1.3 Basics of Quantum field theory

Physical information in QFT (particle spectra, cross sections, time evolution, etc.) can be extracted from correlation functions. In the path integral framework, they are obtained from

$$\langle \phi(x_1) \dots \phi(x_n) \rangle := \langle 0 | T [\underbrace{\phi(x_1,t_1) \dots \phi(x_n,t_n)}_{\text{Heisenberg - operators}}] | 0 \rangle$$

$$= \frac{\int_\Lambda \mathcal{D}\phi \, \phi(x_1) \dots \phi(x_n) \, e^{iS[\phi]}}{\int_\Lambda \mathcal{D}\phi \, e^{iS[\phi]}} \qquad (1.40)$$

where we have indicated a potentially necessary regularization with ultraviolet (UV) cutoff Λ.

All connected correlation functions can be summarized in terms of the <u>Schwinger</u> functional as the generating functional. Let us work in Euclidean space in the following (assuming that Minkowski-valued correlation functions can be obtained by suitable analytic continuation.

$$Z[J] = \exp W[J] = \int_\Lambda \mathcal{D}\chi \; e^{-S[\chi] + \int J\chi} \tag{1.41}$$

$$\Rightarrow \langle \chi(x_1) \dots \chi(x_n) \rangle = \frac{1}{Z[0]} \frac{\delta^n}{\delta J_1(x_1) \dots \delta J_n(x_n)} Z[J] \Big|_{J=0} \tag{1.42}$$

Here, $W[J]$ is the Schwinger functional generating the connected correlation functions whereas $Z[J]$ generates all correlation functions. An even more efficient "information storage" is given by the <u>effective action</u>. For this, we define the "classical field" (= Legendre conjugate to the source J):

$$\phi := \langle \chi \rangle_J = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J} = \frac{\delta W[J]}{\delta J} \tag{1.43}$$

$$\Rightarrow \quad \phi = \phi[J],$$

and define the effective action $\Gamma[\phi]$ by means of a Legendre transformation:

$$\Gamma[\phi] := -W[J] + \int J\phi \qquad (1.44)$$

where $J = J[\phi]$ is implicitely understood as the inverse function of (1.43). From this, we get

$$\underline{\underline{\frac{\delta\Gamma[\phi]}{\delta\phi}}} = -\int \underbrace{\frac{\delta W[J]}{\delta J}}_{\substack{=\phi \\ (1.43)}} \frac{\delta J}{\delta\phi} + \int \frac{\delta J}{\delta\phi}\phi + J = \underline{\underline{J}} \qquad (1.45)$$

As ϕ is the expectation value of a quantum field in the presence of a source J, we can interpret (1.45) as the <u>quantum equation of motion</u>. Hence Γ truly deserves the title "effective action", as it governs the dynamics of the quantum expectation values of the fields much in the same way as a classical action governs the dynamics of classical fields. The difference, of course, is that $\Gamma[\phi]$ contains all information of the full quantum field theory. This becomes also visible from the fact that $\Gamma[\phi]$ is obtained from a full functional integral:

$$e^{-\Gamma[\phi]+\int J\phi} \underset{(1.44)}{=} e^{W[J]} \underset{(1.41)}{=} \int_\Lambda \mathcal{D}\chi \; e^{-S[\chi]+\int J\chi}$$

$$\Rightarrow \underline{\underline{e^{-\Gamma[\phi]}}} = \int_\Lambda \mathcal{D}\chi \; e^{-S[\chi] + \int \frac{\delta\Gamma[\phi]}{\delta\phi}(\chi-\phi)}$$

$$\overset{\chi\to\chi+\phi}{=} \underline{\underline{\int_\Lambda \mathcal{D}\chi \; e^{-S[\chi+\phi] + \int \frac{\delta\Gamma[\phi]}{\delta\phi}\chi}}} \qquad (1.46)$$

which is a functional integro-differential equation for the effective action.

In the present lecture course, we will mostly be concerned with a simple approximation of $\Gamma[\phi]$ in terms of a perturbative loop expansion (to lowest nontrivial order).

For this, we perform a Taylor expansion of $S[\chi+\phi]$:

$$S[\phi+\chi] = S[\phi] + \int \frac{\delta S[\phi]}{\delta\chi}\chi + \frac{1}{2}\iint \frac{\delta^2 S}{\delta\chi\,\delta\chi}[\phi]\,\chi\,\chi + \mathcal{O}(\chi^3), \qquad (1.47)$$

Such that

$$e^{-\Gamma[\phi]} \underset{(1.46)}{=} \int_\Lambda \mathcal{D}\chi \, e^{-S[\phi] - \int \underbrace{\left(\frac{\delta S[\phi]}{\delta \chi} - \frac{\delta \Gamma[\phi]}{\delta \phi}\right)}_{\text{qualitatively} \sim \mathcal{O}(\hbar)} \chi - \frac{1}{2}\int \chi \frac{\delta^2 S[\phi]}{\delta\chi\delta\chi}\chi + \mathcal{O}(\chi^3)}$$

$$\simeq e^{-S[\phi]} \, \det_\Lambda^{-1/2} \underbrace{\frac{\delta S[\phi]}{\delta \chi \, \delta \chi}}_{=: S^{(2)}} \tag{1.48}$$

Taking the logarithm, we obtain the "one-loop" effective action

$$\Gamma[\phi] = S[\phi] + \Gamma^1[\phi] \tag{1.49}$$

where

$$\Gamma^1[\phi] = \frac{1}{2} \ln \det S^{(2)}[\phi] = \frac{1}{2} \operatorname{Tr} \ln S^{(2)}[\phi] \tag{1.50}$$

Here, Eq. (1.48) is the functional generalization of a Gaußian integration. All these formal operations become plausible if one thinks of $S^{(2)}$ as, for instance, a symmetric matrix. All this will become more concrete in the following sections. (Or see my QFT lecture course.)

As an application, we can take a closer look now at the Goldstone theorem, demonstrating that it is not modified by quantum fluctuations.

The effective action is, of course, in general a complicated functional of the fields. Expanding it in terms of derivatives of ϕ in a local fashion, we can write (back in Minkowski space)

$$\Gamma[\phi] = \int d^Dx \left(-V(\phi) + \text{terms with derivatives}\right) \tag{1.51}$$

Here, $V(\phi)$ is not a classical potential, but the so-called "effective potential" obtained after integrating out all quantum fluctuations (e.g. its Taylor expansion will generally not terminate at ϕ^4-order.)

Let us now assume that $V(\phi)$ is minimized by a homogeneous field ϕ_0^a which is constant in space and time. Then

$$\frac{\partial}{\partial \phi^a} V \Big|_{\phi^a(x) = \phi_0^a} = 0 \tag{1.52}$$

Expanding V about this minimum, we get

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi-\phi_0)^a(\phi-\phi_0)^b \frac{\partial^2}{\partial\phi^a\partial\phi^b} V(\phi_0) + \dots \tag{1.53}$$

since the linear term vanishes at ϕ_0. The coefficient of the quadratic term

$$m^2_{ab} := \frac{\partial^2}{\partial\phi^a\partial\phi^b} V(\phi_0) \tag{1.54}$$

is a symmetric matrix the eigenvalues of which specify the masses of the fields. Since ϕ_0 is a minimum, these eigenvalues cannot be negative. Next, we assume that the theory has a continuous symmetry (obeyed by the bare action as well as the quantization procedure); the transformed field has the form

$$\phi^a \rightarrow \phi^a + \delta\phi^a \tag{1.55}$$

where $\delta\phi^a$ can be some function of all fields $\delta\phi^a = \delta\phi^a(\phi)$. Considering only constant fields, invariance of the effective action implies the invariance of the effective potential

$$V(\varphi^a) = V(\varphi^a + \delta\varphi^a)$$

$$\Rightarrow \quad \delta\varphi^a \frac{\partial}{\partial\varphi^a} V(\varphi) = 0 \, . \qquad (1.56)$$

Differentiating with respect to φ^b and setting $\varphi = \varphi_0$, we get

$$0 = \left.\frac{\delta(\delta\varphi^a)}{\delta\varphi^b}\right|_{\varphi_0} \underbrace{\left(\frac{\partial V(\varphi_0)}{\partial\varphi^a}\right)}_{=0} + \delta\varphi^a(\varphi_0)\, m_{ab} \qquad (1.57a)$$

$$= \delta\varphi^a(\varphi_0)\, m_{ab} \, . \qquad (1.57b)$$

If the transformation leaves φ_0 unchanged, then $\delta\varphi^a(\varphi_0) = 0$, and (1.57b) is trivially satisfied. A spontaneously broken symmetry is precisely one for which $\delta\varphi^a(\varphi_0) \neq 0$. In this case $\delta\varphi^a(\varphi_0)$ is an eigenvalue of the mass matrix with eigenvalue zero. This proves Goldstone's theorem: every continuous symmetry of the theory that is not a symmetry of the ground state φ_0 gives rise to a massless excitation corresponding to a Nambu-Goldstone boson.