1.4 Coleman-Weinberg mechanism

In sections 1.1 & 1.2, we have already seen a couple of examples in classical field theory, where the ground state is non trivial and exhibits interesting physical properties. Let us now demonstrate that quantum fluctuations can actually induce such an interesting ground state structure. As a warm-up, we compute the

1.4.1 Effective potential for a ϕ^4 theory

We consider a simple field theory for a single real scalar with bare Euclidean action

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4 . \tag{1.58}$$

The corresponding effective action to one-loop order is

$$\Gamma[\phi] = S[\phi] + \frac{1}{2} \ln \det \frac{\delta^2 S[\phi]}{\delta\phi\delta\phi} =: S[\phi] + \Gamma^1[\phi] \tag{1.59}$$

Assuming that the ground state is homogeneous, it suffices to consider $\phi = \text{const.}$, and to compute the effective potential

$$V_{\text{eff}}(\phi) = V^0(\phi) + V^1(\phi) = \underbrace{\frac{1}{2} m^2\phi^2 + \frac{\lambda}{4!}\phi^4}_{V^0} + \underbrace{\frac{1}{2\Omega} \ln\det S^{(2)}[\phi]}_{V^1} \qquad (1.60)$$

where $\Omega = \int d^4x$ denotes the spacetime volume.

Here, we need

$$S^{(2)}[\phi] = -\partial^2 + m^2 + \frac{\lambda}{2}\phi^2 \qquad (1.61)$$

such that

$$V^1_{\text{eff}} = \frac{1}{2\Omega} \ln\det\left(-\partial^2 + m^2 + \frac{\lambda}{2}\phi^2\right) \qquad (1.62)$$

Actually, in writing down the ln det formula, we have ignored so far the normalization of the functional integral. As this normalization acts multiplicative on the functional integral, it acts additive on the effective action; i.e. we can use it for a zero-point shift of V_{eff}.

For this, we impose the condition

$$V^1_{\text{eff}}(\phi = 0) \stackrel{!}{=} 0 \qquad (1.63)$$

yielding

$$V^1_{eff}(\varphi) = \frac{1}{2\Omega} \ln \det \frac{(-\partial^2 + m^2 + \frac{\lambda}{2}\varphi^2)}{(-\partial^2 + m^2)}$$

$$= \frac{1}{2\Omega} \operatorname{Tr} \ln \frac{(-\partial^2 + m^2 + \frac{\lambda}{2}\varphi^2)}{(-\partial^2 + m^2)} \qquad (1.64)$$

Here we use Frullani's formula

$$\ln \frac{A}{B} = -\int_0^\infty \frac{dT}{T} \left(e^{-TA} - e^{-TB}\right) \qquad (1.65)$$

which is also called the <u>proper time representation</u>.

$$\Rightarrow V^1_{eff}(\varphi) = -\frac{1}{2\Omega}\int_0^\infty \frac{dT}{T} \underbrace{\operatorname{Tr}}_{= \int \frac{d^4p}{(2\pi/L)^4}} \left[e^{-T(-\partial^2 + m^2 + \frac{\lambda}{2}\varphi^2)} - e^{-T(-\partial^2 + m^2)}\right]$$

$\Omega = L^4$

$$= -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2 T}\left(e^{-T\frac{\lambda}{2}\varphi^2} - 1\right) \underbrace{\int \frac{d^4p}{(2\pi)^4} e^{-Tp^2}}_{= \frac{1}{(2\pi T)^2}}$$

$$= -\frac{1}{32\pi^2}\int_0^\infty \frac{dT}{T^3} e^{-m^2 T}\left(e^{-T\frac{\lambda}{2}\varphi^2} - 1\right) \qquad (1.66)$$

The resulting integral is finite at the upper bound $T \to \infty$, but diverges at the lower bound $T \to 0$, as becomes obvious from a Taylor expansion of the integrand:

$$\int^{\infty} \frac{dT}{T^3} e^{-m^2 T} \left(-T \frac{\lambda}{2} \phi^2 + \frac{1}{2} T^2 \frac{\lambda^2}{4} \phi^4 + \mathcal{O}(T^3 \lambda^3 \phi^6) \right) \tag{1.67}$$

Formally, Frullani's formulae is a Laplace transformation. Similarly to a Fourier transformation which interconnects high momenta with short distances $p \leftrightarrow \frac{1}{x}$, the Laplace transformation interconnects high momenta with short propertimes $p \leftrightarrow \frac{1}{T}$. We thus interpret the $T \to 0$ singularity of (1.67) as a high momentum "ultra violet" (UV) singularity. In order to deal with it, we make the integral finite by hand and introduce a "propertime cutoff" $\int_0^{\infty} dT \to \int_{T_{min}}^{\infty} dT$. As T has a mass dimension $[T] = -2$, we can relate T_{min} to a UV momentum cutoff Λ by

$$T_{min} = \frac{1}{\Lambda^2} \tag{1.68}$$

This regularization procedure at this stage corresponds

to an artificial modification of the original theory.
However, it will turn out that carefully defined physical observables are in fact independent of this regularization procedure.

Let us proceed by computing the regularized finite integral:

$$V^1_{eff}(\phi) = -\frac{1}{32\pi^2}\int_{1/\Lambda^2}^{\infty}\frac{dT}{T^3}\, e^{-m^2T}\left(e^{-T\frac{\lambda}{2}\phi^2}-1\right)$$

$$= -\frac{1}{32\pi^2}\int_{1/\Lambda^2}^{\infty}\frac{dT}{T^3}\, e^{-m^2T}\left(e^{-T\frac{\lambda}{2}\phi^2}-1+T\frac{\lambda}{2}\phi^2\right)$$

$$+\frac{1}{32\pi^2}\underbrace{\int_{1/\Lambda^2}^{\infty}\frac{dT}{T^2}\, e^{-m^2T}}_{=\Lambda^2\left(1+\left(\gamma-1+\ln\frac{m^2}{\Lambda^2}\right)\frac{m^2}{\Lambda^2}\right)+\mathcal{O}(1)}\frac{\lambda}{2}\phi^2 \qquad (1.69)$$

Here, we have subtracted (and added back in) the first truly divergent term in the first integral.
One consequence is that the first line does not contain a term $\sim\phi^2$. As this corresponds to

a mass parameter, let us collect all mass-like terms $\sim \varphi^2$ up to one-loop order:

$$V_{\text{eff}}(\varphi)\Big|_{\varphi^2} = \left(V^0_{\text{eff}} + V^1_{\text{eff}}\right)\Big|_{\varphi^2} = \frac{1}{2}\left(m^2 + \frac{\lambda}{32\pi^2}\Lambda^2(1+\ldots)\right)\varphi^2 \qquad (1.70)$$

Now, it is important to realize that m^2 (though being the mass parameter in the classical theory) does not correspond to the mass parameter of the quantum theory. The point is that measurements at laboratory momentum scales cannot disentangle this "classical" or so-called <u>bare</u> mass parameter from the quantum contribution. All, we can do, is to observe the sum of both including the quantum fluctuations. Hence, we write

$$V^{''}_{\text{eff}}(\varphi=0)\Big|_{\mu} = m_R^2 \qquad (1.71)$$

where we call m_R^2 the <u>renormalized</u> mass parameter, and $|_\mu$ shall remind us of the fact, that any measurement is performed at a typical scale μ, corresponding to the resolution of the measurement.

Eq. (1.71) is an example for a <u>renormalization condition</u> that implements the fixing of physical parameters. Comparing (1.71) with (1.70) implies that we can arrange for m_R^2 being independent of the cutoff Λ if we adjust $m^2 = m^2(\Lambda)$ accordingly. We write

$$m_R^2 = m^2 + \delta m^2 \,, \quad \delta m^2 = \frac{\lambda}{32\pi^2} \Lambda^2 (1 + \dots) = \mathcal{O}(\lambda^1) \tag{1.72}$$

where δm^2 is called the mass shift being formally of higher order in the coupling λ. We should think of (1.72) as $m^2(\Lambda)$ being chosen such that the "divergencies" $\sim \Lambda^2$ cancel on the RHS, leaving m_R^2 finite and fixed (possibly to a physical experiment).

This allows to write the full effective potential as

$$V_{eff}(\phi) = \frac{1}{2} m_R^2 \phi^2 + \frac{\lambda}{4!}\phi^4 - \frac{1}{32\pi^2}\int_{1/\Lambda^2}^{\infty} \frac{dT}{T^3} e^{-m_R^2 T}\left(e^{-T\frac{\lambda}{2}\phi^2} - 1 + T\frac{\lambda}{2}\phi^2\right) \tag{1.72}$$

Here, we have also replaced m^2 by m_R^2 in the propertime exponent. This is justified by the fact that the difference δm^2 is of higher-order in λ. This difference hence is of the order of a two-loop calculation (NB: The art of perturbative renormalization corresponds to showing that all potential divergencies can be arranged such that they cancel order by order in perturbation theory. Here, we will assume that this miracle does happen.)

In the following, we will confine ourselves to the case which is most relevant to the Coleman-Weinberg mechanism, namely, $m_R^2 = 0$ (such that the classical potential is non-symmetry breaking)

$$\Rightarrow V_{eff}(\phi) = \frac{\lambda}{4!}\phi^4 - \frac{1}{32\pi^2}\int_{1/\Lambda^2}^{\infty}\frac{dT}{T^3}\left(e^{-T\frac{\lambda}{2}\phi^2} - 1 + T\frac{\lambda}{2}\phi^2\right)$$

$$= \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2\phi^4}{256\pi^2}\left(\ln\frac{\lambda\phi^2}{2\Lambda^2} - \left(\frac{3}{2} - \gamma\right)\right) + \mathcal{O}\left(\frac{1}{\Lambda^2}\right) \tag{1.73}$$

(If we had used a diffrent way of regularizing the integrand, we would have found the same ln-term; by contrast, the term $\sim \left(\frac{3}{2}-\gamma\right)$ does depend on the regularization.) Obviously, there is still a log-like divergence, but also another free bare parameter λ to be fixed. This, we do by fixing the renormalized coupling with the second renormalization condition

$$\lambda_R := \left.\frac{d^4 V_{eff}}{d\phi^4}\right|_{\mu}$$

$$= \lambda + \frac{3\lambda^2}{32\pi^2}\left(\ln\frac{\lambda\mu^2}{2\Lambda^2} + \frac{25}{6} - \left(\frac{3}{2}-\gamma\right)\right) \tag{1.74}$$

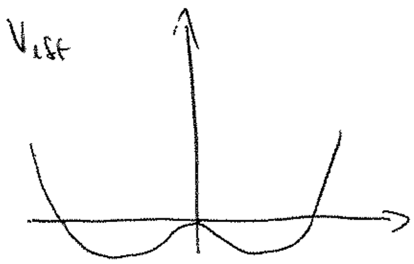
where we have identified the field amplitude ϕ in this definition of λ_R with the measurement scale $\phi \to \mu$ (other choices would also have been legitimate).

Again, arranging $\lambda = \lambda(\Lambda)$ such that λ_R is independent of Λ, we can write (1.73) in the form

$$V_{eff}(\phi) = \frac{\lambda_R}{4!}\phi^4 + \frac{\lambda_R^2}{256\pi^2}\phi^4\left(\ln\frac{\phi^2}{\mu^2} - \frac{25}{6}\right) \qquad (1.75)$$

This is the Coleman-Weinberg potential of ϕ^4 theory. Observations:

1) Plotting the potential, we observe



i.e. an apparently symmetry breaking potential, even though the classical action was symmetric (we will see below that this result should not be trusted).

2) The choice of the renormalization scale μ should be arbitrary. Choosing a different $\mu' \neq \mu$, we would have obtained a different definition of the renormalized coupling $\lambda_R' = \left.\frac{d^4 V_{eff}}{d\phi^4}\right|_{\mu'}$

However, the different couplings are related to each other (because the bare couplings would be the same for the two cases):

$$\lambda_R' = \lambda_R + \frac{3\lambda_R^2}{32\pi^2} \ln \frac{\mu'^2}{\mu^2} \qquad (1.76)$$

The effective potential would have remained form-invariant

$$V_{eff}(\phi) = \frac{\lambda_R'}{4!}\phi^4 + \frac{\lambda_R'^2}{256\pi^2}\phi^4\left(\ln\frac{\phi^2}{\mu'^2} - \frac{25}{6}\right) \qquad (1.77)$$

3) (1.76) indicates that the coupling must be viewed as a scale-dependent object. Eq. (1.76) characterizes this fact to first order in perturbation theory. It can also be rephrased in terms of a differential expression:

$$\beta(\lambda_R) = \mu \frac{\partial \lambda_R}{\partial \mu} = \frac{3\lambda_R^2}{16\pi^2} > 0 \qquad (1.78)$$

This "β function" being > 0 tells us that the coupling increases with increasing scale μ.

Integrating (1.78) yields

$$\lambda_R(\mu) = \frac{\lambda_R(\mu_0)}{1 - \frac{3}{32\pi^2}\lambda_R(\mu_0)\ln\frac{\mu^2}{\mu_0^2}} \qquad (1.79)$$

Obviously (1.76) agrees with (1.79) to order λ_R^2. Eq. (1.79) is called a result to first order in "RG-improved" perturbation theory. It arises from a perturbative computation but contains all orders in the coupling in the sense that leading-order terms have been resummed.

For this resummation in the form of a geometric series to be legitimate, we need $\left|\frac{3}{32\pi^2}\lambda_R(\mu_0)\ln\frac{\mu^2}{\mu_0^2}\right| < 1$. Naively extending (1.79) beyond this, we observe that $\lambda_R(\mu\to\mu_L)\to\infty$ diverges at a finite scale

$$\mu_L^2 = \mu_0^2\, e^{\frac{32\pi^2}{3\lambda_R(\mu_0)}} \qquad (1.80)$$

which is called the Landau pole. μ_L in this spirit is a naive charakterization of a validity scale of perturbation theory (or even of the full theory if the Landau pole existed beyond perturbation theory).

4) Eq. (1.75) predicts a nontrivial minimum to occur at

$$\lambda_R \ln \frac{v^2}{\mu^2} = -\frac{32}{3}\pi^2 + \mathcal{O}(\lambda) \qquad (1.81)$$

This violates the validity considerations of perturbative resummations. In fact, higher loops yield contributions of the form

$$\left(\lambda_R \ln \frac{v^2}{\mu^2}\right)^n$$

such that the loop expansion can not reliably address the question of symmetry breaking.

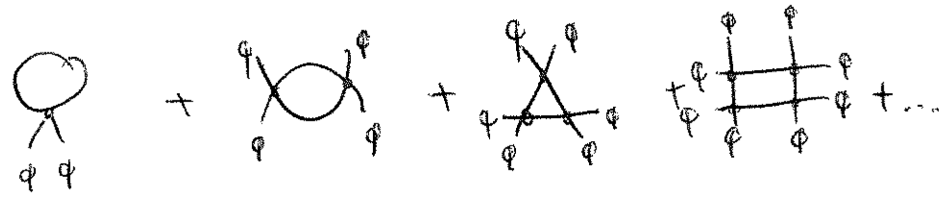
5) The effective potential can straightforwardly be mapped onto a Feynman diagram language,

using the formula

$$V^{1}_{\text{eff}}(\varphi) = \frac{1}{2\Omega} \ln \det \left(\frac{-\partial^2 + m^2 + \frac{\lambda}{2}\varphi^2}{-\partial^2 + m^2} \right)$$

$$\overset{\varphi = \text{const}}{=} \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \ln \left(1 + \frac{\lambda}{2} \frac{1}{p^2 + m^2} \varphi^2 \right)$$

$$= -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{n} \left(-\frac{\lambda}{2} \frac{1}{p^2 + m^2} \varphi^2 \right)^n \qquad (1.82)$$

$=$ 

with

$\frac{1}{p^2 + m^2}$: propagator (inner line)

$-\frac{\lambda}{2}$: vertex $\times$

φ : external line (const. in p)

Hence V^{1}_{eff} summarizes an infinite sum of all 1-loop Feynman diagrams in the limit of vanishing external momentum.