1.2 General scalar field theories

Let us consider scalar fields $\phi(x)$ on a $D = d+1$ dimensional spacetime with Lagrange density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi) \qquad (1.20)$$

The corresponding EoM is

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = -V' + \partial^2\phi \qquad (1.21)$$

Here is a list of examples:

$\boxed{1}$ $V(\phi) = \frac{1}{2}m^2\phi^2$ (1.22)

EoM: $(-\partial^2 + m^2)\,\phi(x) = 0$ Klein-Gordon Eq. (1.23)

Fourier modes: $\phi(p)$ satisfy

$$(p^2 + m^2)\,\phi(p) = 0 \qquad (1.24)$$

$\Rightarrow$ wave packets move relativistically with dispersion relation $p_\mu p^\mu + m^2 = 0$

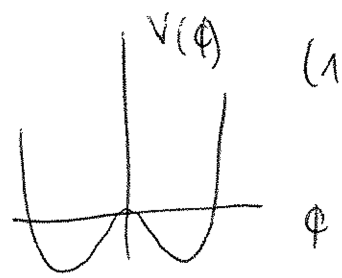
This describes a free theory with a trivial vacuum $\phi = 0$

$\boxed{2.}$ $$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \qquad (1.25)$$

EoM: $$\left(-\partial^2 + m^2 + \frac{\lambda}{3!} \phi^2\right) \phi = 0 \qquad (1.26)$$

describes an interacting theorie, still with a trivial vacuum $\phi = 0$

$\boxed{3.}$ $$V(\phi) = -\frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \qquad (1.27)$$



describes an interacting theory with a potential minimum at

$$\phi_0 \equiv v = \pm \sqrt{\frac{6\mu^2}{\lambda}} \qquad (1.28)$$

In its ground state, we expect a field value at $\phi^2 = v^2$

It is instructive to study the excitations on top of the vacuum:

$$\phi(x) = v + \sigma(x) \quad , \quad v = +\sqrt{\frac{6\mu^2}{\lambda}} \qquad (1.29)$$

$$\Rightarrow \mathcal{L} = -\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \left[\frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{3!}\lambda v\sigma^3 + \frac{1}{4!}\lambda\sigma^4\right] \qquad (1.30)$$

We conclude that σ corresponds to a scaler "particle" with $\text{mass}^2 = 2\mu^2$ and cubic and quartic self-interactions.

The following observation is very important: whereas the original potential (1.27) is symmetric under $\phi \to -\phi$, i.e. $V(-\phi) \equiv V(\phi)$, the potential term in square brackets in (1.30), $V_\sigma(\sigma) = [\%]$ is not (symmetric under $\sigma \to -\sigma$) because of the cubic piece. This is obvious, once we have noted that we have made a particular choice in (1.29): we have decided to pick the $+\sqrt{\ }$. Of course, picking $v = -\sqrt{\ }$ would have lead to the same conclusion that $V_{\bar{\sigma}}(-\bar{\sigma}) \neq V_{\bar{\sigma}}(\bar{\sigma})$. The mere fact that $v \neq 0$ is in conflict with the symmetry in the sense that the symmetry no longer holds for the excitations on top of the non trivial vacuum expectation value.

It is useful to introduce some nomenclature: if the ground state of a field corresponds to a non zero amplitude, we say that the field condenses. The value v of the amplitude in the ground state is called a condensate.

As the ground state no longer respects the symmetry of the Lagrangian, we talk about

Spontaneous symmetry breaking.

The symmetry transformations in the present case are $\phi \to \phi$ (trivially) and $\phi \to -\phi$, associated with the set $\{1, -1\}$. This set forms the group $\mathbb{Z}_2$. The presence of $v \neq 0$ breaks this symmetry group spontaneously.

4.] Let us promote the field ϕ of example 3.] to an N-component vector field ϕ^a, $a = 1 \dots N$.
The corresponding potential analogous to (1.30) is

$$V(\phi) = -\frac{1}{2}\mu^2 \phi^a \phi^a + \frac{\lambda}{4!}(\phi^a \phi^a)^2 \qquad (1.31)$$

It is easy to see that $V(\phi)$ is invariant under orthogonal transformations

$$\phi^a \to U^{ab}\phi^b \qquad \text{with } U^{ab}U^{ac} = (U^T U)^{bc} = (\mathbb{1})^{bc} = \delta^{bc} \qquad (1.32)$$

The set of $N \times N$ matrices U^{ab} satisfying (1.32) forms the group $O(N)$ (more precisely: corresponds to a matrix representation of this group). In fact, the full Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - V(\phi) \qquad (1.33)$$

is invariant under this $O(N)$ symmetry as long as the U's do not depend on spacetime. I.e. the fields $\phi(x)$ at different spacetime points have to be transformed with one and the same

U = const. In this case, we talk about a <u>global</u> symmetry.

(NB: by contrast, if we chose $U = U(x)$, the Lagrangian would not be invariant under this <u>local</u> symmetry, as the kinetic term would not be invariant in general.)

The vacuum expectation value, i.e. the minimum of $V(\phi)$ satisfies

$$\phi^a \phi^a = v^2 = \frac{6\mu^2}{\lambda} \,. \tag{1.34}$$

Let us choose to fulfil (1.34) with a field ϕ^a pointing into the N-direction:

$$\phi_0^a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ v \end{pmatrix} \,, \quad v = \sqrt{\frac{6\mu^2}{\lambda}} \tag{1.35}$$

Then the $O(N)$ symmetry is spontaneously broken, since a generic $O(N)$ transformation would rotate ϕ_0^a. However, it is important to note that a subset of $O(N)$ transformations leaves ϕ_0^a invariant. This is the subset of rotations about the axis defined by ϕ_0^a in field space.

One can convince oneself that this subset forms again a group, namely $O(N-1)$. We say that the ground state breaks $O(N)$ spontaneously to $O(N-1)$.

Now, it is interesting to study the excitations on top of the vacuum, which we parameterize by

$$\phi^a(x) = \begin{pmatrix} \pi^i(x) \\ v + \sigma(x) \end{pmatrix} \quad , \; i = 1, \ldots, N-1 \tag{1.36}$$

In terms of the fields $\pi^i(x)$ and $\sigma(x)$, the Lagrangian (1.33) reads

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \pi^i)^2 - \frac{1}{2}(\partial_\mu \sigma)^2 - V(\sigma, \pi^i) \tag{1.37}$$

where

$$V(\sigma,\pi^i) = \frac{1}{2}(2\mu^2)\sigma^2 + \sqrt{\frac{\lambda}{6}}\,\mu\,\sigma^3 + \sqrt{\frac{\lambda}{6}}\,\mu\,(\pi^i)^2\sigma$$
$$+ \frac{\lambda}{4!}\sigma^4 + \frac{\lambda}{12}(\pi^i)^2\sigma^2 + \frac{\lambda}{4!}\left[(\pi^i)^2\right]^2 \tag{1.38}$$

The following facts can be read off from (1.38):

- There is a scalar excitation $\sigma(x)$ with mass $m_\sigma^2 = 2\mu^2$ (c.f. (1.30)).

- The π^i and σ fields are interacting, as well as self-interacting.

- The Lagrangian is invariant under transformations of π^i by orthogonal $(N-1)\times(N-1)$ matrices

$$\pi^i \to U^{ij}\pi^j \quad \text{with } U^{ij} \in \mathcal{O}(N-1)$$

$\Rightarrow \mathcal{O}(N-1)$ residual symmetry.

- The π field <u>remains massless</u>, as there is no pure quadratic term in π^i.

The last point is particularly important: the spontaneous breaking of a <u>continuous</u> global symmetry $\mathcal{O}(N) \to \mathcal{O}(N-1)$, yields $N-1$ massless bosons. The latter are called <u>Nambu-Goldstone bosons</u> (often only "Goldstone bosons"). We will show below that the # of Goldstone bosons is equal to the

of "broken generators"

($\hat{=}$ number of generators of $O(N)$ that would generate transformations that would not leave the chosen ground state invariant.)

Let us verify this statement in the present example:

of $O(N)$ generators: $\quad n_{O(N)} = \frac{1}{2}N(N-1)$

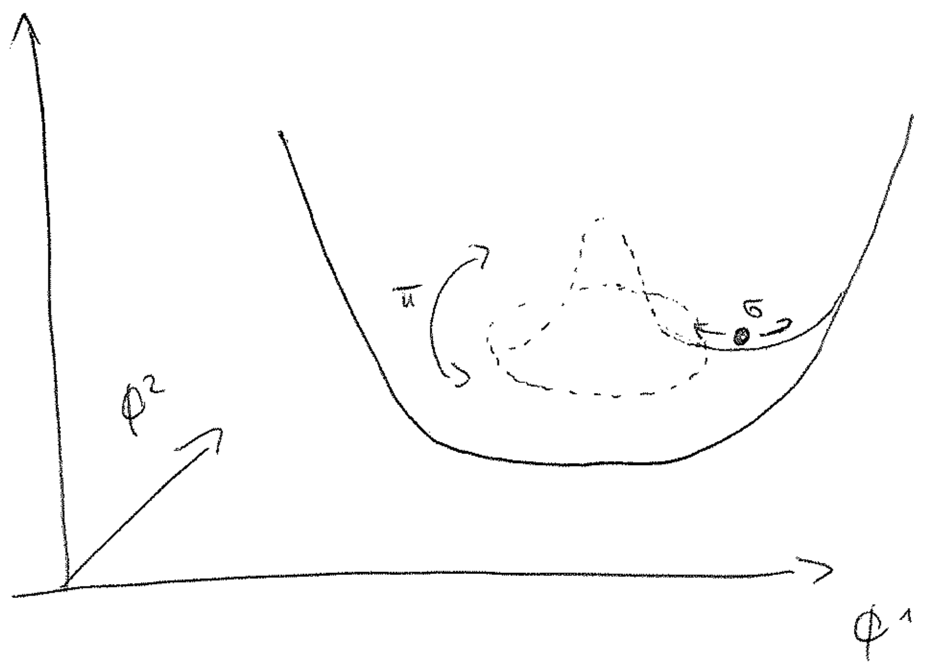
of $O(N-1)$ generators: $\quad n_{O(N-1)} = \frac{1}{2}(N-1)(N-2)$

$$\Rightarrow n_{O(N)} - n_{O(N-1)} = N - 1 \qquad (1.39)$$

$\hat{=}$ # of massless π^i fields.

The present example is a special case of the more general Goldstone Theorem to be discussed below. The notation in terms of σ and π fields is taken over from low-energy models of QCD: the QCD-Lagrangian (in terms of quarks & gluons) has an approximate chiral symmetry. In the case, where we consider only the up and down quark, the symmetry corresponds to independent "flavor" rotations, i.e. unitary transformations, of the left and right-handed components of the Dirac spinor fields. The symmetry group is

$$SU(2)_L \times SU(2)_R \underset{\substack{\uparrow \\ \text{isomorphic to}}}{\simeq} O(4)$$



The σ-field is also often called a "radial" excitation. In QCD, it corresponds to a heavy scalar mesonic resonance ($\sim O(1\,\text{GeV})$), whereas the π-fields, π^1, π^2, π^3 describe the light pions (mass ~ 135 MeV).

The fact that the pions are not completely massless is due to the violation of the chiral symmetry arising from the current quark masses ($\mathcal{O}$(3-5 MeV)).

The present model is also often called a "linear sigma model".

We will come back to the Goldstone theorem in the fully quantized version of field theory.