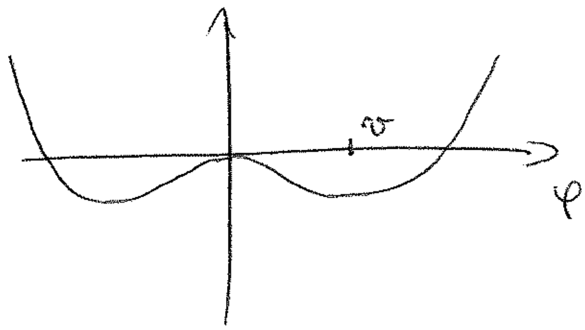
On the level of counting parameters, we started out with 2 parameters λ_R, e and now ended up with 2 parameters e, v. However, while λ_R and e are dimensionless, v is dimensionful. It simply reflects the fact that the definition of the theory required a regularization scale Λ. Even though the value of Λ has no physical meaning (at least it is not necessarily related to a physical scale), the initial occurrence of this scale leaves a remnant in the system ($\Lambda \to \mu$) which ultimately is converted into v. This is known as "dimensional transmutation".

(More conceptually, this is related to the fact that the scale invariance of the massless classical theory is not preserved on the quantum level due to the scale anomaly of the functional integral.)

Eliminating λ_R using (1.103) to first order, yields

$$V_{eff}(\varphi) = \frac{3e^4}{64\pi^2}\varphi^4\left(\ln\frac{\varphi^2}{v^2} - \frac{1}{2}\right) \qquad (1.104)$$

which exhibits a non-zero minimum.



The perturbative excitations on top of the condensate are

- a scalar boson with mass $m_\sigma^2 = V''(\varphi = v) = \frac{3e^4}{8\pi^2}v^2$ (1.105a)
- a vector-particle with mass $m_A^2 = e^2 v^2$ (1.105b)

and a corresponding mass ratio of

$$\frac{m_\sigma^2}{m_A^2} = \frac{3}{2\pi}\,\frac{e^2}{4\pi}. \qquad (1.106)$$

In the present model, we indeed find the generation of a nontrivial vacuum state (& corresponding excitations on top of the condensate) induced by quantum fluctuations. Note that the result holds for an arbitrarily weak-coupling e^2 such that the counter-argument of the purely scalar case does not apply.

Beyond any concrete application, the Coleman-Weinberg mechanism is characteristic for the influence of quantum fluctuations on the properties of the vacuum. Aside from the applications of this model in $d=3$ in the context of superconductivity, it has initially been speculated that a similar mechanism in a non abelian version of the model could be responsible for the value of the Higgs mass (analogous to (1.105a)).

However, the corresponding Higgs mass values would be on the order of a few GeV and are thus ruled out by experiment.

Of course, even if the Higgs mass came out correctly, there is no a-priori reason known, why the renormalized mass parameter $\sim \varphi^2$ should vanish identically.

Still, the Coleman-Weinberg mechanism is active in the standard model, as – even for finite values of m_R – fluctuations induce finite contributions to the effective potential of the scalar Higgs field (being an $SU(2)_L$ doublet complex scalar)

In the standard model, the effective potential of the Higgs field receives contributions from scalar, (non-abelian) gauge boson and fermion fluctuations:

$$\text{Higgs}: \quad V_H(H) \sim \frac{m_h^4}{16\pi^2 v^4} H^4 \ln \frac{H^2}{v^2}$$

$$\text{gauge}: \quad V_{W,Z}(H) \sim \frac{m_{W,Z}^4}{16\pi^2 v^4} H^4 \ln \frac{H^2}{v^2} \tag{1.107}$$

$$\text{fermions}: \quad V_f(H) \sim -\frac{m_f^4}{16\pi^2 v^4} H^4 \ln \frac{H^2}{v^2}$$

From measurements, we know that

$$\text{VEV}: \quad v \simeq 246 \text{ GeV}$$
$$\text{gauge boson masses}: \quad m_{W,Z} \simeq 80, 91 \text{ GeV} \tag{1.108}$$

In the fermion sector, the top quark dominates because of its heavy mass:

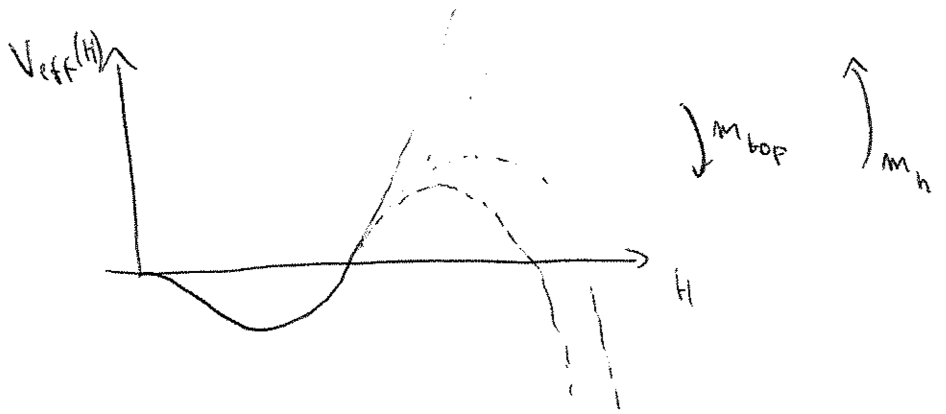
fermions:

$$m_{top} \simeq 173 \pm 1.5 \text{ GeV } (\pm \text{ scheme dependence } \mathcal{O}(5\text{–}10 \text{ GeV})) \quad (1.109)$$

Recently, also indications for a Higgs particle (being the excitation on top of the VEV) have been collected:

$$m_h \simeq 125 \text{ GeV} . \quad (1.110)$$

For the full effective potential, this yields naively



where an instability for large Higgs amplitudes seems to possibly occur if the Higgs is too light (or the top too heavy). In fact, this instability is likely to be an artefact of perturbation theory (similar to the Coleman-Weinberg minimum in pure Φ^4 theory). Still, this instability has been widely used in the literature to argue that the Higgs mass should be bounded from below. The currently detected Higgs mass seems to be near this lower bound. As this bound depends on the precise mass of m_{top} (and other uncertainties), the final answer has still to be given.

A proper theoretical analysis of the Higgs mass bounds requires renormalization theory and is beyond the scope of this lecture course.

Still, it is a very interesting (and perhaps even pressing) question, how a QFT behaves, if it sits in the "wrong" vacuum.